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HARPER'S
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HARPER'S EDUCATIONAL SERIES

° ADVANCED ARITHMETIC

FOR

SECONDARY SCHOOLS

(HIGH SCHOOLS, ACADEMIES, SEMINARIES, AND NORMAL SCHOOLS)

A COMPREHENSIVE TREATISE ON THE SCIENCE OF
NUMBERS AND THEIR APPLICATION TO
BUSINESS TRANSACTIONS

BY

JOHN H. FRENCH, LL.D.

AUTHOR OF "HARPER'S GRADED ARITHMETICS" ETC.

NEW YORK

HARPER & BROTHERS, FRANKLIN SQUARE

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PUBLISHERS' NOTE.

HARPER'S ADVANCED ARITHMETIC, by John H. French, LL.D., is the last work of that distinguished mathematician and educator, having been completed by him only a few days before his death. It embodies the results of life-long study and observation made serviceable by habits of critical analysis and a rare mathematical judgment. The principles and methods which it expounds have been evolved and perfected through an experience of nearly half a century in the practical work of the school-room and the teachers' institute. It is at once scholarly and complete, embracing the whole science of Arithmetic. The order of arrangement of topics, as well as the general plan of the work, may be understood by a study of the table of contents.

In view of the distinguished services which were rendered by Dr. French to the cause of education in this country, it is deemed fitting to preface this volume with the following brief sketch of his life:

JOHN H. FRENCH was born in Batavia, Genesee County, New York, July 7, 1824. While he was yet a child his father died, leaving a family of four children entirely dependent upon the labor and good management of their mother. As may be inferred, his early opportunities for obtaining an education were extremely limited. Working on a farm in the summer, and attending a district school for a few months each winter, he nevertheless acquired such a fund of information that at the age of seventeen he was able to teach a successful term of school in the town of Alabama, Genesee County. His life from this time, it may be said, was spent in the school-room or in work directly connected with it. In 1845, while teaching at Seneca Castle, in Ontario County, he made his first attempt at mathematical authorship by undertaking the revision of Adams's Arithmetic, at that time the standard work on that subject. Two years afterwards he went to Keene, N. H., where he wrote, for Adams's Series of Mathematics, a Mental Arithmetic, a work on Mensuration, and one on Book-keeping. In 1849 he was elected Principal of the High School at Clyde, N. Y., a position which he held for three years, and resigned to accept the principalship of the Newtown (Conn.) Academy. Returning to his native State in 1855, he engaged in the preparation and publication of town and city maps from actual surveys,

beginning with Waterloo, N. Y., and continuing with Geneva, Oswego, Auburn, Niagara Falls, and other places. In the following year, in connection with a Philadelphia publisher, he began the preparation of a map and gazetteer of the State of New York—a work which required four years' labor, the aid of numerous assistants, and an outlay of funds equal to the entire receipts. While prosecuting this enterprise Dr. French was also assisting in the revision of Robinson's Series of Mathematics. The algebras of this series were chiefly his own work. At about the same time he began the preparation of a series of Arithmetics, which was afterwards (1866-69) published by Harper & Brothers, and is known as French's Series. In 1860 the degree of LL.D. was conferred upon him by Iowa University, in recognition of his distinguished services as an educator. In 1865 he was appointed superintendent of the schools at Syracuse, a position which he soon afterwards resigned in order to become Superintendent of the Experimental Department of the State Normal School at Albany. In 1870 he was appointed Secretary of the State Board of Education for the State of Vermont, a position which he held five years. From 1878 to 1881 he was Principal of the State Normal School at Indiana, Pennsylvania, during which time he prepared the first two books of Harper's Graded Series of Arithmetics. Retiring, on account of ill health, from the principalship of the normal school, he was soon afterwards appointed Institute Conductor for the State of New York, a position which he retained during the remainder of his life. Dr. French was a man of varied attainments—a mathematician of national reputation, the discoverer of "French's Binomial Theorem," a teacher and organizer of recognized ability, an accomplished musician, an inventor of many articles of utility for schools, and withal a studious investigator and an indefatigable worker. He was a corresponding member of the New York Historical Society and of the Albany Institute, and for many years a member of the American Association for the Advancement of Science. He died at his home in Rochester, New York, December 23, 1888.

HARPER & BROTHERS.

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ADVANCED ARITHMETIC.

INTRODUCTION.

The four classes of numbers used in arithmetical computations are integers, decimals, fractions, and compound numbers; and the four fundamental processes by which arithmetical computations are made are addition, subtraction, multiplication, and division.

A preliminary knowledge of reductions and of some of the properties of numbers, is essential to a complete knowledge of any one of these four fundamental processes, in its application to the four classes of numbers.

Students of advanced arithmetic are familiar with the principles and methods found in the elementary text-books, and may therefore profitably study each of these subjects—as here scientifically arranged—in its application to the four classes of numbers.

PRELIMINARY DEFINITIONS.

Art. 1. *Quantity* is any thing that has size or value.

2. A *unit* is a *single thing*, a *one*, or any definite quantity regarded as a *one*.

One, one apple, one mile, one ten, one thousand, one thousandth, one third are units.

3. An *integral unit* is *one*, or a whole thing.

4. *Decimal parts* are parts obtained by dividing an integral unit into tenths, hundredths, thousandths, and so on.

5. *Fractional parts* are parts obtained by dividing an integral unit into any number of parts.

6. A *decimal unit* is one of the decimal parts into which an integral unit is divided.

7. A fractional unit is one of the equal fractional parts into which an integral unit is divided.

Any quantity whatever may be expressed by a unit or units.

Give examples | 2. Of fractional parts. | 4. Of a decimal unit.
 1. Of decimal parts. | 3. Of an integral unit. | 5. Of a fractional unit.

8. A number is a unit, or two or more units.

One, twelve, six hundred fifty-four men, four tenths, seven ninths, eight miles are numbers.

9. An integer is a number that consists of one or more integral units.

10. A decimal is a number that consists of one or more decimal units.

11. A fraction is a number that consists of one or more fractional units.

(See Art. 80, *a* and *b*).

12. A mixed number is a number that consists of an integer and a decimal, or an integer and a fraction.

Give examples | 7. Of a decimal. | 9. Of a mixed decimal number.
 6. Of an integer. | 8. Of a fraction. | 10. Of a mixed fractional number.

13. An abstract unit is a unit not associated with a thing or object.

Unity is the abstract unit 1.

14. A concrete unit is a unit associated with a thing or object.

15. A denominate unit is a concrete unit that expresses measure, weight, or money value.

Give examples

11. Of an abstract unit. | 12. Of a concrete unit. | 13. Of a denominate unit.

16. An abstract number is a number whose unit is abstract.

17. A concrete number is a number whose unit is concrete.

18. A denominate number is a concrete number whose unit is denominate.

Denomination is the name of the unit of a denominate number; *e. g.*, 75 gallons is a denominate number, the denomination is gallons, and the unit is 1 gallon.

Give examples of

14. An abstract number. | 15. A concrete number. | 16. A denominate number.

17. State the difference between a concrete number and a denominate number.

19. A *simple number* is a number that consists of a unit, or of units of the same kind.

20. A *compound number* is a number that consists of units of two or more denominations of the same kind of measure, weight, or money value.

a. Compound numbers are sometimes called *denominate numbers*; but a denominate number is properly a number of but one denomination of measure, weight, or money value.

b. A simple number may be abstract, concrete, or denominate.

c. A compound number is always denominate.

d. A denominate number may be an integer, as 7 bushels; a decimal, as .75 of a mile; a mixed number, as 5.125 gallons or $5\frac{1}{8}$ gallons; or a compound number, as 4 days 9 hours 20 minutes.

e. An abstract number is also called a *pure number*, and a concrete number an *applied number*.

Give examples

18. Of a simple number.

20. Of a pure number.

19. Of a compound number.

21. Of an applied number.

21. The *unit of a number* is *one* or a single thing, of the kind of which the number consists.

a. The *unit of an integer* is *one* or a single thing, of the kind of which the integer consists.

b. The *unit of a decimal* is *one* or a single thing, of the kind of which the decimal consists.

c. The *unit of a fraction* is *one* or a single thing, of the kind of which the fraction consists.

d. When any number is abstract, its unit is abstract; when the number is concrete, its unit is concrete; and when the number is denominate, its unit is denominate.

What is the unit

22. Of 25!

27. Of $19\frac{1}{12}$ dozen eggs!

23. Of 862 pounds of beef!

28. Of 54 thousand bricks!

24. Of \$2,050,019!

29. Of 72.41 meters of silk!

25. Of $\frac{1}{4}$ of a week!

30. Of 4,900 feet of timber!

26. Of .039 of a mile!

31. Of 987.056 tons of iron!

22. *Like numbers* are numbers that have the same unit.

23. *Unlike numbers* are numbers that have different units.

Give examples of like numbers; of unlike numbers.

24. A *cardinal number* is any abstract integer.

25. An *ordinal number* is a cardinal number modified to express an equal part, relative order or value.

a. Five, thirty-one, sixteen; 87, 352, 95, are cardinal numbers.

b. Fifth, thirty-first, sixteenth; 87th, 352d, 95th, are ordinal numbers.

Give

32. Five cardinal numbers, each less than 100.

33. Six cardinal numbers, each greater than 100.

34. Two ordinal numbers, expressing relative order.

35. Two ordinal numbers, expressing relative values.

36. Three ordinal numbers, in the plural, each expressing equal parts.

26. The *four classes of numbers* are integers, decimals, fractions, and compound numbers.

27. A *scale* is an established order of increase or decrease of the successive orders of units in a number.

28. A *uniform scale* is a scale in which the order of increase or decrease is always the same.

29. A *varying scale* is a scale in which the order of increase or decrease is not always the same.

a. A *decimal scale* is a scale in which the values of the successive orders of units increase by the constant multiplier 10, and decrease by the constant divisor 10.

b. The scale of integers is a decimal scale.

c. The scales of most compound numbers are varying scales.

Give examples

37. Of a scale. | 38. Of a uniform scale. | 39. Of a varying scale.

30. A *problem* is a statement of such conditions or relations existing between given numbers, requiring a result that can be obtained only by computation.

31. A *principle* is a general truth upon which a method of computation is based.

32. An *example* is a problem used to illustrate a principle, or in explaining a method of computation.

33. An *explanation* is a statement of each step in a process or computation.

34. An *analysis* is a statement, in logical order, of the reasons upon which a process or a method of computation is based.

35. A *demonstration* is a course of reasoning by which a truth is established.

36. A *proposition* is a statement of a truth to be demonstrated, or of a problem to be solved.

37. A *rule* is a brief direction for performing a computation.

38. *Arithmetical symbols* are *figures* and *letters*, used to express numbers; and *signs*, used to indicate operations, relations, or denominations.

I. Symbols of number.

Values :	naught or zero	one	two	three	four	five	six	seven	eight	nine
1. Figures :	0	1	2	3	4	5	6	7	8	9
2. Letters :	I	V	X	L	C	D	M			
3. The decimal point (.), followed by figures.										

II. Symbols of operation.

1. Addition,	+	4. Subtraction,	—
2. Multiplication,	×	5. Division,	÷
3. Involution,	7 ⁴	6. Evolution; { Square root,	√
		{ Cube root,	∛

III. Symbols of relation.

1. Equality,	=
2. Ratio,	:
3. Proportion,	: :: :

IV. Symbols of aggregation.

1. Vinculum,	—
2. Parenthesis,	()
3. Brackets,	[]

a. When numbers are connected by the sign of multiplication or division, the whole expression thus formed is always to be taken as one number.

$$\text{Thus, } 5 \times 2 \times 7 + \frac{1}{2} - 12 \div 4 = 70 + 5 - 3 = 72$$

b. Any numerical expression included in or under a sign of aggregation is considered as one number.

$$\text{Thus, } (7 + 8 - 3) \times 2 = 12 \times 2 = 24,$$

$$\text{and } 7 + \overline{8 - 3 \times 2} = 7 + \overline{8 - 6} = 7 + 2 = 9$$

$$\text{Whereas, } 7 + (8 - 3) \times 2 = 7 + 5 \times 2 = 17;$$

$$\text{and } 7 + 8 - 3 \times 2 = 15 - 6 = 9$$

The several signs of aggregation here given are used interchangeably.

c. Compound brackets are one sign of aggregation within another in the same numerical expression.

$$\text{Thus, } 15 + 2 \times [36 - (4 + 3 \times 7)] - 3 \times 8$$

Combining the expression in the compound brackets, we have

$$15 + 2 \times \overline{36 - 25} - 3 \times 8, \text{ and}$$

$$15 + 2 \times \overline{36 - 25} - 3 \times 8 = 15 + 22 - 24 = 13$$

$$\text{Whereas, } 15 + 2 \times 36 - 4 + 3 \times 7 - 3 \times 8 = 15 + 72 - 4 + 21 - 24 = 80$$

(See Supplement, page 365.)

V. Symbols of denomination.

Symbols of denomination are abbreviations or characters used to indicate denominations. They are given in the tables of compound numbers, in the Supplement.

39. An *equation*, in arithmetic, is an expression of the equality existing between one number or combination of numbers and another.

a. The statement *the sum of 15 and 9 equals the product of 4 and 6*, or *15 plus 9 equals 4 times 6*, is an arithmetical equation, and is written thus :

$$15 + 9 = 4 \times 6$$

b. Every written equation contains the sign of equality.

c. The *members of an equation* are the parts separated by the sign of equality.

d. The *first member* is the part preceding the sign.

e. The *second member* is the part following the sign.

f. Either member may consist of one number or of a combination of numbers.

Write six equations, as follows :

40. Two with the sign + or the sign — in one member of each.

41. Two with the sign + or the sign — in both members of each.

42. Two with the signs +, —, ×, and ÷ in both members of each.

40. *Arithmetic* is the science of numbers, and the art of using them.

As a *science*—or *pure* arithmetic—it investigates, classifies, and arranges the principles and rules of the subject.

As an *art*—or *applied* arithmetic—it employs the principles and rules for computation in the practical affairs of life.

CHAPTER I. NOTATION AND NUMERATION.

SECTION I. INTEGERS AND DECIMALS.

41. Numbers are commonly expressed by figures or letters. They may also be expressed by words.

42. *Notation* is the writing of numbers; and

Numeration is the reading of numbers.

The two methods of notation are the *Roman* and the *Arabic*.

43. The *Roman method of notation* is the method of expressing numbers by letters.

The principal uses of the Roman notation are for numbering volumes; the chapters, sections, and other principal divisions of books; heads of written discourses, and of general outlines or tabular views; physicians' prescriptions, etc. (See Supplement, page 365.)

44. The *Arabic method of notation* is the method of expressing numbers by figures.

45. The Arabic method of notation and numeration embraces

1. Figures and the values they express;
2. Orders of units, and periods; and
3. Principles and laws governing orders of units, periods, and values.

I. FIGURES AND VALUES.

46. The symbols of number used in the Arabic method, and the values they express, are as follows:

Figures:	0	1	2	3	4	5	6	7	8	9
Values:	naught	one	two	three	four	five	six	seven	eight	nine

a. The naught is often called *cipher* or *zero*.

b. The other figures are called *digits*.

c. Each digit expresses the number of *ones* indicated by its name.

47. To express an integer greater than nine, two or more figures are used.

Ten ones taken together are a *ten*, ten tens a *hundred*, ten hundreds a *thousand*, ten thousands a *ten-thousand*, 10 ten-thousands a *hundred-thousand*, 10 hundred-thousands a *million*, and so on. Hence,

48. A figure at the left of ones expresses *tens*; at the left of tens, *hundreds*; at the left of hundreds, *thousands*; at the left of thousands, *ten-thousands*; at the left of ten-thousands, *hundred-thousands*; at the left of hundred-thousands, *millions*; and so on.

49. The value expressed by a digit in a number is determined by two elements,—a *simple* or *independent* element, and a *local* or *place* element.

a. The *simple element* is the value expressed by the name of the digit; and

b. The *local element* is the value of a unit of the place the digit occupies in the number. Hence,

50. A digit expresses units of the place it occupies, and the place determines the value of these units.

i. The place being constant, the value expressed in it varies with the figure; and

ii. The figure being constant, the value expressed by it varies with the place.

1. Give examples illustrating i. | 2. Give examples illustrating ii.

II. ORDERS OF UNITS, AND PERIODS.

51. The *order of a unit* in an integer takes its name from the place it occupies.

A digit written in the first or right-hand place expresses *units of the first order* (ones); in the second place it expresses *units of the second order* (tens); in the third place, *units of the third order* (hundreds); in the fourth place, *units of the fourth order* (thousands), and so on, to higher orders.

52. A *period of figures* in an integer consists of three places—ones, tens, and hundreds.*

The first or right-hand period consists of *ones, tens, and hundreds*; the second of *ones, tens, and hundreds* of thousands; the third of *ones, tens, and hundreds* of millions; the fourth of *billions*; the fifth of *trillions*; the sixth of *quadrillions*; and so on, as shown below :

of sextillions.	of quintillions.	of quadrillions.	of trillions.	of billions.	of millions.	of thousands.	of units or ones.
1 2 4,	1 2 4,	1 2 4,	1 2 4,	1 2 4,	1 2 4,	1 2 4,	1 2 4
hundreds tens ones	hundreds tens ones	hundreds tens ones	hundreds tens ones	hundreds tens ones	hundreds tens ones	hundreds tens ones	hundreds tens ones
8th period.	7th period.	6th period.	5th period.	4th period.	3d period.	2d period.	1st period.

53. Facts concerning orders of units and periods.

1. 10 units of any order are 1 unit of the next higher order.
2. 1,000 units of any period are 1 unit of the next higher period.
3. The left-hand period may contain one, two, or three figures.
4. Periods of figures are separated by commas.

54. Derivation of names of periods.

Latin numerals.	Meaning.	Names of periods.	Numbers of periods.
—	—	units	1st
—	—	thousands	2d
M.	—	millions	3d
bis	twice	billions	4th
tres	three	trillions	5th
quatuor	four	quadrillions	6th
quinque	five	quintillions	7th
sex	six	sextillions	8th
septem	seven	septillions	9th
octo	eight	octillions	10th
novem	nine	nonillions	11th
decem	ten	decillions	12th
undecim	eleven	undecillions	13th
duodecim	twelve	duodecillions	14th

From examination of this table, it will be seen

1st. That the names of all periods higher than thousands consist of two parts—a principal part or root, derived from *mille* (the Latin word for thousand), and a Latin numeral prefix, that indicates the number of times *thousand* has been used as a factor.

2d. That in the name of any period, the number expressed by the Latin numeral prefix, plus two, is the number of the period.

What is the name of the

- | | | | |
|-------------------|--------------------|---------------------|-------------------|
| 3. Second period? | 5. Third period? | 7. Fifth period? | 9. Sixth period? |
| 4. Eighth period? | 6. Seventh period? | 8. Eleventh period? | 10. Ninth period? |

Reading from the left, what are the names of the successive periods of a number expressed by

- | | | | |
|---------------------|--------------------|---------------------|-----------------------|
| 11. Five figures? | 13. Ten figures? | 15. Six figures? | 17. Fourteen figures? |
| 12. Twelve figures? | 14. Seven figures? | 16. Eleven figures? | 18. Nineteen figures? |

What is the highest order of units in a number expressed by

- | | | | |
|-------------------|--------------------|----------------------|---------------------|
| 19. Six figures? | 21. Eight figures? | 23. Eleven figures? | 25. Ten figures? |
| 20. Nine figures? | 22. Five figures? | 24. Sixteen figures? | 26. Twenty figures? |

How many figures are required to express

- | | | |
|------------------------|-----------------|---------------------------|
| 27. Ten-millions? | 29. Trillions? | 31. Ten-billions? |
| 28. Hundred-thousands? | 30. Octillions? | 32. Hundred-quintillions? |

55. Orders of decimal units, and their values.

Since a thousand is ten times a hundred, a hundred is ten times a ten, and a ten is ten times a one, it follows that one tenth of a thousand is a hundred, one tenth of a hundred is a ten, and one tenth of a ten is a one; *i. e.*,

1. The value of a unit in any place is one tenth of the value of a unit of the next higher place; and

2. One tenth of the value of a unit in any place is a unit of the next lower place (or next place to the right). Hence,

56. I. The first place to the right of ones is the place of tenths of ones, or *tenths*; and a figure written in this place expresses tenths.

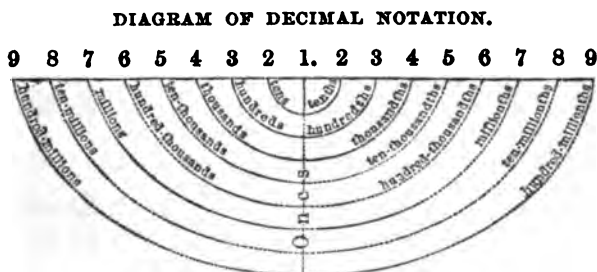
II. The second place to the right of ones, or the first place to the right of tenths, is the place of tenths of tenths, or *hundredths*; and a figure written in this place expresses hundredths.

III. The third place to the right of ones, or the first place to the right of hundredths, is the place of tenths of hundredths, or *thousandths*; and a figure written in this place expresses thousandths.

IV. A figure at the right of thousandths expresses *ten-thousandths*; at the right of ten-thousandths, *hundred-thousandths*; at the right of hundred-thousandths, *millionths*; and so on.

57. In writing a decimal, the decimal point (.) must always be placed before tenths.

58. The names of the first eight places to the left of ones, and the names of the corresponding orders of decimal units, are shown in the following



From this diagram it will be seen that

59. Orders of units at equal distances to the left and right of ones have corresponding names.

60. Reading decimals.

.3 is 3 tenths.

.37 is 37 hundredths. For .37 is 3 tenths and 7 hundredths; 3 tenths = 30 hundredths, and 30 hundredths + 7 hundredths = 37 hundredths.

.378 is 378 thousandths. For .378 is 37 hundredths and 8 thousandths; 37 hundredths = 370 thousandths, and 370 thousandths + 8 thousandths = 378 thousandths.

I. Tenths and hundredths are read together as hundredths; and tenths, hundredths, and thousandths are read together as thousandths.

II. When the right-hand figure of a decimal is ten-thousandths, the whole decimal is read as ten-thousandths; when the right-hand figure is hundred-thousandths, the decimal is read as hundred-thousandths; when the right-hand figure is millionths, the decimal is read as millionths. And in general,

61. *A decimal is read the same as an integer, the name of the place occupied by its right-hand figure being given to the whole decimal.*

.5 is read 5 tenths.		.725 is read 725 thousandths.
.25 is read 25 hundredths.		.00015 is read 15 hundred-thousandths.

What is the name of each of the following decimal places :

33. The second?		35. The fifth?		37. The ninth?		39. The fourth?
34. The eleventh?		36. The tenth?		38. The twelfth?		40. The seventh?

How many figures are required to express

41. Millionths?		43. Billionths?		45. Hundred-millionths?
42. Trillionths?		44. Ten-thousandths?		46. Ten-billionths?

What is the lowest order of units in a decimal expressed

47. By five figures?		49. By three figures?		51. By thirteen figures?
48. By eight figures?		50. By ten figures?		52. By six figures?

How is a decimal read that is expressed

53. By four figures?		55. By six figures?		57. By seven figures?
54. By nine figures?		56. By ten figures?		58. By thirteen figures?

62. *Reading mixed decimal numbers.*

186.25 is read 186 and 25 hundredths.
4,972.015 is read 4,972 and 15 thousandths. And in general

The integral and decimal parts of a mixed decimal number are read separately, with the word *and* between them. Hence,

In reading a mixed decimal number, read and in place of the decimal point.

Various readings.

.237 may be read 237 thousandths; or 2 tenths 3 hundredths and 7 thousandths; or 2 tenths and 37 thousandths; or 23 hundredths and 7 thousandths.

Again, 2.37 may be read 2 and 37 hundredths; or 2 ones 3 tenths and 7 hundredths; or 237 hundredths; or 23 tenths and 7 hundredths.

But for uniformity and convenience, mixed decimal numbers are read as stated in the beginning of this Article.

How many figures are required to express, in one number,

- | | |
|-----------------------------------|---|
| 59. Thousands and ten-millionths? | 62. Ten millions and hundred-thousandths? |
| 60. Millions and hundredths? | 63. Hundreds and ten-trillionths? |
| 61. Ten-thousands and millionths? | 64. Billions and ten-thousandths? |

Write and read a number expressed by

- 65. Eight figures, five of which are decimal.
- 66. Twelve figures, eight of which are decimal.
- 67. Seventeen figures, six of which are decimal.
- 68. Eleven figures, nine of which are decimal.
- 69. Nineteen figures, twelve of which are decimal.
- 70. Twenty-one figures, ten of which are decimal.

63. In extending the decimal scale below ones, it will be seen that

1. One is ten tenths, one tenth is ten hundredths, one hundredth is ten thousandths, and so on; and that

2. Ten thousandths are one hundredth, ten hundredths are one tenth, and ten tenths are one. Hence,

Integers and decimals form one general class of numbers in the decimal scale, and they are subject to the same

III. PRINCIPLES AND LAWS.

A. Of orders of units.

64. I. *The unit 1 is the base of all numbers. It is called a one, a unit of the first order, the fundamental unit, or unity.*

II. *The place of ones and all places to the left of ones are places of integral units; and all places to the right of ones are places of decimal units.*

III. *Ten ones taken together form a larger unit, or a unit of the second order, called a ten; ten tens taken together form a still larger unit, or a*

unit of the third order, called a hundred; ten hundreds taken together form a unit of the fourth order, called a thousand; and so on.

IV. One of the ten equal parts of the fundamental unit is a smaller unit, or a unit of the first decimal order, called a tenth; one of the ten equal parts of a tenth is a unit of the second decimal order, or a hundredth; one of the ten equal parts of a hundredth is a unit of the third decimal order, or a thousandth; and so on.

V. Ten units of any order are one unit of the next higher order; and one unit of any order is ten units of the next lower order. Hence,

65. Fundamental law of decimal notation.

The values of the successive orders of units of a number written in the decimal scale increase by the constant multiplier 10, and decrease by the constant divisor 10.

In written numbers, decimal orders are also called decimal places.

B. Of periods.

66. I. *In expressing integers by figures, every three places, counting from the right, form a period; and the places in each period consist of ones, tens, and hundreds.*

II. *The name of each period to the left of the period of thousands is determined by the number of the period, counting from the right.*

III. *The successive orders of units in the decimal scale increase by ten, and the successive periods by one thousand.*

C. Of the values of figures.

67. I. *The same figure always expresses the same number of units; and the place it occupies determines its order or local value.*

II. *The value expressed by any figure of a number written in the decimal scale, is one tenth of the value expressed by the same figure standing in the next place to the left; and ten times the value expressed by the same figure standing in the next place to the right.*

68. On these principles and laws are based

1. Rules for Arabic notation.

I. INTEGERS.—*Begin at the left and write the figures that express the hundreds, tens, and ones of each period, in order, separating the periods by commas.*

II. DECIMALS.—*Write the decimal point, and after it the figures that express the units of the successive orders.*

2. Rules for Arabic numeration.

I. INTEGERS.—*Begin at the left and read each period, omitting the name of the last period, and the names of all places filled with ciphers.*

II. DECIMALS.—*Read the decimal as an integer, and give to it the name of its lowest order.*

EXERCISES IN NOTATION AND NUMERATION OF INTEGERS AND DECIMALS.

1. Integers.

Express in figures and read the following numbers :

1. Three units of the fourth order, five of the third, four of the first.
2. Six units of the seventh order, three of the fifth, one of the third, one of the second, nine of the first.
3. Three units of the tenth order, five of the eighth, two of the sixth, one of the fourth, two of the third, three of the first.
4. Seven units of the sixth order, two of the fourth, eight of the second.
5. Nine units of the thirteenth order, seven of the eleventh, three of the seventh, six of the fifth, five of the third, two of the first.
6. Four units of the tens' place in each of the first six periods.

Read

7. 20,059.	11. 42,000,000.	15. 240,325,000,070.
8. 90,602.	12. 800,030,704.	16. 4,084,008,200,000.
9. 701,809.	13. 5,236,481,279.	17. 73,800,000,000,003.
10. 5,008,600.	14. 10,000,500,600.	18. 365,982,579,719,612.
19. 3,000,000,000,000,000.	22. 13,017,005,000,000,106,800,475.	
20. 748,847,933,000,000,602.	23. 8,025,963,877,300,080,191,811,070.	
21. 900,800,070,006,050,400,000.	24. 909,090,909,090,909,090,909,090,909,090.	

Express by Arabic notation

25. Forty-five billion one hundred fifteen million sixty-four thousand eighty-nine.

26. Eight trillion three hundred billion forty-five thousand.

27. Sixty-five quadrillion four hundred trillion fifteen million.

28. Eighty-five quintillion three billion.

29. Four hundred fifty-nine septillion ninety quadrillion four trillion seven hundred billion seventy-five million.

30. Thirty-three octillion three million three.

31. 72 septillion 308 thousand 40.

32. 999 octillion 551 quintillion 17 trillion 6 million.

33. 149 nonillion 284 sextillion 914 billion 84 thousand.

34. 600 decillion 257 septillion 53 quadrillion.

35. 8 undecillion 80 nonillion 800 octillion 8 quintillion 8 billion 80 thousand 8 hundred.

36. 1 duodecillion 11 undecillion 10 nonillion 110 octillion 111 sextillion 2 quintillion 20 trillion 202 million 22.

Express by Roman notation

37. The number of days in this calendar year.

38. The number of the year in which America was discovered.

39. The number of the year in which the independence of the United States was declared.

40. The number of the last page of this book.

2. Decimals.

The name of the right-hand place of a decimal may be determined *at sight*, by prefixing to the decimal the unit 1, and regarding the number thus formed as an integer. For example, take the number .087458, omit the decimal point and prefix the unit 1, and we have the number 1,087,458, in which the 1 expresses 1 million. The decimal place corresponding to millions is millionths; hence, the decimal expresses millionths.

Express in figures and read

41. Seven units of the first decimal order.

42. Four units of the 1st decimal order, nine of the 2d.

43. Eight units of the 1st decimal order, three of the 2d, six of the 3d.

44. Five units of the 2d decimal order, one of the 4th.
 45. Nine units of the 3d decimal order, five of the 4th, two of the 5th.
 46. Three units of the 1st decimal order, seven of the 4th, four of the 5th, nine of the 6th.

Read

47. .41793	50. .0476398	53. .0060054	56. .0070876941
48. .000004	51. .72000017	54. .200000002	57. .755340010631
49. .8400	52. .405980172	55. .000000001	58. .000006001078502

Express by figures

59. Nine hundredths. 62. Two hundred eleven thousandths.
 60. Ninety hundredths. 63. Thirty-two thousandths.
 61. Ninety-nine hundredths. 64. Nine hundred twelve ten-thousandths.
 65. Six hundred seven thousand fifty-nine millionths.
 66. One hundred seventy-two hundred-millionths.
 67. Seventeen ten-billionths.
 68. Nine hundred eight thousand seven hundred fifty-one quadrillionths.
 69. Four million twenty thousand ten-millionths.
 70. Four million ten thousand ninety-seven ten-millionths.
 71. Forty-seven ten-thousandths. 72. 243 hundred-millionths.
 73. 297 million 641 thousand 879 trillionths.
 74. 700 thousand 849 ten-billionths.
 75. 900 millionths; 9 hundred-millionths.

3. Mixed numbers.

Write and read the number expressed by

76. Three units of the 5th order, seven of the 3d, four of the 2d, and eight of the 1st decimal order, two of the 4th, five of the 7th.
 77. Five units of the 8th order, one of the 4th, and three of the 1st decimal order, eight of the 3d, one of the 5th, seven of the 9th.
 78. Fifty-six thousand ninety and five hundred sixty-nine thousandths.
 79. Two thousand and thirteen thousandths.
 80. Two thousand thirteen ten-thousandths.
 81. Two hundred and two thousandths; two hundred two thousandths.

82. One million and two hundred fifty-one millionths.

83. Three hundred seven thousand and three hundred seven thousandths.

84. Six hundred six and six hundredths; six hundred and six hundredths.

Read

85. 317.108	87. 5,080.06	89. 525,000.525	91. 6,180.00002
86. 51.051	88. 72.9008	90. 35.20604	92. 832,005.16000406
93. 9,000,000.000009	95. 380,965.800000965		
94. 506,001,014,300,000.4005006007	96. 9,024,000,000.000024		

Write and read

97. 506 billion and 81 millionths.

98. 50 thousand and 43 ten-thousandths.

99. 7 hundred thousand and 7 hundred-thousandths.

100. 9 thousand 4 hundred-thousandths.

101. 9 thousand and 4 hundred-thousandths.

102. 9 thousand and 4 hundred thousandths.

103. 3 million 3 hundred and 33 thousand ten-billionths.

104. 6 million and 175 thousand 203 millionths.

105. 10 and 10 ten-billionths.

106. 90 thousand 10 and 1 million 1 thousand billionths.

69. The *French method of notation*—by periods of three figures each—is the one in general use in the United States.

The *English method*—by periods of six figures each, as shown below—is the one used in Great Britain and in most of her colonies.

hundred-thousands ten-thousands thousands hundreds tens ones	hundred-thousands ten-thousands thousands hundreds tens ones	hundred-thousands ten-thousands thousands hundreds tens ones	hundred-thousands ten-thousands thousands hundreds tens ones
0 0 0 0 0 0,	0 0 0 0 0 0,	0 0 0 0 0 0,	0 0 0 0 0 0,
4th period	3d period	2d period	1st period
Trillions	Billions	Millions	Ones

SECTION II.

FRACTIONS.

70. Define

- | | |
|---|--|
| 1. Fractional parts (5).
2. A fractional unit (7). | 3. A fraction (11).
4. The unit of a fraction (21). |
|---|--|

71. The notation and numeration of fractions embraces

- | | |
|--|---|
| 1. The <i>terms</i> of a fraction;
2. The <i>values</i> of fractions; | 3. The <i>kinds</i> of fractions;
4. The <i>analysis</i> of fractions. |
|--|---|

1. Terms of a fraction.

72. The *terms of a fraction* are the two integers used to express it.

73. The *denominator* is that term which names the parts expressed by the fraction.

74. The *numerator* is that term which numbers the parts expressed by the fraction.

In expressing a fraction by figures, the numerator is always written above, and the denominator below a horizontal line.

75. The denominator

1. *Denominates* or names the parts expressed by the fraction;
2. *Expresses* the number of equal parts in a 1;
3. *Determines* the size or value of each part, and of the fractional unit;
4. *Is written* below the horizontal line.

76. The numerator

- | | |
|--|--|
| <ol style="list-style-type: none"> 1. <i>Numerates</i> or numbers the parts expressed by the fraction; 2. <i>Expresses</i> the number of equal parts in the fraction; 3. <i>Determines</i> the number of fractional units expressed; 4. <i>Is written</i> above the horizontal line. | <ol style="list-style-type: none"> 1. <i>Numerates</i> or numbers the parts expressed by the fraction; 2. <i>Expresses</i> the number of equal parts in the fraction; 3. <i>Determines</i> the number of fractional units expressed; 4. <i>Is written</i> above the horizontal line. |
|--|--|

77. In reading a fraction, the numerator is read as a cardinal number, and the denominator as an ordinal number.

e. g., $\frac{5}{8}$, $\frac{17}{25}$, $\frac{63}{71}$ are read 5 eighths, 17 twenty-fifths, 63 seventy-firsts.

EXERCISES.

Write in figures

- | | |
|-----------------------------|-----------------------------------|
| 1. Four ninths. | 5. 231 seven thousandths. |
| 2. Eight fifteenths. | 6. 3 five hundred thirty-seconds. |
| 3. Nineteen sixteenths. | 7. 41 fiftieths. |
| 4. Twenty-seven hundredths. | 8. 700 eight hundred firsts. |

Read

- | | | | | |
|-------------------|---------------------|-----------------------|-------------------------------|-----------------------|
| 9. $\frac{7}{9}$ | 11. $\frac{2}{15}$ | 13. $\frac{44}{127}$ | 15. $\frac{488}{10884}$ | 17. $\frac{31}{8}$ |
| 10. $\frac{2}{3}$ | 12. $\frac{11}{13}$ | 14. $\frac{124}{384}$ | 16. $\frac{1388246}{8871108}$ | 18. $\frac{2272}{14}$ |

2. Values of fractions.

78. The *value of a fraction* is the amount or number expressed by it.

The numerator of any fraction is a dividend, the denominator is a divisor, and the value expressed by the fraction is the quotient.

Since the denominator of a fraction expresses the number of equal parts in a 1, and the numerator expresses the number of equal parts in the fraction, it follows that

The value of a fraction depends upon the relative values of its numerator and denominator. Hence,

1. The value of a fraction is 1, when the terms are equal.
2. The value of a fraction is less than 1, when the numerator is the less term.
3. The value of a fraction is more than 1, when the numerator is the greater term. And in general,

79. *The greater the number of equal parts into which a 1 is divided, the less is the value of each part; and the less the number of equal parts into which a 1 is divided, the greater is the value of each part.* Hence, in comparing the values of fractions,

1. Of two fractions which have the same denominator, that one having the greater numerator expresses the greater value;
2. Of two fractions which have the same numerator, that one having the greater denominator expresses the less value.

3. Kinds of fractions.

80. Any number that consists of one or more of the equal parts of an integral unit is a fraction (**7, 11**).

a. The term *fraction* is now generally applied to the expressions of fractional units by numerator and denominator ;

b. But decimal units as well as fractional units are equal parts of a thing, or of an integral unit. Hence,

(1) *Decimal parts are fractional parts, and decimal units are fractional units.*

(2) *Decimals are also called decimal fractions.*

81. Fractions are classified, in respect to their values, as *proper* and *improper*; and in respect to their denominators, as *similar* and *dissimilar*.

82. A *proper fraction* is a fraction whose value is less than 1.

83. An *improper fraction* is a fraction whose value is 1 or more than 1.

The numerator of a proper fraction is *always* a less number than the denominator. The numerator of an improper fraction is *never* a less number than the denominator (**78**; 1, 8).

84. *Similar fractions* are fractions that have the same fractional unit.

85. *Dissimilar fractions* are fractions that have different fractional units.

a. *Least similar fractions* are fractions that have the greatest common fractional unit possible.

b. A *common denominator* is the denominator of each of two or more similar fractions.

c. A *least common denominator* is the denominator of least similar fractions.

86. Fractions are also classified as *simple*, *compound*, and *complex*.

a. A *simple fraction* has only one numerator and one denominator, and each is an integer.

b. A *compound fraction* is two or more numbers connected by the word *of*, the first at least of the numbers being a fraction.

c. A *complex fraction* has a fraction or a mixed number in one or both terms.

Neither a compound nor a complex fraction answers to the definition of a fraction (**11**); neither of them has a unit, integral or fractional (**2, 7**); and neither can be read as a fraction is read (**77**). Therefore, the terms *compound fraction* and *complex fraction* are of doubtful propriety.

87. In reading a mixed number, read *and* between the integer and the fraction.

EXERCISES.

Of the numbers in the margin

- | | | |
|--|-----------------|-------------------|
| 19. Which are proper fractions, and why? | $\frac{3}{8}$ | $\frac{487}{37}$ |
| 20. Which are improper fractions, and why? | $\frac{75}{8}$ | $\frac{368}{19}$ |
| 21. Which are similar fractions, and why? | | |
| 22. Which are dissimilar fractions, and why? | $54\frac{1}{2}$ | $7\frac{1}{3}$ |
| 23. Which are mixed numbers, and why? | $\frac{6}{5}$ | $306\frac{1}{11}$ |
| 24. Which have common denominators, and why? | | |
| 25. Which have a common fractional unit? | $\frac{75}{8}$ | $\frac{7}{8}$ |

26, 27. Of those having a common fractional unit, which expresses the greatest value, and why? Which the least value, and why?

28, 29. Of those having the same number of fractional units, which expresses the greatest value, and why? Which the least value, and why?

30. Write in figures and read five proper fractions.

31. Five improper fractions.

33. Four similar fractions.

32. Five mixed fractional numbers.

34. Four dissimilar fractions.

4. Analysis of fractions.

88. To *analyze a fraction* is to name and describe all its parts, its units, its kind, and its value.

Ex. Analyze the fraction $\frac{3}{4}$ of a mile.

ANALYSIS.— $\frac{3}{4}$ of a mile is a *fraction*, because it expresses fractional units. 3 and 4 are the *terms*, because they are the two numbers used to express the fraction; 4 is the *denominator*, because it shows that 1 mile is divided into 4 equal parts; and 3 is the *numerator*, because it shows that 3 of the 4 equal parts are taken. 1 mile is the *unit of the fraction*, because it is the *one* divided to form the fraction; and 1 fourth of a mile is the *fractional unit*, because it is one of the four equal parts into which the unit, 1 mile, is divided. The fraction is a *proper fraction*, because the numerator is the less term; and the *value* expressed by it is three fourths of 1 mile.

EXERCISES.

Write and analyze

35. Five ninths of a month.

39. Eleven fifteenths.

36. Three eighths of a dollar.

40. Seven twelfths.

37. Twenty sevenths of a week.

41. Thirteen fifths.

38. 381 5,163ds of a square mile.

42. Twenty-five thirty-seconds.

SECTION III.

COMPOUND NUMBERS.

89. Measure is that by which extent, dimension, quantity of matter, or money value is determined.

Measure may be length, breadth, thickness, area, volume, capacity, number, amount, or value.

90. Weight is a measure of the quantity of matter in a body.

Weight and measure are determined by processes called *weighing* and *measuring*.

These processes consist in comparing the thing to be weighed or measured with some fixed standard.

91. Money value is the equivalent, in money units, for other units of measure.

92. Define { 1. A denominate unit (15). | 3. A denominate number (18).
2. Denomination (18). | 4. A compound number (20).

93. Like compound numbers are numbers expressed in denominations of the same table.

94. The notation and numeration of compound numbers embraces *units, tables, scales, and symbols.*

95. A unit of measure is a number of fixed value, used as a standard of comparison in measuring other like numbers, to determine their amount or value.

A number is measured by finding the number of times it contains the unit of measure.

96. A standard unit is that unit which law or custom has established, as the one from which the other units in the same scale of numbers are derived.

97. The nine classes of units are units of

- | | | |
|----------------------|--------------|---------------------------------|
| 1. Abstract numbers; | 4. Capacity; | 7. Angular measure; |
| 2. Concrete numbers; | 5. Weight; | 8. Enumeration, count, or tale; |
| 3. Extension; | 6. Time; | 9. Currency or money value. |

Any unit whatever, used to express number, belongs to some one of these nine classes of units.

EXERCISES.

Write

- | | |
|--|---------------------------------------|
| 1. Three abstract numbers of four, eight, and eleven places, respectively. | 6. Three concrete numbers. |
| 2. Three unlike concrete numbers. | 7. Three numbers expressing distance. |
| 3. Four like compound numbers. | 8. Three numbers expressing area. |
| 4. Four unlike denominate numbers. | 9. Three numbers expressing volume. |
| 5. Three abstract numbers. | |
10. Write two numbers expressing solidity, or cubic units.
- 11, 12. Write two numbers expressing amounts of liquids; and two numbers expressing amounts of grains, fruit, or root crops.
- 13, 14, 15. Write two numbers each, expressing the weight of gold, of hay, and of medical prescriptions.
16. Write two numbers expressing time.
17. Write three numbers expressing angular measure.
18. Write numbers expressing three kinds of articles sold by count.
19. Write three numbers expressing money value.

98. A *table*, in compound numbers, is a regular arrangement of the denominations used to express any measure, weight, or money value; with the number of units of each denomination equal in value to a unit of the next higher denomination.

99. Define a scale (27). Define an arithmetical symbol (38).

100. *Magnitude* is extent, size, or quantity.

101. *Dimensions* are length, width, and thickness.

The term magnitude is applied to objects that have one or more of these dimensions.

102. A *point* is that which has position, but has not magnitude.

103. A *line* is a magnitude that has one dimension—length.

a. A *straight line* is the shortest distance between two points.

b. A *curved line* is a line that changes its direction at every point.

c. A straight line is commonly called a *line*; and a curved line, a *curve*.

d. Length is expressed in units of measure for lines.

104. *Linear measure* is the measure of lines.

105. A *surface* or *superficies* is a magnitude that has two dimensions—length and width.

106. *Area* is the extent of any surface.

Area is expressed in units of measure for surfaces.

107. *Square measure* or *superficial measure* is the measure of surfaces.

108. A *solid* or *body* is a magnitude that has three dimensions—length, width, and thickness.

109. *Volume* is any limited portion of space that is, or may be, occupied by a body.

Volume is expressed in units of measure for cubic contents.

110. *Extension* is either length, area, or volume.

111. *Cubic contents* is the extent of any solid, or of any limited portion of space.

a. Solidity is the volume of any solid or body.

b. Capacity is the volume of any limited portion of space.

c. Contents is a general term for areas and volumes.

112. *Solid measure* or *cubic measure* is the measure of volume.

113. A *plane* is a surface any two points in which being taken, the straight line that joins them lies wholly in that surface.

114. An *angle* is the difference of direction of two lines that emanate from a common point in a plane.

115. An *angle of one degree* is one of the 360 equal angles that exactly fill the space about a common point in a plane.

From this definition it is evident that the sum of all the angles that can be formed about a point in a plane is 360 degrees.

116. If two intersecting lines form four equal angles at their point of intersection, the two lines are perpendicular to each other, and the four angles are right angles. Hence,

A *right angle* is an angle whose sides are perpendicular to each other. It is an angle of 90 degrees. And

Two lines are perpendicular to each other when they form a right angle at their point of meeting.

117. A *parallelogram* is a figure whose four sides are straight lines, and whose opposite sides are parallel.

The opposite sides of a parallelogram are equal.

118. A *rectangle* is a parallelogram whose angles are right angles.

119. A *square* is a rectangle that has four equal sides.

120. A *rectangular solid* is a body bounded by six rectangles.

121. A *cube* is a rectangular solid whose sides are equal squares.

Describe	2. A cubic inch.	5. A square yard.	8. A cubic rod.
	3. A square foot.	6. A cubic yard.	9. A square mile.
1. A square inch.	4. A cubic foot.	7. A square rod.	10. A cubic mile.

Draw 11. Two lines perpendicular to each other.

12. A right-angle.	14. A rectangle.	16. A square.
13. A parallelogram.	15. A rectangular solid.	17. A cube.

122. A *circle* is a surface bounded by a curved line, every part of which is equally distant from a point within, called the centre.

a. The *circumference* of a circle is the line that bounds it.

b. An *arc* of a circle is any part of its circumference.

c. The *diameter* of a circle is the distance across it, through its centre.

d. The *radius* of a circle is the distance between its centre and circumference.

The radius of a circle equals one half of its diameter.

e. A *sector* is a portion of a circle bounded by two radii and the included arc.

f. A *quadrant* is a sector whose arc is one fourth of the circumference of a circle.

g. A *sextant* is a sector whose arc is one sixth of the circumference of a circle.

123. If the surface about a point in a plane be divided into 360 equal parts or spaces, by lines drawn from the point, 360 equal angles will be formed. The lines that form the sides of these angles may be of any length; and if, about their common point of meeting as a centre, a circumference be drawn cutting all these lines, it will be divided into 360 equal parts, and one of these parts will be a degree, or the measure of the angle at the centre of the circle. Hence,

124. A *degree* is one of the 360 equal parts of a circumference.

125. The *measure of an angle* at the centre of a circle is that part of the circumference included between the sides of the angle.

a. Angles are at a centre, and degrees (the measures of angles) are parts of a circumference. Hence, *angles are not degrees.*

b. Since circles may be great or small, the degrees in their circumferences will be correspondingly great or small. An angle of 1 degree is constant, while the measure of the angle—*i.e.*, 1 degree in a circumference—varies with every change in the circumference.

c. In geography and astronomy a circumference is called a circle. Hence,

A *geographical* or *astronomical circle* is a mathematical circumference.

Draw { 18. A curve. | 20. An arc. | 22. A radius. | 24. A quadrant.
 { 19. A circle. | 21. A diameter. | 23. A sector. | 25. A sextant.

126. MEASURES OF EXTENSION.

KIND.	USE.	STANDARD UNIT.
i. Linear.	For measuring distances and dimensions.	Yard.
ii. Cloth.	For measuring goods sold by the linear yard.	Yard.
iii. Mariners'.	For estimating marine lengths and distances.	
iv. Square.	For computing areas.	{ Square foot, square yard.
v. Surveyors'.	{ <i>Linear.</i> For measuring roads and dimensions of land.	{ Chain. Acre.
	{ <i>Square.</i> For computing area of land.	
vi. { Solid or cubic.	{ For computing solidity or volume.	{ Cubic foot, cubic yard.
vii. Dry.	{ For measuring grains, fruits, vegetables, seeds, lime, coal, and other dry substances.	{ Bushel.
viii. Liquid.	{ <i>Commercial.</i> For measuring liquids.	{ Gallon.
	{ <i>Pharmacists'.</i> For compounding medicines in liquid form.	

(See Supplement, page 368.)

EXERCISES.

Read

20. 13 mi. 5 yd. 3 in.	23. 32 mi. 71 ch. 29.3 l.	26. 7 gal. 1.75 pt.
21. 84 rd. 7½ ft.	24. 5 mi. 31 ch. 71 l.	27. 3 pk. 1½ qt.
22. 75 yd. 3 qr.	25. 235 bu. 3.75 qt.	28. 352 cd. 76 cu. ft.

Read

- | | |
|--|-----------------------------------|
| 29. 107 mi. 39 rd. 4 yd. 2 ft. 11 in. | 33. 4,080 bu. 1 pk. 7 qt. 1 pt. |
| 30. 13 sq. rd. 19 sq. yd. 5 sq. ft. 64.9 sq. in. | 34. 124 gal. 3 qt. 1 pt. 1 gi. |
| 31. 1 Tp. 13 sq. mi. 432 A. 7 sq. ch. 21.15 sq. l. | 35. 29 cu. yd. 1,028.625 cu. in. |
| 32. 29 cu. yd. 11 cu. ft. 802 cu. in. | 36. Cong. v O vi f3 xi f3 iii xiv |

Write the following compound numbers, using the proper abbreviations :

37. Ninety-five acres five square chains one hundred one square links.
38. Four hundred and four hundredths gallons.
39. One thousand miles seventy-one chains fifty-three and eighty-one hundredths links.
40. Seventy-five miles two hundred three rods one yard two feet eleven inches.
41. Twenty-three cords two and five twelfths cubic feet.
42. 2,064 bushels 1 peck 6 quarts 1 and 8 tenths pints.
43. Thirty-three square rods fourteen square yards seven square feet sixty and five tenths square inches.
44. 41 cubic yards 19 cubic feet 1,606 cubic inches.
45. 31 square miles 950 square rods 179½ square feet.
46. Fifteen gallons three pints five fluid ounces three fluid drachms nineteen minims.

127. MEASURES OF WEIGHT.

KINDS.	USES.	STANDARD UNITS.
ix. Troy.	{ For weighing the precious metals and jewels, and in philosophical experiments.	{ Troy pound of 5,760 grains.
x. Apothecaries'.	{ By physicians in prescribing, and apothecaries and pharmacists in mixing medicines that are dry.	
xi. Assayers'.	For weighing gold and silver.	
xii. Lapidaries'.	For weighing diamonds.	{ Diamond grain = .8 Troy grain.
xiii. Avoirdupois.	For all the ordinary purposes of weighing.	{ Avoirdupois pound of 7,000 Troy grains.

(See Supplement, page 369.)

EXERCISES.

Read

- | | |
|--------------------------------------|------------------------|
| 47. 2 T. 13 cwt. 47 lb. 10 oz. | 50. 12 cwt. 3 oz. |
| 48. 17 lb. 11 oz. 15 pwt. 24.375 gr. | 51. 91 T. 1,625.75 lb. |
| 49. 1b ii 3 viij 3 iv 3 i gr. xvij. | 52. 807 lb. 13.25 oz. |

Write, using proper abbreviations,

53. Two thousand ten pounds three and three fourths ounces.
 54. Eleven tons seventeen hundred-weight.
 55. Six pounds five ounces twenty-one pennyweights twenty grains.
 56. 5 ounces 6 drams 1 scruple 15 grains.

128. ANGULAR MEASURES.

KINDS.	USES.	STANDARD UNITS.
xiv. { Surveyors' and navigators'.	In surveying, civil engineering, and navigation. In astronomical observations and calculations.	Degree.
xv. Astronomers'.		

(See Supplement, page 370.)

129. MEASURES OF TIME.

The day and the year are the natural divisions of time; the other denominations, except centuries, are parts of one or the other of these units.

Units. Day and Year.

(See Supplement, page 371.)

130. MEASURES OF MONEY VALUE.

KINDS.	WHERE USED.	STANDARD UNITS.
xvi. U. S. money,	{ In United States. In Dominion of Canada.	{ Dollar, \$1.00
xvii. Canada money,		
xviii. French money.	In France.	Franco, \$.193
xix. English money.	Sterling; In Great Britain.	Pound, \$4.866½
xx. German money.	German; In German Empire.	{ Reichsmarken (mark) \$.2385

(See Supplement, page 373.)

131. ENUMERATION OR COUNTING.

KINDS.	USES.
xxi. Counting.	In marketing certain classes of articles.
xxii. Paper.	In the paper trade.

(See Supplement, page 374.)

Read

EXERCISES.

- | | |
|---|--|
| 57. 2 wk. 5 da. 24 min. 7.4 sec. | 62. 10 yr. 1 mo. 27 da. 3 h. 14 min. 49 sec. |
| 58. 11 gt. gro. 5 gro. $3\frac{3}{4}$ doz. | 63. $103^{\circ} 25' 49''$; $7^{\circ} S. 24^{\circ} 30.21''$. |
| 59. 95 dol. 3s. 9 cts. 60. 17 fr. | 64. \$235.67; \$40.059; \$24,007.40 $\frac{1}{2}$. |
| 61. £274 16s. 9d. 2 far. | 65. 31 B. 3 bun. 1 rm. 17 quires 18 sheets. |
| 66. Jan. 15, 1756. 67. July 4, 1776. 68. Sept. 29, 1880. 69. Dec. 21, 1897. | |

Write, using the proper abbreviations,

70. Thirty-one years seven months twenty days nineteen hours forty-five and seventy-five hundredths minutes.
71. One hundred seventy-one degrees forty and twenty-five hundredths seconds.
72. Ninety-seven degrees fifty-five and two hundred thirty-one thousandths geographic miles.
73. Six great gross eleven gross nine and four twelfths dozens.
74. Eighteen dollars and one half-cent.
75. Six pounds ten shillings five pence three farthings.
76. Twenty-five dollars two shillings ten pence.
77. Five Napoleons thirteen francs two and five tenths decimes.

132. METRIC SYSTEM OF MEASURES AND WEIGHTS.

In the year 1866 Congress passed an Act authorizing the use of a *decimal system* of measures and weights. The one standard unit of this system, from which all the other units are derived, is the *meter*; hence, the system is known as the *metric system*.

This system is gradually coming into use among chemists, pharmacists, scientists, and artists. It is also used at the United States Mints, and in invoices of imported goods from most foreign countries. It is *not* used in ordinary commercial transactions.

(See Supplement, page 378.)

SECTION IV.

REVIEW OF PRELIMINARY DEFINITIONS AND NOTATION.

133. TEST QUESTIONS IN THE THEORY OF NOTATION.

Explain and illustrate the two uses of the cipher

I. In integers. | II. In decimals.

III. Show that the fraction $\frac{327}{1000}$ is also a decimal.

IV. Show 1. That all denominate numbers are concrete; and
2. That some concrete numbers are not denominate.

V. What denominate numbers are simple, and what are compound?

How is a number in a decimal or a varying scale affected by transferring a unit

VI. From a higher order or denomination to the next lower? Why?

VII. From a lower order or denomination to the next higher? Why?

VIII. What order or place of a decimal gives name to the whole decimal?

IX. Show that one of the nine equal parts of the denominator of the fraction $\frac{1}{9}$ equals one of the four equal parts of the numerator.

X. In writing compound numbers, what abbreviations or signs of denominations are placed before the numbers?

Show the three offices

XI. Of the denominator of a fraction. | XII. Of the numerator of a fraction.

XIII. Show the difference between like fractions and similar fractions.

XIV. In notation of integers, what periods must be full, and what periods may be partial, and why?

XV. What determines the value of the unit of any figure in a written number?

XVI. What determines the value of a fraction?

XVII. What determines the value of a decimal?

XVIII. Show that, in expressing a given value by a fraction

(1) If the parts are large, the number of parts is small; and

(2) If the parts are small, the number of parts is large.

XIX. What two things determine the value expressed by any figure in a decimal number?

XX. What tables in compound numbers have uniform scales?

BLACKBOARD OUTLINES.

134. To the Teacher.—Require the pupils to copy the outline of each subject, and to complete it by filling each blank with the appropriate term, phrase, or symbol.

The completed outline (either on the blackboard or in the hands of the pupils) will enable you to conduct a review thoroughly and rapidly.

135. OUTLINES OF PRELIMINARY DEFINITIONS.

NUMBERS.	A. ELEMENTS OF.	I. PARTS.	1. _____	2. _____		
		II. UNITS.	1. <i>As wholes or parts.</i>	a. _____		
				b. _____		
			2. <i>As pure or applied.</i>	c. _____		
				a. _____		
				b. _____		

		III. SCALES.	1. _____	2. _____		
		B. CLASSIFICATION OF.	I. FROM SCALES.	1. _____	2. _____	
			II. FROM EQUAL PARTS.	1. _____	2. _____	
III. AS PURE AND APPLIED.	1. _____		2. _____			
IV. FROM UNIT VALUES.	1. a. _____		b. _____			
	2. a. _____		b. _____			
V. FROM USE AS NUMERALS.	1. _____		2. _____			
C. SYMBOLS.	I. OF NUMBER.		1. _____	2. _____	3. _____	
	II. OF OPERATION.		1. _____	4. _____		
			2. _____	5. _____		
			3. _____	6. _____		
	III. OF RELATION.	1. _____		IV. OF AGGREGATION.	1. _____	
		2. _____			2. _____	
		3. _____		3. _____		
	UNCLASSIFIED TERMS.	1. _____	5. _____	9. _____	a. _____	
		2. _____	6. _____		b. _____	
		3. _____	7. _____	10. _____		
4. _____		8. _____	_____			

C. COMPOUND NUMBERS.	I. TERMS.	1. _____	14. _____	27. _____
		2. _____	15. _____	28. _____
		3. _____	16. _____	29. _____
		4. _____	17. _____	30. _____
		5. _____	18. _____	31. _____
		6. _____	19. _____	32. _____
		7. _____	20. _____	33. _____
		8. _____	21. _____	34. _____
		9. _____	22. _____	35. _____
		10. _____	23. _____	36. _____
		11. _____	24. _____	37. _____
		12. _____	25. _____	38. _____
		13. _____	26. _____	39. _____

C. COMPOUND NUMBERS.	II. MEASURES, WEIGHTS, AND MONEY VALUES.	1. Extension.	a. _____	4. Time.	a. _____
			b. _____		b. _____
			c. _____		c. _____
			d. _____		
			e. _____		
			1st. _____		
		2. Weight.	a. _____	5. Money.	a. _____
			b. _____		b. _____
			c. _____		c. _____
			d. _____		d. _____
			e. _____		e. _____
			1st. _____		1st. _____
		3. Angular measures.	a. _____	6. Counting.	a. _____
			b. _____		b. _____
			c. _____		
			d. _____		
			e. _____		
			1st. _____		
7. Metric system.	a. _____	7. Metric system.	a. _____		
	b. _____		b. _____		
	c. _____		c. _____		
	d. _____				
	e. _____				
	1st. _____				
2d. _____					

CHAPTER II.

PROPERTIES OF NUMBERS.

137. *Properties of numbers* are those elements or facts that are inseparable from them.

138. Properties of numbers depend upon these two primary facts :

1. All integers are even or odd. | 2. All integers are composite or prime.

139. Properties of numbers embrace

- | | | |
|---|--|--------------------|
| I. Definitions and axioms. | | IV. Prime numbers. |
| II. Properties of even and odd numbers. | | V. Divisors. |
| III. Properties of composite numbers. | | VI. Multiples. |
| VII. Cancellation. | | |

SECTION I.

DEFINITIONS AND AXIOMS.

140. The *factors* of a number are those numbers of which the given number is the product.

141. A *power* is the product of equal factors.

142. A *root* is one of the equal factors that produce a power.

a. The ordinal name of a power or root is determined from the number of factors used to produce the power.

e. g. The product of five equal factors is the *fifth* power, and one of these factors is the *fifth* root of that power.

b. The *square* of a number is its second power, and the *cube* is its third power.

c. The *square root* of a number is its second root, and the *cube root* is its third root.

143. Division is *exact*, when the quotient is an integer.

Division is *complete* when all the dividend is divided.

e. g. In $32 \div 5 = 6\frac{2}{5}$ the division is *complete* but not *exact*.

144. An *exact divisor* of an integer is any integral factor of that integer.

145. A *common divisor* of two or more integers is any exact divisor common to those integers.

146. The *greatest common divisor* of two or more integers is the greatest exact divisor common to those integers.

147. *Commensurable numbers* are numbers that have a common divisor.

148. A *multiple* is any number of which a given number is an exact divisor.

149. A *common multiple* of two or more numbers is any number of which each of the given numbers is an exact divisor.

150. The *least common multiple* of two or more numbers is the least number of which each of the given numbers is an exact divisor.

a. In the subsequent discussion of properties of numbers, the terms *divisor*, *division*, and *divided* signify *exact divisor*, *exact division*, and *exactly divided*.

b. *Measure* and *sub-multiple* are other names for exact divisor.

EXERCISES.

1. What are the integral factors of 15? Why?
2. What are the ten integral factors of 84?
3. What six integers are factors of 12? Why?
4. Name three exact divisors of 64.
5. What integer is a common divisor of 18, 45, and 80? Why?
6. What is the greatest common divisor of 24, 72, 40, and 16?
7. Name five multiples of the number 7.
8. Give an example of a common multiple of three numbers.
9. What is the least common multiple of 4, 6, 15, and 20? Why?

151. An *even number* is a number that is exactly divisible by 2.

152. An *odd number* is a number that is not exactly divisible by 2.

153. A *composite number* is a number that can be separated into integral factors each greater than 1.

154. A *prime number* is a number that can not be separated into integral factors each greater than 1.

A multiple must be a composite number; but a divisor may be either composite or prime.

155. A *composite factor* is a factor that is itself a composite number.

156. A *prime factor* is a factor that is a prime number.

157. *Relatively prime numbers* are numbers that have no common integral factor.

Two composite numbers may be relatively prime ; as 15 and 16.

EXERCISES.

Of the twelve numbers in the margin,	8	22	89
10. Which are even numbers? Why?	13	29	96
11. Which are odd numbers? Why?	15	48	104
12. Which are composite numbers? Why?	20	54	209

Write

- | | |
|--|---|
| 13. Four even numbers. | 16. The composite factors of 72. |
| 14. Five composite numbers. | 17. All the odd numbers less than 20. |
| 15. The prime factors of 45. | 18. All the prime numbers from 50 to 100. |
| 19. Which of the ten factors of 84 are prime factors? Which are composite factors? | |
| 20. Are 18 and 35 relatively prime? Why? | |
| 21. Are 60 and 144 relatively prime? Why? | |
| 22. Give six examples of numbers that are relatively prime. | |

AXIOMS.

158. An *axiom* is a truth so obvious that no demonstration can make it plainer.

As bases of arithmetical demonstrations, use is often made of the following axioms :

AXIOM 1.—*A whole equals the sum of all its parts.*

AXIOM 2.—*A whole is greater than any of its parts.*

AXIOM 3.—*If the same number or equal numbers be added to each of two numbers, the difference between the sums will be the same as the difference between the numbers.*

AXIOM 4.—*If the same number or equal numbers be subtracted from each of two numbers, the difference between the remainders will be the same as the difference between the numbers.*

AXIOM 5.—*If equal numbers be multiplied by the same number, the products will be equal.*

AXIOM 6.—*If equal numbers be divided by the same number, the quotients will be equal.*

AXIOM 7.—*If each of two numbers is equal to a third number, the two numbers are equal to each other.*

SECTION II.

PROPERTIES OF EVEN AND ODD NUMBERS.

159. Demonstrate the truth of each property here given :

PROPERTY 1.—*All even numbers except 2 are composite.*

PROPERTY 2.—*All prime numbers except 2 are odd, but an odd number is not necessarily prime.*

PROPERTY 3.—*The sum of any number of even numbers is even.*

PROPERTY 4.—*The sum of an even number and an odd number is odd.*

PROPERTY 5.—*The sum of an even number of odd numbers is even.*

PROPERTY 6.—*The sum of an odd number of odd numbers is odd.*

PROPERTY 7.—*The difference between any two even numbers, or between any two odd numbers, is even.*

PROPERTY 8.—*The difference between an even number and an odd number is odd.*

PROPERTY 9.—*Any odd number except 1 is either 1 greater or 1 less than some multiple of 4.*

PROPERTY 10.—*The product of any number of even numbers is even.*

PROPERTY 11.—*The product of any number of odd numbers is odd.*

PROPERTY 12.—*The product of any number of factors is even, if one of the factors is even.*

PROPERTY 13.—*If an exact divisor is odd and the dividend is even, the quotient is even.*

PROPERTY 14.—*If both exact divisor and dividend are odd, the quotient is odd.*

PROPERTY 15.—*An even number is not an exact divisor of an odd number.*

PROPERTY 16.—*Any even number except 2 has at least one even factor.*

PROPERTY 17.—*All the factors of an odd number are odd.*

SECTION III.

TEN PROPERTIES OF COMPOSITE NUMBERS.

160. PROPERTY 1.—*A number is divisible by 2, if the right-hand figure of the number is 0, 2, 4, 6, or 8.*

DEMONSTRATION.—The right-hand figure of any written even number is 0, 2, 4, 6, or 8; and in dividing any even number by 2, the ones of the last partial dividend are 0, 2, 4, 6, or 8. Hence, the Property.

PROPERTY 2.—*A number is divisible by 5, if the right-hand figure of the number is 0 or 5.*

DEMONSTRATION.—The right-hand figure of the product of an even number of times 5 is 0; thus, $4 \times 5 = 20$, $8 \times 5 = 40$, $16 \times 5 = 80$. The right-hand figure of any odd number of times 5 is 5; thus, $3 \times 5 = 15$, $5 \times 5 = 25$, $17 \times 5 = 85$. Hence, the Property.

PROPERTY 3.—*A number is divisible by 3 or 9, if the sum of its digits is divisible by 3 or 9.*

DEMONSTRATION.—The number 67,854, and the sum of the numbers 6, 7, 8, 5, 4 (the digits of 67,854), are divisible by 3. The number 21,843, and the sum of the numbers 2, 1, 8, 4, 3 are divisible by 3 and also by 9. Hence, the Property.

PROPERTY 4.—*Any factor of a number is also a factor of any multiple of that number.*

DEMONSTRATION.—2 and 3, the factors of 6, are factors of 12 (2×6), of 18 (3×6), of 30 (5×6), of 54, of 96, or of any number of times 6. Hence, the Property.

PROPERTY 5.—*A number is divisible by the product of two or more of its factors, if it is divisible by those factors successively.*

DEMONSTRATION.— $84 \div 3 = 28$, and $28 \div 7 = 4$; also $84 \div 21$ (or 3×7) = 4. Again, $756 \div 4 = 189$, $189 \div 3 = 63$, and $63 \div 7 = 9$; also $756 \div 84$ (or $4 \times 3 \times 7$) = 9. Hence, the Property.

A number may be divisible by two or more of its factors, and not be divisible by their product. Thus, 48 is divisible by 8 and by 12, but not by their product, 96.

PROPERTY 6.—*A composite number is divisible by the product of any two or more of its prime factors.*

DEMONSTRATION.—If any composite number be divided by one of its prime factors, the result be divided by another prime factor, and so on, until the quotient is 1, all the prime factors of the number will be used for divisors. Thus, $84 \div 2 = 42$; $42 \div 2 = 21$; $21 \div 3 = 7$; $7 \div 7 = 1$. The factors used as divisors are 2, 2, 3, and 7; and their product 84 is divisible by the product of any number of these factors. Hence, the Property.

In a similar manner, demonstrate

PROPERTY 7.—*A composite number is divisible by the product of any two or more of its relatively prime factors.*

PROPERTY 8.—*Any even number divisible by an odd number, is also divisible by 2 times that odd number.*

DEMONSTRATION.—Any even number is divisible by 2 (Property 1); and if the quotient of an even number divided by 2 is also divisible by an odd number, the number is divisible by the factors 2 and the odd number successively; and it is also divisible by 2 times the odd number (Property 5). Hence, the Property.

PROPERTY 9.—*Any factor common to two numbers is also a factor of their sum, and of their difference.*

DEMONSTRATION.—Since 45 is 5 times 9, and 27 is 3 times 9, $45 + 27 (= 72)$ is 5 times 9 + 3 times 9 (= 8 times 9); and $45 - 27 (= 18)$ is 5 times 9 - 3 times 9 (= 2 times 9). Hence, the Property.

PROPERTY 10.—*A number is divisible by any power of 2 or 5, if the number expressed by as many of its right-hand figures as equal the index of the power is divisible by the same power of 2 or 5.*

DEMONSTRATION.—Any number expressed by more than one figure may be separated into two parts, one of which is a multiple of some power of 10, and the other is ones. Any factor common to two numbers is a factor of their sum (Property 9). The truth of that part of this Property relating to any power of 2 is evident from examples 1, 2, 3, below:

$$\text{Ex. 1. } 56 = 5 \times 10^1 + 6; \frac{5 \times 10^1}{2^1} = 25, \text{ and } \frac{6}{2^1} = 3.$$

$$\text{Ex. 2. } 256 = 2 \times 10^2 + 56; \frac{2 \times 10^2}{2^2} = 50, \text{ and } \frac{56}{2^2} = 14.$$

$$\text{Ex. 3. } 3,256 = 3 \times 10^3 + 256; \frac{3 \times 10^3}{2^3} = 375, \text{ and } \frac{256}{2^3} = 32.$$

And the truth of that part of the Property relating to any power of 5, is readily seen from examples 4, 5, 6, which follow:

$$\text{Ex. 4. } 75 = 7 \times 10^1 + 5; \frac{7 \times 10^1}{5^1} = 14, \text{ and } \frac{5}{5^1} = 1.$$

$$\text{Ex. 5. } 875 = 8 \times 10^2 + 75; \frac{8 \times 10^2}{5^2} = 32, \text{ and } \frac{75}{5^2} = 3.$$

$$\text{Ex. 6. } 3,875 = 3 \times 10^3 + 875; \frac{3 \times 10^3}{5^3} = 24, \text{ and } \frac{875}{5^3} = 7;$$

and similarly of other numbers. Hence, the Property.

TESTS OF DIVISIBILITY OF NUMBERS.

161. Since it is frequently necessary in arithmetical computations to determine whether a number is composite or prime, a few of the more common tests of divisibility, deduced from the preceding properties of composite numbers, are here given.

2 is an exact divisor of any even number (Property 1).

3 and **9** are exact divisors of a number, if they are divisors of the sum of the digits of that number (Property 3).

5 is an exact divisor of any number the right-hand figure of which is 0 or 5 (Property 2).

4 is an exact divisor of a number, if it is a divisor of the number expressed by the two right-hand figures of that number (Property 10).

6 is an exact divisor of any even number that is divisible by 3 (Property 8).

8 is an exact divisor of a number, if it is a divisor of the number expressed by the three right-hand figures of that number (Property 10).

14 is an exact divisor of any even number that is divisible by 7 (Property 8).

16 is an exact divisor of a number, if it is a divisor of the number expressed by the four right-hand figures of that number (Property 10).

18 is an exact divisor of any even number that is divisible by 9 (Property 8).

25 is an exact divisor of a number the two right-hand figures of which express 25, 50, 75, or 00 (Property 10).

For tests of divisibility by **7**, **11**, and **13**, see **Supplement**, pages 384, 385.

PROBLEMS.

Which of the numbers in the margin are divisible, and why,

1. By 2?	7. By 9?	54	96	219	576	1,728	1,944
2. By 3?	8. By 14?	72	100	486	381	5,219	2,016
3. By 4?	9. By 16?	65	244	745	759	9,008	18,225
4. By 5?	10. By 18?	90	560	360	972	6,072	174,960
5. By 6?	11. By 25?	88	168	827	999	3,808	725,760
6. By 8?	12. By 72?	69	825	784	1,200	7,254	2,707,200

D

SECTION IV.

FIVE PROPERTIES OF PRIME NUMBERS.

162. PROPERTY 1.—*The right-hand figure of a prime number is 1, 3, 7, or 9.*

DEMONSTRATION.—All even numbers except 2, and all odd numbers ending in 5, except 5, are composite. Hence, the Property.

PROPERTY 2.—*Numbers relatively prime have no common divisor.*

DEMONSTRATION.—A common divisor of two or more numbers is any factor common to those numbers (**145**); and numbers that are relatively prime have no common integral factor (**157**). Hence, the Property.

PROPERTY 3.—*Any integer is either a prime number, or is the product of prime numbers.*

DEMONSTRATION.—Any integer that is not a prime number is a composite number; and a composite number can be separated into integral factors. If any of these factors are composite numbers, they can be separated into other factors; and so on until factors are obtained, all of which are prime numbers (**160**, Property 6). Hence, the Property.

PROPERTY 4.—*When any prime number is divided by 4, the remainder is 1 or 3.*

DEMONSTRATION.—All the remainders that can be obtained by dividing numbers by 4, are 1, 2, and 3; and when the remainder is 2, the number divided is even, and is therefore composite (**160**, Property 1). Hence, the Property.

PROPERTY 5.—*When any prime number is divided by 6, the remainder is 1 or 5.*

DEMONSTRATION.—All the remainders resulting from dividing integers by 6 are 1, 2, 3, 4, and 5.

Since 2 is a factor of 4 and of 6, it is a factor of $2 + 6$, and of $4 + 6$ (**160**, Property 10), and of 2 or 4 + any number of times 6 (**160**, Property 4); and since 3 is a factor of 6, it is a factor of $3 + 6$ (**160**, Property 10), and of 3 + any number of times 6 (**160**, Property 4); i. e., if the remainder is 2, 3, or 4, the number divided is composite. Hence, the Property.

Properties 4 and 5 are true of all prime numbers, but they are not tests for determining whether a number is prime or composite. e. g., $21 \div 4 = 5, + 1$ remainder; and $15 \div 4 = 3, + 3$ remainder. $49 \div 6 = 8, + 1$ remainder; and $35 \div 6 = 5, + 5$ remainder.

For Table of Prime Factors, see **Supplement**, pages 886, 887.

163. To find all the prime numbers less than a given number,

RULE.

I. Write, in their order, the number 2 and all the odd numbers between 2 and the given number.

II. Beginning with the prime number 3, cancel every third number from that place.

III. Then, successively, cancel every fifth number above the prime number 5, every seventh number above the prime number 7, every eleventh number above the prime number 11, and so on.

The numbers not cancelled are the required prime numbers.

DEMONSTRATION.—Since all the even numbers are omitted, every third number above 3 is divisible by 3, every fifth number above 5 is divisible by 5, every seventh number above 7 is divisible by 7, every eleventh number above 11 is divisible by 11, and so on.

In cancelling every third number above 3 and 9, all the composite numbers above 9, of which numbers 3 is a factor, are cancelled. So, also, in cancelling every fifth number above 5, every seventh number above 7, and every eleventh number above 11, every composite number of which 5, 7, and 11 are respectively factors, is cancelled.

In the process every composite number, of the odd numbers given, is cancelled. Hence, the remaining numbers must be prime numbers.

This method of finding prime numbers is called "Eratosthenes's Sieve," from its discoverer.

Ex. Find all the prime numbers less than 60.

PROCESS.

2, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 49, 51, 53, 55, 57, 59.

Prime numbers; { 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59.

PROBLEMS.

1. Find all the prime numbers between 60 and 100.
2. Find all the prime numbers from 100 to 200.
3. Use Eratosthenes's Sieve, and construct a table of prime numbers to 500.

SECTION V.

DIVISORS.

CASE I. Factoring.

164. Factoring is the process of finding the integral factors of a number. It consists in resolving or separating the number into its factors.

165. The process of factoring is based on the following

PRINCIPLES OF DIVISORS.

PRINCIPLE I.—*Any composite number is divisible by any of its factors (144).*

PRINCIPLE II.—*Any composite number is divisible by the product of any two or more of its prime factors (160, Property 6).*

PRINCIPLE III.—*A composite number that is divisible by two or more of its factors successively, is divisible by their product (160, Property 5).*

PRINCIPLE IV.—*A composite number is equal to the product of all its prime factors (160, Property 6).*

PRINCIPLE V.—*The only divisors of a composite number are its prime factors, and the factors that are themselves composite numbers.*

166. RULES FOR FACTORING COMPOSITE NUMBERS.

I. For prime factors :

1. *Divide the number by any prime factor.*
2. *Divide the quotient in the same manner; and so on, till a quotient is obtained that is a prime number.*

The divisors and the last quotient will be the prime factors required.

II. For composite factors :

Multiply together any number of the prime factors less than the whole number of prime factors in the given number.

The product will be a composite factor of the given number.

a. Proceed similarly for other composite factors.

b. The possible number of composite factors of any number equals the number of different combinations of its prime factors by multiplication, taken in sets of at least one factor less than the number of prime factors in the given number.

Ex. 1. What are the prime factors of 630?

EXPLANATION.—Since the right-hand figure of 630 is 0, I divide by the prime number 5 (**161**). Since the right-hand figure of the first quotient (126) is 6, I next divide by the prime number 2 (**161**). Since the sum of the digits of the second quotient (63) is divisible by 3, I divide this quotient by 3 (**161**); and for the same reason, I divide the third quotient (21) by 3. The last quotient (7) is a prime number. The product of the divisors 5, 2, 3, 3, and the last quotient 7, is 630; hence, the numbers 5, 2, 3, 3, 7 are all the prime factors of that number (**165**, Principle IV).

PROCESS.

630	126	63	21	7
5	2	3	3	7

Ex. 2. What are all the exact divisors of 36?

EXPLANATION.—All the prime factors of 36 are 2, 2, 3, and 3. 4 or 2 times 2, 6 or 2 times 3, 9 or 3 times 3, 12 or 2 times 2 times 3, and 18 or 2 times 3 times 3, are composite factors of 36 (**165**, V). Hence, all the factors or exact divisors of 36 are 2, 2, 3, 3, 4, 6, 9, 12, 18.

A number that has but two prime factors has no composite factors.

PROCESS.

$36 = 2 \times 2 \times 3 \times 3$
$2 \times 2 = 4$
$2 \times 3 = 6$
$3 \times 3 = 9$
$2 \times 2 \times 3 = 12$
$2 \times 3 \times 3 = 18$

PROBLEMS.

- | | |
|--|------------------------------------|
| Find | 2. Eight prime factors of 189,072. |
| 1. The twelve prime factors of 25,600. | 3. Seven prime factors of 6,840. |
| Find the six prime factors | Find all the prime factors |
| 4. Of 1,836. | 24. Of 882,882. |
| 5. Of 2,500. | 25. Of 540. |
| 6. Of 40,250. | 26. Of 252. |
| 7. Of 3,327,835. | 27. Of 34,550. |
| Find the five prime factors | 28. Of 5,005. |
| 8. Of 1,650. | 29. Of 91,001. |
| 9. Of 945. | 30. Of 317. |
| 10. Of 3,990. | 31. Of 509. |
| 11. Of 33,915. | Find all the factors |
| Find the four prime factors | 32. Of 87. |
| 12. Of 8,595. | 33. Of 210. |
| 13. Of 59,890. | 34. Of 525. |
| 14. Of 39,736. | 35. Of 37. |
| 15. Of 315. | 36. Of 1,341. |
| 16. Of 798. | 37. Of 75. |
| 17. Of 350. | 38. Of 120. |
| Find the three prime factors | 39. Of 144. |
| 18. Of 741. | 40. Of 1,000. |
| 19. Of 245. | 41. Of 1,755. |
| 20. Of 51,127. | 42. Of 90,447. |
| 21. Of 6,629,053. | 43. Of 101,101. |
| 22. Of 26,059. | 44. Of 17,727. |
| 23. Of 116,405. | 45. Of 666,666. |
| | 46. Of 164. |
| | 47. Of 2,500. |

NOTE.—In solving many of the problems in this chapter, the process may be shortened by using the Table of Prime Factors (Supplement, pages 386, 387).

CASE II. Common divisors of integers.

167. Define a common divisor (145).

The common factors—prime and composite—of two or more numbers are common divisors of those numbers.

168. The methods of finding a common divisor in this Case may readily be deduced from the following

PRINCIPLES OF COMMON DIVISORS.

PRINCIPLE I.—*All the prime factors of a common divisor of two or more numbers are also prime factors of the numbers.*

The truth of this Principle is apparent from the definitions of common divisor and prime factors (145, 156).

PRINCIPLE II.—*The product of any two or more of the prime factors common to any given numbers is a common divisor of those numbers.*

The reason for the truth of this Principle is readily inferred from 160, Property 6.

PRINCIPLE III.—*A common divisor of two numbers is their difference, or some factor of that difference.*

DEMONSTRATION.—The less of two numbers plus their difference equals the greater. Hence, if the difference between two numbers is a factor of the less number, it is also a factor of the less number plus the difference, *i. e.*, of the greater number. And any factor of the difference that is also a factor of the less number, is a factor of the less number plus the difference, *i. e.*, of the greater number. Hence, the Principle.

PRINCIPLE IV.—*If the less of two numbers is not their common divisor, then, all their common divisors are common divisors of the less number and the remainder obtained by dividing the greater number by the less.*

DEMONSTRATION.—The dividend is always equal to the remainder plus the divisor, or to the remainder plus some number of times the divisor; consequently, the remainder is either the difference between the two numbers, or between the greater number and some number of times the less number.

A number that is a factor of the remainder plus the divisor, is also a factor of the remainder plus any number of times the divisor, *i. e.*, it is a factor of the dividend.

Hence, the Principle.

If the difference between two numbers is a prime number, this difference is the common divisor of the two numbers, or the two numbers are relatively prime.

Ex. Find all the common divisors of 84, 126, and 168.

EXPLANATION.—The common divisors of two or more numbers are the common prime factors of those numbers (Principle I), and the product of any two or more of these prime factors (Principle II).

PROCESS.				
84	42	14	2	$2 \times 3 = 6$
126	63	21	3	$2 \times 7 = 14$
168	84	28	4	$3 \times 7 = 21$
2	3	7		$2 \times 3 \times 7 = 42$

To find the common prime factors, I write the numbers in columns, as for addition, and divide successively by the common prime factors 2, 3, and 7, writing the quotients in columns, at the right of the numbers divided.

Since the last quotients 2, 3, and 4 have no common factor, the factors 2, 3, and 7 are all the prime numbers that are common divisors of the given numbers (Principle I).

To find the common composite factors, I multiply together the prime factors 2 and 3, 2 and 7, 3 and 7, and 2, 3, and 7; and since no other combination of these factors can be made by multiplication, the products 6, 14, 21, and 42 are all the composite numbers that are common divisors of the given numbers (Principle II). Hence, 2, 3, 7, 6, 14, 21, and 42 are all the common divisors of 84, 126, and 168.

PROBLEMS.

48. Find a common divisor of 21 and 36.

49. Find the common prime factors of 20, 32, 56, and 18.

Find all the common divisors

50. Of 36, 42, and 90.

51. Of 64, 112, 48, and 144.

52. Of 72, 96, and 216.

53. Of 240, 320, and 480.

54. Of 18,900, 11,970, 6,090, and 21,630.

55. Of 1,518, 858, 5,874, and 8,382.

CASE III. Greatest common divisor of integers.

169. Define the greatest common divisor (**146**).

170. The methods of finding a greatest common divisor in this Case are based on the following

PRINCIPLES OF GREATEST COMMON DIVISORS.

PRINCIPLE I.—The greatest common divisor of two or more numbers is the product of all their common prime factors.

DEMONSTRATION.—The greatest exact divisor common to two or more integers is their greatest common divisor (**146**); this greatest common divisor is the product of all its prime factors (**165**, Principle IV); and these prime factors are all the prime factors common to the given numbers. Hence, the Principle.

PRINCIPLE II.—*The greatest common divisor of the less of two numbers and the remainder obtained by dividing the greater number by the less, is the greatest common divisor of the two numbers.*

DEMONSTRATION.—A common divisor of the less of two numbers and the remainder obtained by dividing the greater number by the less, is a common divisor of the two numbers (168, Principle IV); and

Since no number greater than the remainder can be a common factor of the two numbers, their greatest common divisor must be the remainder, or some factor of that remainder; consequently, it must be the greatest common divisor of the remainder and the less number. Hence, the Principle.

PRINCIPLE III.—*The greatest common divisor of two numbers is the less number, or the difference between the numbers, or some factor of that difference.*

DEMONSTRATION.—1st. Since the greatest common divisor of two numbers can not be greater than the less number, when the less number is an exact divisor of the greater, it is the greatest common divisor.

2d. Since the greatest common divisor of two numbers can not be greater than their difference, when the difference is a common divisor of the two numbers, it is their greatest common divisor

3d. And when neither the less of two numbers, nor their difference, is a common divisor of the two numbers, their greatest common divisor must be some exact divisor of that difference. Hence, the Principle.

171. First Method.

Ex. Find the greatest common divisor of 336, 420, and 504.

EXPLANATION.—I first find the common prime factors of the given numbers, as in Case II. Then, since the product of all these common prime factors (2, 3, 7, 2) is the greatest common divisor of the given numbers (Principle I), I multiply these common prime factors together, and obtain 84, which is the greatest common divisor of the given numbers.

PROCESS.				
336	168	56	8	4
420	210	70	10	5
504	252	84	12	6
$2 \times 3 \times 7 \times 2 = 84$				

172. Second Method.

Ex. Find the greatest common divisor of 63 and 231.

EXPLANATION.—I divide 231 by 63 (the greater number by the less), and obtain a remainder of 42; hence, 63 is not a divisor of the two numbers.

The greatest number that is a common divisor of 63 and 42 is the greatest common divisor of 63 and 231 (Principle II); and since this divisor can not be greater than 42, I divide 63 by 42, and obtain a remainder of 21; hence, 42 is not a common divisor of the two given numbers.

PROCESS.		
	231	
63	189	3
	42	1
21	42	2

Again, the greatest number that is a common divisor of 42 and 21 is also the greatest common divisor of 42 and 63 ($63 = 42 + 21$), and of 63 and 231 ($231 = 42 + 8 \times 63$). 21 is the greatest common divisor of 42 and 21; and hence, it is the greatest common divisor of the given numbers.

In the written work by this method, observe

1. That the various terms are so placed that they may readily be used as either dividend or divisor;
2. That the *division* is *first* from left to right, *then* from right to left;
3. That the *divisor* is *always* one line lower than the *dividend*; and
4. That the *quotient figure* is *always* on the line with, and at the right of *divisor* and *product*.

173. On the Principles of Divisors, given in this case, are based the

RULES FOR FINDING A GREATEST COMMON DIVISOR.

I. For First Method:

1. *Divide the given numbers by any common prime factor; divide the results in the same manner; and so continue, till results are obtained that have no common prime factor.*

2. *Multiply together all the numbers used as divisors.*

The product will be the greatest common divisor.

II. For Second Method:

Divide the greater number by the less, the first divisor by the first remainder, the second divisor by the second remainder, and so on, until an exact divisor is obtained.

This divisor will be the greatest common divisor.

a. By the Second Method, if more than two numbers are given, first find the greatest common divisor of two of the numbers, then of their greatest common divisor and another of the numbers; and so on, till all the given numbers have been used.

The last common divisor obtained will be the greatest common divisor of all the given numbers.

*b. The First Method is also called the method by factoring; and
The Second Method, the method by division.*

174. PROPOSITIONS.

I.—*Prove that the only numbers that can have a common divisor are abstract numbers, or like concrete numbers of the same denomination.*

II.—*Prove that the common divisor of two or more concrete numbers may be either an abstract number or a concrete number.*

PROBLEMS.

Find the greatest common divisor

- | | |
|--|---|
| 56. Of 112 and 168. | 65. Of 2,014 and 2,124. |
| 57. Of 72, 120, 196. | 66. Of 3,535 and 8,989. |
| 58. Of 225, 350, 175. | 67. Of 3,895; 82,164; 41,041. |
| 59. Of 2,040; 4,080; 1,628. | 68. Of 793; 1,950; 2,171. |
| 60. Of 1,232 and 1,628. | 69. Of 3,013; 2,231; 2,047. |
| 61. Of 3,975 and 4,611. | 70. Of 153 and 494. |
| 62. Of 1,088; 1,632; 3,808. | 71. Of 5,775; 8,547; and 10,325. |
| 63. Of 121, 87, 112, 72. | 72. Of 4,328; 6,492; and 11,361. |
| 64. Of 10,800 and 11,400. | 73. Of 5,946; 9,910; and 18,829. |
| 74. Of 1,649; 5,423; 1,981; 323; and 2,198. | |
| 75. Of 47,648; 1,843,420; and 2,254,764. | |

76. What is the length of the longest line that will exactly measure the sides of two fields, one 192 rods and the other 152 rods long?

77. The dimensions of my orchard are 504 ft., 1,120 ft., 546 ft., and 756 ft. What is the length of the longest boards that I can use in fencing it, without cutting any of them?

78. How many fruit-trees can be set on a piece of ground 264 ft. long, and 231 ft. wide, placing them at the greatest equal distance from one another, both ways in rows, and allowing 1 rod on every side for the fence?

79. Three regiments of soldiers of 819, 910, and 1,092 men, are formed into companies,—the largest that will admit the same number of men in each without uniting soldiers of different regiments. What is the number of soldiers in each company?

80. A merchant buys three lots of muslins measuring, respectively, 280 yd., 440 yd., and 520 yd., in pieces of the greatest equal length possible in yards. What is the length of each piece?

81. What is the greatest price that I can pay per head for cattle, and expend either \$972, \$2,400, \$1,404, or \$3,636?

SECTION VI.

MULTIPLES.

CASE I. Common Multiples.

175. The method of finding a common multiple is based on the two following principles :

PRINCIPLE I.—*Any multiple of a number is a multiple of any factor of that number.*

DEMONSTRATION.—A multiple is a composite number (**148**); and any composite number is divisible by any of its factors (**165**, I). Hence, the Principle.

The product of two or more numbers, or any multiple of their product, is a common multiple of the numbers.

PRINCIPLE II.—*A common multiple of two or more numbers is a multiple of all the factors of each of those numbers.*

DEMONSTRATION.—A common multiple of two or more numbers is exactly divisible by each of the numbers (**149**); and any number that is divisible by two or more numbers, is divisible by any factor of those numbers (**160**, Property 4). Hence, the Principle.

Either the greatest of two or more numbers, or some multiple of this greatest number, is a common multiple of the numbers.

Ex. Find a common multiple of 12 and 21.

EXPLANATION.—A multiple of 12 contains any factor of 12, and a multiple of 21 contains any factor of 21 (Principle I). A common multiple of 12 and 21 contains all the factors of both 12 and 21 (Principle II); and the product of two numbers contains all the factors of those numbers.

PROCESS.

$$12 \times 21 = 252$$

I therefore multiply 12 and 21 together, and obtain 252, which is a common multiple of the given numbers.

PROPOSITION.—*Show that 2 times 252, 5 times 252, 8 times 252, or any other number of times 252 is also a common multiple of the given numbers (Principle I, Remark).*

PROBLEMS.

Find a com-	{ 1. Of 5, 7, 15, and 16.	3. Of 2, 15, and 85.
mon multiple	{ 2. Of 9, 48, and 525.	4. Of 89 and 761.

5. Find three common multiples of 36, 24, and 12.

CASE II. Least Common Multiple.

176. PRINCIPLES.

PRINCIPLE I.—*The least common multiple of two or more numbers is a multiple of all the prime factors of each of those numbers, and of no other prime factors.*

DEMONSTRATION.—Any multiple of a number contains all the prime factors of that number (175, Principle I); it may contain other prime factors.

A common multiple of two or more numbers contains all the prime factors of those numbers (175, Principle II); it may contain other prime factors.

The least common multiple of two or more numbers, being exactly divisible by each of them, contains all their prime factors; and being the least number that is exactly divisible by each of them, it contains no other prime factors; i. e., it contains only the prime factors necessary to form each of the given numbers taken separately. Hence, the Principle.

The product of two or more relatively prime numbers is their least common multiple.

PRINCIPLE II.—*The least common multiple of two or more numbers contains every prime factor of each of the numbers the greatest number of times it occurs in any of the numbers.*

DEMONSTRATION.—Since the least common multiple of two or more numbers contains the prime factors necessary to form each of the given numbers, taken separately, it must contain any prime factor of each of the numbers as many times as the greatest number of times that factor is a factor in any of the given numbers. Hence, the Principle.

PRINCIPLE III.—*The least common multiple of two or more numbers is either the greatest of the numbers, or the product of this greatest number and those prime factors of the other numbers that are not factors of this greatest number.*

DEMONSTRATION.—If each of the less numbers is a factor of the greatest number, then the greatest number is evidently the least common multiple of all the numbers (150).

The least common multiple must be a common multiple of all the prime factors of all the numbers (Principle I). Since the greatest number is a multiple of all its factors, it is also a multiple of any of the same factors of any of the other numbers. If the other numbers contain other factors than those of the greatest number, then the least common multiple must be a multiple of those factors also; i. e., of all the factors of the greatest number, and of all the factors of all the other numbers not factors of this greatest number. Hence, the Principle.

177. On the Principles of Multiples given in 175, 176, are based the

RULES FOR FINDING MULTIPLES.

I. For a common multiple:

Multiply the numbers together.

The product, or any number of times the product, will be a common multiple.

II. For the least common multiple:

First Method.

1. *Separate the numbers into their prime factors.*
2. *Multiply together the largest given number and those prime factors of the other given numbers not found among the factors of this largest number.*

The product will be the least common multiple.

Second Method.

1. *Cancel all the numbers that are factors of any of the greater given numbers, and write the numbers not cancelled in a column, in order, from least to greatest.*

2. *Divide the remaining numbers by any composite number that is a factor of all of them, or by any prime number that is a factor of two or more of them, and write the quotients and undivided numbers in a column at the right.*

3. *Divide these numbers in the same manner; and continue to divide, until results are obtained that are relatively prime numbers.*

4. *Multiply together the divisors and the numbers in the right-hand column.*

The product will be the least common multiple.

The only numbers that can have a common multiple are abstract numbers or like concrete numbers of the same denomination.

178. Ex. Find the least common multiple of 12, 16, and 20.

DEMONSTRATION.—The least common multiple required is 20 or some multiple of 20 (**176**, Principle III). Since 20 is not a multiple of 12 and 16, the factors of the required least common multiple are 20 and those prime factors of 12 and 16 that are not factors of 20 (**176**, Principle III).

PROCESS BY FIRST METHOD.

$$12 = 2 \times 2 \times 3$$

$$16 = 2 \times 2 \times 2 \times 2$$

$$20 = 2 \times 2 \times 5$$

$$20 \times 3 \times 2 \times 2 = 240$$

Comparing separately the prime factors of 12 and 16 with the prime factors of 20, I find that I have all the prime factors of 12 but 3, and all the prime factors of 16 but 2 and 2; and I write these numbers and the 20 in a line, as factors of the required least common multiple (**176**, Principle I).

Multiplying these factors together, I obtain the least common multiple of the given numbers.

179. Ex. Find the least common multiple of 15, 18, 54, 90, 105, and 315.

DEMONSTRATION. — I write the numbers in a line, in order, from least to greatest; and since 90 is a multiple of 15 and 18, and 315 is a multiple of 105, I cancel 15, 18, and 105, and write the remaining numbers in a column.

PROCESS BY SECOND METHOD.

15, 18, 54, 90, 105, 315					
54	27	9	3	3	
90	45	15	5		
315	315	115	35	7	
$2 \times 3 \times 3 \times 5 \times 3 \times 7 = 1,890$					

Or, I cancel 15 and 18 (which are factors of 90), and 105 (which is a factor of 315), and write the remaining numbers in a column.

Since 2 is a prime factor of one or more of the remaining numbers, it is a factor of the least common multiple (**176**, Principle I). I therefore divide by 2, and write the quotients and undivided numbers in a column at the right.

For a like reason, I divide successively by the prime numbers 3, 3, and 5, and write the results as before. The numbers 3 and 7 in the last column being prime numbers, further exact division is impossible.

The divisors 2, 3, 3, and 5, and the numbers 3 and 7 in the last column, are all of the prime factors of the given numbers; each is taken as many times as it is used as a factor in any of the given numbers (Principle II); and hence, these numbers are all of the prime factors of the least common multiple of all the given numbers (**176**, Principle I).

Multiplying these factors together, I obtain 1,890, the required least common multiple. (See **Supplement**, pages 888, 889.)

PROBLEMS.

- | | |
|--|---|
| 6. Find the least common multiple of 9, 15, and 18. | |
| 7. Of \$14 and \$18. | 12. Of 324 and 240. |
| 8. Of 2, 3, 4, 5, 6, 7, 8, and 9. | 13. Of 4,230 and 3,807. |
| 9. Of 4, 16, 20, 48, 60 and 72. | 14. Of 487; 1,461; 5,357. |
| 10. Of 125, 200, and 375. | 15. Of 126, 154, 280, and 560. |
| 11. Of 45, 9, 100, and 15. | 16. Of 315, 626, 1,565, and 937. |
| 17. Of 9, 60, 45, 72, 15, 35, 18, and 12. | |

18. What is the smallest sum of money with which a fruit-vender can buy apples at 1¢, peaches at 2¢, oranges at 3¢, and melons at 4¢ each?

19. What is the shortest distance that can be exactly measured by lines either 18 ft., 12 ft., 24 ft., 15 ft., or 21 ft. long?

20. A certain mowing-machine can cut 9 acres of grass in a day, another 10 acres, and another 12 acres. Required the least number of acres that will furnish an integral number of days' work for any of the machines.

21. What is the smallest sum of money that may be in either 2-cent, 3-cent, 5-cent, 10-cent, or 25-cent pieces?

22. What is the smallest quantity of flour that will exactly fill bags holding 4 lb., 5 lb., 6 lb., 12 lb., and 25 lb.? How many bags will there be of each kind?

23. A can build 14 rd. of fence in a day, B 25 rd., C 8 rd., and D 20 rd. What is the least number of rods that will furnish a number of whole days' work to any one of the four men?

24. What is the smallest sum of money with which I can buy chickens at \$.40 each, ducks at \$.85 each, or turkeys at \$1.50 each? How many of each kind can I buy for this sum?

25. What is the smallest sum of money with which I can buy pears at 4 cents, lemons at 6 cents, oranges at 8 cents, or bananas at 10 cents each, and have no money left?

26. What is the smallest sum of money with which I can buy either sheep at \$3.50 per head, calves at \$10.50, cows at \$35, oxen at \$70, or horses at \$105?

27. What is the least distance that can be accurately measured with a foot rule, a yard-stick, or a rod pole?

28. Allowing 222 5-penny nails for a pound, 104 8-penny nails or 8's, 68 10's, 54 12's, or 34 20's, what is the least number of nails of any of the five sizes that will make an exact number of pounds?

29. What is the least number of spikes that will make an exact number of pounds of either 4-inch spikes, at 16 to the pound; $4\frac{1}{2}$ -inch, 12 to the pound; 5-inch, 10 to the pound; 6-inch, 7 to the pound; or 7-inch, 5 to the pound?

SECTION VII.

CANCELLATION.

180. Cancellation is the process of omitting or striking out equal factors from the dividend and divisor.

181. The process is based upon the following

PRINCIPLES.

PRINCIPLE I.—*Cancelling a factor from a number divides the number by that factor.*

DEMONSTRATION.—Since any composite number equals the product of its factors (165, Principle IV), if one of the factors be cancelled or omitted, the product of the remaining factors will equal the quotient of the composite number divided by the factor cancelled. Hence, the Principle.

PRINCIPLE II.—*Cancelling a common factor from dividend and divisor does not change the value of the quotient.*

DEMONSTRATION.—Any part of a divisor—as one half, one third, or one fourth of it—is contained in an equal part of the dividend as many times as the entire divisor is contained in the entire dividend; i. e., both dividend and divisor may be divided by the same factor without affecting the value of the quotient. Hence, the Principle.

Ex. Divide $12 \times 40 \times 3 \times 18.9$ by $6.3 \times 72 \times 4$.

EXPLANATION.—I write the factors of the dividend above, and those of the divisor below a horizontal line. I then cancel all the factors common to the numbers on opposite sides of the dividing line, and have no factor remaining for a divisor, and the factors 5 and 3 for a dividend. Hence, $15 (= 5 \times 3)$ is the required result.

The factors of the dividend may be written on the right, and those of the divisor on the left of a vertical line, as in the Second Process.

a. The quotient of a number divided by itself is 1; hence, when either dividend or divisor is cancelled, a 1 belongs in its place.

FIRST PROCESS.

$$\begin{array}{r}
 5 \qquad \qquad \qquad 3 \\
 12 \quad 40 \quad 3 \quad 18.9 \\
 \hline
 6.3 \times 72 \times 4
 \end{array} = 5 \times 3 = 15$$

SECOND PROCESS.

$$\begin{array}{r|l}
 6.3 & 12 \\
 2 \ 6 \ 72 & 40 \ 10 \ 5 \\
 4 & 3 \\
 \hline
 & 18.9 \ 3 \\
 & \hline
 & 5 \times 3 = 15
 \end{array}$$

If 1 is the only remaining factor of the dividend, it should be written in the quotient. In all other cases it may be omitted.

b. Any factor in either term—dividend or divisor—cancels only one factor of equal value in the other term.

c. When a number is cancelled, an oblique line should be drawn across it.

PROBLEMS.

1. Divide 975 by 7 times 25.

2. The factors of the dividend are 12, 18, 15, 10, and 21; and the factors of the divisor are 30, 45, 8, 9, and 7. What is the quotient?

3. $\frac{12 \times 5 \times 153 \times 35 \times 18 \times 2}{3 \times 14 \times 9 \times 5 \times 17 \times 20 \times 6} = \text{what number?}$

4. Find the quotient of $30 \times 85 \times 462 \times 27,720 \times 242 \times 1,155$, divided by $72 \times 126 \times 455 \times 198 \times 2,057 \times 630$.

5. How many tons of iron at \$45 a ton must be given in exchange for 160 tons of coal at \$5 a ton?

6. In how many days can 486 men do as much work as 63 men can do in 270 days?

7. If a machinist can earn \$110 in 35 days, how much can he earn in the working days of 7 weeks?

8. I rent a hall 16 months for \$1,800. What is the rent per year?

9. If 480 acres of land produce 10,800 bushels of wheat, how much wheat will 176 acres produce at the same rate?

10. A garrison of 98 men has provisions enough to last it 84 days. If the garrison be increased to 126 men, how long will the same provisions last?

11. In erecting a building, 11 bricklayers laid 515,880 bricks in 50 days. At the same rate, how many bricks can 15 bricklayers lay in 55 days?

12. If a stenographer can report 11,880 words in 198 minutes, how many words can he report in 24 minutes?

SECTION VIII.

REVIEW.

182. TEST QUESTIONS IN PROPERTIES OF NUMBERS.

I. Prove that the greatest common divisor of two or more numbers contains only the prime factors common to those numbers.

II. Prove that the least common multiple of two or more numbers contains no other prime factors than the prime factors of those numbers.

III. Prove that 2 and 5 are the only prime factors of an integral unit of any order.

IV. In finding a least common multiple, when *may* a composite number be used for a divisor; and when *must* a prime number be used?

V. On what principle is the process of cancellation based?

Of what practical use

VI. Is least common multiple?

VIII. Are tests of divisibility?

VII. Is greatest common divisor?

IX. Is cancellation?

X. Show that the greatest common divisor of two numbers is their difference or some factor of their difference.

183. BLACKBOARD OUTLINES.

A. GENERAL DEFINITIONS AND AXIOMS.	{	I. TERMS.	{ 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____	{ 7. _____ 8. _____ 9. _____ 10. _____ 11. _____
		II. PRIMARY FACTS.	{ 1. _____ 2. _____	
		III. THE SEVEN AXIOMS.		
B. PROPERTIES OF EVEN AND ODD NUMBERS.	{	I. COMPOSITE AND PRIME.	{ 1. _____ 2. _____	
		II. SUMS.	{ 1. _____ 2. _____ 3. _____ 4. _____	
		III. DIFFERENCES.	{ 1. _____ 2. _____ 3. _____	
		IV. PRODUCTS.	{ 1. _____ 2. _____ 3. _____	
		V. QUOTIENTS.	{ 1. _____ 2. _____ 3. _____	
		VI. FACTORS.	{ 1. _____ 2. _____	

C. PROPERTIES OF COMPOSITE NUMBERS. {

1.	_____
2.	_____
3.	_____
4.	_____
5.	_____
6.	_____
7.	_____
8.	_____
9.	_____
10.	_____

D. THE 11 (or 14) TESTS OF DIVISIBILITY.

E. PRIME NUMBERS. {

{	I. TERMS.	1.	_____
		2.	_____
	II. PROPERTIES.	1.	_____
		2.	_____
		3.	_____
4.		_____	
III. RULE.	5.	_____	

F. DIVISORS. {

{	I. TERMS.	1.	_____	8.	_____	5.	_____
		2.	_____	4.	_____	6.	_____
	II. PRINCIPLES.	1. <i>Of Divisors.</i>	a.	_____			
			b.	_____			
			c.	_____			
			d.	_____			
			e.	_____			
		2. <i>Of Common Divisors.</i>	a.	_____			
			b.	_____			
			c.	_____			
3. <i>Of Greatest Common Divisors.</i>		a.	_____				
	b.	_____					
III. RULES.	1. <i>For Factoring.</i>	a.	_____				
		b.	_____				
	2. <i>For Greatest Common Divisor.</i>	a.	_____				
		b.	_____				

G. MULTIPLES. {

{	I. TERMS.	1.	_____	3.	_____
		2.	_____		_____
	II. PRINCIPLES.	1. <i>Of Common Multiples.</i>	a.	_____	
			b.	_____	
		2. <i>Of Least Common Multiple.</i>	a.	_____	
			b.	_____	
			c.	_____	
			d.	_____	
	III. RULES.	1.	a.	_____	
			b.	_____	
2. <i>For the Least Common Multiple.</i>		a.	_____		
		b.	_____		

H. CANCELLATION. {

{	I. TERM.	_____	
	II. PRINCIPLES.	1.	_____
		2.	_____
	III. IMPORTANT FACTS.	1.	_____
2.		_____	
IV. RULE.	_____		

CHAPTER III.

REDUCTIONS.

In arithmetical computations it is often necessary to change numbers and their unit values, without changing the values expressed by the numbers.

184. Reduction is the process of changing numbers from given to required units, without changing their values.

Reduction is of two kinds—*descending* and *ascending*.

185. Reduction descending is the process of changing a number to another of less unit value.

186. Reduction ascending is the process of changing a number to another of greater unit value.

Which kind of reduction is used in changing

- | | |
|--|--------------------------------|
| 1. Ones to hundredths, thousandths, millionths, and so on? | 7. Gallons to gills? |
| 2. Tenths to millionths? | 8. Billionths to tenths? |
| 3. Cents to dollars? | 9. Dollars to cents and mills? |
| 4. Halves to eighths? | 10. An integer to a decimal? |
| 5. Twelfths to ones? | 11. A decimal to an integer? |
| 6. Pints to bushels? | |

SECTION I.

REDUCTIONS OF NUMBERS IN THE DECIMAL SCALE.

187. Similar decimals are decimals that have the same number of decimal places.

188. Dissimilar decimals are decimals that have different numbers of decimal places.

189. PRINCIPLE OF DECIMAL REDUCTIONS.

Annexing decimal ciphers to a number, or removing decimal ciphers from the right of a number, does not change its value.

DEMONSTRATION.—

- .5 is 5 tenths ;
- .50 is 5 tenths and 0 hundredths, or 50 hundredths ;
- .500 is 5 tenths, 0 hundredths, and 0 thousandths, or 500 thousandths ; *i.e.*,
- .5, .50, and .500 all express the same value, namely, 5 tenths.

This illustration shows that

Ciphers on the right of a decimal do not change the places of the other figures.
Hence, the Principle.

190. On the principle above given are based the

RULES FOR THE REDUCTIONS OF DECIMAL NUMBERS.

I. To lower decimal units:—*Annex decimal ciphers, till the number has the required number of decimal places.*

II. Dissimilar to similar decimals:—*Annex decimal ciphers to the several given numbers, till all have the same number of decimal places.*

III. To higher decimal units:—*Remove decimal figures from the right of the given number, till the lowest order of units expressed by it is of the order required.*

1. How are dollars reduced to cents? Why? To mills? Why?
2. How are mills reduced to cents? Why? To dollars? Why?
3. How are cents reduced to dollars? Why?

Which of the three Rules will be used to reduce

- | | |
|--|-----------------------------------|
| 4. An integer to thousandths? | 7. Cents to dollars? |
| 5. Hundredths to ten-millionths? | 8. Dollars to mills? |
| 6. Millionths to tenths? | 9. Ten-thousandths to an integer? |
| 10. Tenths, ten-thousandths, and hundred-millionths to similar decimals? | |

Reduce

PROBLEMS.

- | | |
|---------------------------------|-------------------------------|
| 1. .5 to hundredths. | 9. 629 to ten-billionths. |
| 2. .8 to thousandths. | 10. 25.402 to millionths. |
| 3. .3 to hundredth-thousandths. | 11. 10.08 to ten-thousandths. |
| 4. .06 to ten-millionths. | 12. .0103 to quadrillionths. |
| 5. .31 to billionths. | 13. .014 to trillionths. |
| 6. .057 to hundred-billionths. | 14. 810 to hundredths. |
| 7. .090 to hundred-millionths. | 15. 8.1 to hundredths. |
| 8. .1 to ten-trillionths. | 16. 1.004 to millionths. |

17. .0001, .1000, 75.01, 102, 1,000.030, and 5,010 to ten-quadrillionths

18. 901, .901, 3, and .3 to trillionths.

Change {	19. .200 to tenths.	22. 26.000 to tenths.
	20. .350000 to ten-thousandths.	23. 869.00000 to hundredths
	21. .7900000 to thousandths.	24. 105.000 to ones.

25. 10.000 to millionths; to hundredths; to ones.

26. .79, 1,001, 99.011, 1.480000 and .0001000 to ten-thousandths.

27. 10,010.25, .01007, .00050000, .0655000000, 91, 50.05, and 1,000,000 to similar decimals having one millionth for their common decimal unit.

28. Reduce thirty-one thousand millionths, two hundred sixty-five tenths, and three thousand fifty hundredths to thousandths.

29. Change seven hundred twenty hundredths to a decimal having one tenth for a decimal unit.

30. Reduce .75 of an acre to millionths of an acre.

31. 49 miles are how many ten-millionths of a mile?

32. Change 1,300, 19.1, 100.010, and 64 to similar decimals.

33. Reduce 77, .2, 15, .1001, 3, 125, and 3.35 to similar decimals.

What is the greatest common decimal unit

34. Of 80.057, 98, 9.8, 1,000.1, .9000, .076000, and 1.00330000?

35. Of 60,000, .006, .22, and 41.300508?

36. Of 4.4, .2408, .18009, 752, and 6,604.000031?

37. Reduce nine hundred nine hundred-thousandths, nine hundred and nine thousandths, and nine hundred and nine hundred-thousandths to decimals having a common decimal unit.

38. Reduce two hundred thousand, two hundred-thousandths, and two hundred two thousandths to similar decimals.

Reduce {	39. 36 dollars to mills.	43. 5,000 dollars 18 cents to cents.
	40. \$12.15 to cents.	44. 958,427 cents to dollars.
	41. 75 cents 9 mills to mills.	45. 485,260 mills to dollars.
	42. 10,495 mills to cents.	46. 93 dollars $\frac{1}{4}$ cent to mills.

47. Change 125 dollars 62 cents 5 mills to mills.

48. Change 68,475 mills to cents; to dollars.

49. Change 3,005,805 mills to dollars, cents, and mills.

SECTION II.

REDUCTIONS OF FRACTIONS.

Reductions of fractions are used for preparing numbers for addition, subtraction, multiplication, and division; and in reducing final results to their simplest forms. The processes in the six Cases given in this Section depend upon the following

191. PRINCIPLES.

I. The value of a fraction is not changed by dividing its terms by any common factor.

For demonstration of this Principle, see Cancellation, page 64, Prin. II.

II. The value of a fraction is not changed by multiplying its terms by any number.

DEMONSTRATION.—The terms of a fraction correspond to dividend and divisor in division.

The quotient obtained by dividing one of two numbers by the other is the same as the quotient obtained by dividing any number of times the one number by the same number of times the other; consequently, the value of any given fraction is the same as that of another fraction obtained by multiplying the terms of the given fraction by the same number. Hence, the Principle.

CASE I. A fraction to lowest terms.

192. A fraction is in its *lowest terms* when its terms have no common factor.

EX. Reduce the fraction $\frac{108}{162}$ to its lowest terms.

FIRST PROCESS.

EXPLANATION.—I divide 108 and 162, the terms of the given fraction, by the common factor 9 (Principle I) and obtain $\frac{12}{18}$; I divide 12 and 18, the terms of this result, by the common factor 6, and obtain $\frac{2}{3}$, as shown in the First Process.

$$\frac{108}{162} = \frac{12}{18} = \frac{2}{3}$$

SECOND PROCESS.

Since the terms 2 and 3 have no common factor, they are the lowest terms of the fraction $\frac{108}{162}$; and $\frac{2}{3}$ is the result required.

$$\frac{108}{162} = \frac{2}{3}$$

Or, I divide the terms 108 and 162 by their greatest common divisor 54, and obtain $\frac{2}{3}$, the result required; as shown in the Second Process.

PROBLEMS.

Reduce to their lowest terms

1. $\frac{48}{120}$, $\frac{84}{105}$, and $\frac{875}{1000}$.

2. $\frac{36}{112}$, $\frac{32}{108}$, and $\frac{108}{352}$.

3. $\frac{51}{252}$, $\frac{825}{2000}$, and $\frac{10444}{11111}$.

4. $\frac{5228}{10028}$, $\frac{7243}{39408}$, and $\frac{1008000}{10019000}$.

5. $\frac{5340}{33300}$, $\frac{174882}{174882}$, and $\frac{222}{222}$.

6. $\frac{77}{111}$, $\frac{122}{111}$, and $\frac{873}{2078}$.

CASE II. An integer, a fraction, or a mixed number to a fraction having a given denominator.

193. Ex. 1. Reduce the fraction $\frac{13}{7}$ to 175ths.

PROCESS.

$$175 \div 25 = 7$$

EXPLANATION.—I first divide 175 the required denominator, by 25 the given denominator, and obtain 7, the number by which the terms of the fraction $\frac{13}{7}$ must be multiplied to reduce it to 175ths (191, Principle II). I then multiply the terms 13 and 25 by 7, and obtain $\frac{91}{175}$, the required result. (See Supplement, page 390.)

Ex. 2. Change 29 to 8ths, and $9\frac{1}{5}$ to 75ths.

PROCESSES.

$$29 = \frac{29}{1}$$

$$\frac{29}{1} \times \frac{8}{8} = \frac{232}{8}$$

$$75 \div 15 = 5$$

$$9\frac{1}{5} = \frac{45}{5}, \text{ and } \frac{45}{5} \times \frac{5}{5} = \frac{225}{5}$$

Reduce

PROBLEMS.

- | | |
|--|--|
| <p>7. 379 to fourteenths.</p> <p>8. 59 to ninths.</p> <p>9. 764 to fifty-ninths.</p> <p>10. 907 to 125ths.</p> | <p>11. $\frac{3}{8}$ to tenths; to twenty-fifths; to 140ths.</p> <p>12. $\frac{2}{15}$ to 169ths; to 299ths; to 442ds.</p> <p>13. $93\frac{1}{11}$ to 594ths, and $51\frac{1}{5}$ to 7,921sts.</p> <p>14. $\frac{10}{11}$ is how many 95,375ths?</p> <p>15. Change $\frac{3}{5}$ and $\frac{1}{7}$ to fractions whose denominators are 105.</p> <p>16. $\frac{3}{8}$ qt. is how many 5,000ths of a quart?</p> <p>17. 2,222$\frac{1}{2}$ sq. yd. are how many 819ths of a square yard?</p> |
|--|--|

CASE III. Dissimilar fractions to similar fractions.

194. Ex. Reduce $\frac{2}{5}$, $\frac{1}{7}$, and $\frac{3}{8}$ to similar fractions.

PROCESS.

EXPLANATION.—Since the product of the denominators 5, 7, and 8 is 210, the given fractions may be reduced to 210ths. I do this by multiplying the terms of $\frac{2}{5}$ by 7 and 6, the terms of $\frac{1}{7}$ by 5 and 6, and the terms of $\frac{3}{8}$ by 5 and 7; i. e., I multiply the terms of each fraction by the denominators of the other fractions.

The results $\frac{84}{210}$, $\frac{30}{210}$, $\frac{45}{140}$ are similar fractions.

$$\frac{2}{5} \times \frac{7}{7} \times \frac{6}{6} = \frac{84}{210}$$

$$\frac{1}{7} \times \frac{5}{5} \times \frac{6}{6} = \frac{30}{210}$$

$$\frac{3}{8} \times \frac{5}{5} \times \frac{7}{7} = \frac{105}{280}$$

From this example and explanation it will be seen that

1. A common denominator of the similar fractions equivalent to two or more given dissimilar fractions is a common multiple of the denominators of all the given fractions; and

2. The denominator of each of the dissimilar fractions is a factor of the common denominator of the equivalent similar fractions. Hence,

A fraction can be changed to an equivalent fraction having a different denominator, if the denominator of the required fraction is a multiple of the denominator of the given fraction, and not otherwise.

Changing dissimilar to similar fractions is sometimes called *reducing fractions to equivalent fractions having a common denominator*.

PROBLEMS.

Reduce to similar fractions

18. $\frac{2}{3}$, $\frac{4}{5}$, $\frac{1}{6}$, and $\frac{1}{12}$. | 20. $\frac{1}{11}$, $\frac{1}{13}$, $\frac{1}{17}$, $\frac{1}{19}$, and $\frac{1}{23}$. | 22. $\frac{1}{11}$, $\frac{1}{15}$, $\frac{1}{18}$, and $\frac{1}{24}$.
 19. $\frac{1}{11}$, $\frac{1}{13}$, $\frac{1}{17}$, and $\frac{1}{19}$. | 21. $\frac{1}{13}$, $\frac{1}{17}$, and $\frac{1}{19}$. | 23. $\frac{9}{15}$, $\frac{4}{18}$, $\frac{1}{15}$, and $\frac{1}{24}$.

Reduce to fractions having a common denominator

24. $\frac{4}{5}$, $\frac{2}{15}$, $\frac{1}{12}$, and $\frac{1}{40}$. | 26. $\frac{2}{3}$, $\frac{1}{4}$, and $\frac{1}{15}$.
 25. $\frac{4}{15}$, $\frac{1}{16}$, $\frac{1}{17}$, and $5\frac{1}{2}$. | 27. $\frac{1}{11}$ h., $\frac{1}{12}$ h., $3\frac{1}{15}$ h.
 Find a common denominator { 28. Of $\frac{4}{5}$, $\frac{1}{15}$, and $\frac{1}{12}$. | 30. Of $\frac{1}{15}$, $\frac{1}{3}$, $\frac{4}{5}$, and $\frac{1}{11}$.
 { 29. Of $\frac{1}{15}$, $11\frac{1}{2}$, and $\frac{4}{5}$. | 31. Of $\frac{2}{3}$ yd., $\frac{4}{5}$ yd., and $\frac{1}{3}$ yd.

CASE IV. Dissimilar fractions to least similar fractions.

195. The denominator of the least similar fractions equivalent to two or more given dissimilar fractions is the least common multiple of the denominators of all the given fractions.

Ex. Reduce $\frac{1}{3}$, $\frac{1}{15}$, and $\frac{1}{21}$ to least similar fractions.

PROCESS.

3		$105 \div 3 = 35$	$\frac{1}{3} \times \frac{35}{35} = \frac{35}{105}$
15	5	$105 \div 15 = 7$	$\frac{1}{15} \times \frac{7}{7} = \frac{7}{105}$
21	7	$105 \div 21 = 5$	$\frac{1}{21} \times \frac{5}{5} = \frac{5}{105}$
$3 \times 5 \times 7 = 105$			

Hence, $\frac{1}{3}$, $\frac{1}{15}$, $\frac{1}{21} = \frac{35}{105}$, $\frac{7}{105}$, $\frac{5}{105}$.

EXPLANATION.—I first find the least common multiple of all the given denominators 3, 15, and 21, to be 105. Hence, the several fractions can be reduced to 105ths. I next divide 105, the least common denominator of the equivalent similar fractions, by the given denominators 3, 15, and 21, and obtain 35, 7, and 5, the several numbers by which the terms of the respective given fractions must be multiplied to reduce them to 105ths (*191*, Principle II). I then multiply the terms of $\frac{1}{3}$ by 35, the terms of $\frac{1}{15}$ by 7, and the terms of $\frac{1}{21}$ by 5; *i. e.*, I multiply the terms of each fraction by the quotient obtained by dividing the least common multiple of all the given denominators by the denominator of the fraction.

The results $\frac{35}{105}$, $\frac{7}{105}$, and $\frac{5}{105}$, are the least similar fractions required.

PROBLEMS.

Reduce to least similar fractions

32. $\frac{1}{8}$, $\frac{1}{12}$, and $\frac{1}{15}$.

33. $\frac{1}{12}$ yd., $\frac{1}{16}$ yd., and $\frac{1}{20}$ yd.

34. $\frac{1}{12}$, $\frac{1}{16}$, $\frac{1}{20}$, and $\frac{1}{24}$.

35. $\frac{1}{8}$, $\frac{1}{12}$, $\frac{1}{15}$, and $\frac{1}{24}$.

36. £4 $\frac{1}{2}$, £7 $\frac{3}{4}$, and £1 $\frac{1}{2}$.

37. $\frac{1}{8}$, $\frac{1}{12}$, $\frac{1}{15}$, $\frac{1}{20}$, and $\frac{1}{24}$.

Reduce to fractions having their least common denominator

38. $\frac{1}{8}$, $\frac{1}{12}$, $\frac{1}{15}$, and $\frac{1}{20}$.

39. $\frac{1}{12}$, $\frac{1}{16}$, $\frac{1}{20}$, and $\frac{1}{24}$.

40. $\frac{1}{8}$, $\frac{1}{12}$, $\frac{1}{15}$, and $\frac{1}{20}$.

41. $\frac{1}{8}$ A., $\frac{1}{12}$ A., $\frac{1}{15}$ A., and $\frac{1}{20}$ A.

What is the greatest common fractional unit

42. Of 3ds, 24ths, 60ths, and 15ths?

43. Of $\frac{1}{12}$, $\frac{1}{16}$, $\frac{1}{20}$, and $\frac{1}{24}$?

44. Of $\frac{1}{12}$, $\frac{1}{16}$, $\frac{1}{20}$, and $\frac{1}{24}$?

45. Of $\frac{1}{12}$, $\frac{1}{16}$, $\frac{1}{20}$, and $\frac{1}{24}$?

CASE V. An improper fraction to an integer or a mixed number.

196. Ex. Reduce $\frac{1991}{221}$ to integers or mixed numbers.

EXPLANATION.—The numerator of a fraction is a dividend, and the denominator is a divisor (78). I therefore perform the indicated divisions, and obtain 9 and 266 $\frac{1}{2}$, the required results.

PROCESS.

$$221 \overline{) 1991} = 9$$

$$1991 = 266\frac{1}{2}$$

PROBLEMS.

Reduce to integers or mixed numbers

46. $\frac{1}{12}$, $\frac{1}{16}$, and $\frac{1}{20}$.

47. $\frac{1}{12}$, $\frac{1}{16}$, and $\frac{1}{20}$.

48. $\frac{1}{12}$, $\frac{1}{16}$, and $\frac{1}{20}$.

49. $\frac{1}{12}$ yd., £24 $\frac{1}{2}$, and $\frac{1}{12}$ doz.

50. £10 $\frac{1}{2}$, 48 $\frac{1}{2}$ wk., and 24 $\frac{1}{2}$ cu. ft.

51. 120 $\frac{1}{2}$ ft., 24 $\frac{1}{2}$ da., and 48 $\frac{1}{2}$ bu.

CASE VI. An integer or a mixed number to an improper fraction.

197. Ex. Reduce 83 $\frac{1}{2}$ to an improper fraction.

EXPLANATION.—I multiply 7, the number of sevenths in 1, by 83, and obtain 581, the number of sevenths in 83. To this result I add the 4 sevenths of the given number, and obtain 585, the improper fraction required.

FIRST PROCESS.

$$7 \text{ sevenths}$$

$$83$$

$$\hline 581 \text{ sevenths}$$

$$4 \text{ sevenths}$$

$$\hline 585 \text{ sevenths}$$

SECOND PROCESS.

$$83\frac{1}{2}$$

$$7$$

$$\hline 581 + 4 = 585$$

$$\text{Hence, } 83\frac{1}{2} = \frac{585}{2}$$

The work is commonly written as shown in the Second Process.

PROBLEMS.

Reduce to improper fractions

52. $19\frac{3}{4}$, $14\frac{3}{8}$, and $12\frac{7}{8}$. | 54. 1,001,001 $\frac{1}{11}$, 700,007 $\frac{1}{7}$, and $99\frac{1}{9}$.
 53. $3,225\frac{3}{4}$, $1,201\frac{14}{101}$, and 8,550. | 55. 365 $\frac{1}{4}$ da., $3,719\frac{1}{8}$ yd., $574\frac{1}{8}$ lb.
 56. $825\frac{7}{8}$ pk., $493\frac{2}{3}$ cd., and $11,111\frac{4}{11}$ T.
 57. $583\frac{2}{100}$ ch., $47\frac{4}{3\frac{1}{2}}$ mi., and $\$1,003\frac{1}{2}$.
 58. Express 85,649, and 7,421 in fractional form.
 59. Express $\$231$, 3,156 ft., and 17,295 lb. in fractional form.

198. RULES FOR REDUCTIONS OF FRACTIONS.

I. A fraction to lowest terms:—

Cancel all the factors common to the terms of the fraction; or, cancel their greatest common factor.

II. A fraction to another fraction having a given denominator:—

Divide the given denominator by the denominator of the fraction, and multiply both terms of the fraction by the quotient.

III. Dissimilar fractions to similar fractions:—

Multiply the terms of each fraction by the denominators of all the other fractions.

IV. Dissimilar fractions to least similar fractions:—

1. For the least common denominator;—*Find the least common multiple of all the denominators.*

2. For each new numerator;—*Divide the least common multiple by the denominator of each fraction, and multiply the numerator by the quotient.*

V. An improper fraction to an integer or a mixed number:—

Divide the numerator by the denominator.

VI. An integer or a mixed number to an improper fraction:—

1. *Multiply the integer by the denominator, and if there be a numerator, add it to the product.*

2. *Write this result for the numerator, and the given denominator for the denominator of the required fraction.*

PROBLEMS.

Reduce to lowest terms

60. $\frac{16807}{15188}$, $\frac{72}{384}$, and $\frac{4477}{444}$.
 61. $\frac{343}{44}$, $\frac{105105}{1050}$, and $\frac{74}{44}$.
 62. $\frac{1034}{1188}$, $\frac{105}{1050}$, and $\frac{4444}{1188}$.
 63. $\frac{210}{6087}$, $\frac{444}{144447}$, and $\frac{1070076}{16466475}$.

Reduce to integers or mixed numbers

64. $\frac{1037003}{1000}$, $\frac{50000}{1000}$, and $\frac{5000001}{1000}$.
 65. $\frac{225764}{111}$ T., and $\frac{7100}{11}$ mi.
 66. $\frac{7472}{11}$, $\frac{4231444}{11}$, and $\frac{22333322}{11}$.
 67. $\frac{5136}{100}$ A., $\frac{710000}{100}$ bar., and $\frac{1000}{100}$ lb.

Reduce to im-
proper fractions

68. $43\frac{1}{7}$, $2,000\frac{1}{100}$, and $264\frac{1}{7}$.
 69. $596,103\frac{1}{10}$, $91743\frac{1}{10}$, and $1,000\frac{1}{1000}$.
 70. $130\frac{7}{10}$ mi., $2,371\frac{3}{10}$ A., and $30,015\frac{3}{10}$ cu. ft.
 71. $934\frac{1}{10}$ yr., $20,202\frac{1}{10}$ sq. mi., and $2374\frac{1}{10}$ gal.

72. Reduce 8,213 to 643ds.

73. Reduce 51,213 to 728ths.

74. Reduce $\frac{1}{11}$ to 95ths; to 171sts; to 209ths.75. Change $5,026\frac{2}{11}$ to a fraction whose denominator is 15,357.76. $2,045\frac{3}{4}$ bushels are how many 75ths of a bushel?77. $400\frac{1}{11}$ days are how many thousandths of a day?78. Reduce $\frac{444}{11}$ to three fractions whose denominators are, respectively, 1,052, 3,156, and 4,734.79. Change $\frac{4}{11}$ sq. yd., $1\frac{1}{11}$ bu., $97\frac{1}{11}$ wk., and $1,010\frac{1}{11}$ T. to fractions whose several denominators are 6,048.80. Reduce $\frac{7}{8}$, $\frac{7}{10}$, $\frac{7}{11}$, and $\frac{7}{12}$ to similar fractions.81. Reduce $\frac{7}{8}$, 15, $\frac{7}{10}$, $7\frac{1}{10}$, $\frac{1}{12}$, and 20 to similar fractions.

Reduce to fractions having a common denominator

82. $\frac{1}{3}$, $\frac{1}{6}$, and $\frac{1}{2}$.85. $\frac{1}{3}$, $\frac{1}{11}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, and $\frac{1}{7}$.83. $2\frac{1}{2}$, $3\frac{1}{3}$, $4\frac{1}{4}$, and $5\frac{1}{5}$.86. $5\frac{1}{2}$ bu., $4\frac{1}{3}$ bu., and $\frac{1}{4}$ bu.84. $125\frac{1}{2}$ and $\frac{1}{4}$.87. $\frac{1}{2}$ wk., $\frac{1}{4}$ wk., and $3\frac{1}{4}$ wk.88. Reduce $\frac{1}{3}$, $\frac{1}{11}$, $\frac{1}{12}$, $\frac{1}{10}$, $\frac{1}{15}$, $\frac{1}{18}$, and $\frac{1}{20}$ to least similar fractions.89. Reduce $\frac{3}{100}$ lb., $\frac{1}{44}$ lb., $\frac{1}{150}$ lb., and $\frac{1}{300}$ lb. to least similar fractions.

Reduce to fractions having the least common denominator

90. $\frac{1}{10}$, $\frac{1}{20}$, $2\frac{1}{10}$, and $104\frac{1}{10}$.92. $\frac{1}{10}$, $\frac{1}{11}$, and $\frac{1}{12}$.91. $\frac{1}{11}$, $\frac{1}{12}$, $\frac{1}{13}$, and $\frac{1}{14}$.93. $\$16\frac{1}{10}$, $\$21\frac{1}{10}$, $\$21\frac{1}{10}$, and $\$11\frac{1}{10}$.

94. What is the greatest common fractional unit of 26ths, 39ths, 12ths, and 52ds?

95. Find the greatest common fractional unit of $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, and $\frac{1}{8}$.

SECTION III.

INTERCHANGING DECIMAL AND FRACTIONAL UNITS.

In arithmetical computations it is sometimes necessary to find a fraction equal in value to a given decimal, or a decimal equal in value to a given fraction; i.e., to change a decimal to a fraction, or a fraction to a decimal.

CASE I. A decimal to a fraction.

199. Any decimal may be written in two forms. Thus, 7 tenths is .7 or $\frac{7}{10}$; 59 thousandths is .059 or $\frac{59}{1000}$; 3 ten-thousandths is .0003 or $\frac{3}{10000}$, etc. In the decimal form the unit is indicated by the position of the decimal point; in the fractional form it is expressed by the denominator, which is 1 with as many ciphers annexed as there are decimal places in the decimal.

Ex. 1. Express .025 in the fractional form.

PROCESS.

EXPLANATION.—I write the number without the decimal point, for a numerator, and under it I write its denominator, 1,000.

$$.025 = \frac{25}{1000} = \frac{1}{40}$$

PROCESS.

Ex. 2. Reduce $.083\frac{1}{3}$ to the fractional form.

$$.083\frac{1}{3} = \frac{83\frac{1}{3}}{1000} = \frac{1}{12}$$

PROBLEMS.

Reduce to fractions

1. .6; .84; .08; and .75.

2. .125; .065; .004; and .875.

3. .6875; .0436; .0037; .000075.

4. $4\frac{1}{2}$; $.028\frac{1}{2}$; $17\frac{1}{2}$; $.666\frac{2}{3}$.

5. $.1256\frac{2}{3}$ A.; .000095 gal.; $.5000\frac{1}{11}$ yr.

Reduce to mixed fractional numbers

6. 6.45; 5.08; 18.048.

7. 1.5625; 37.01975; $1.0\frac{1}{10}$.

8. $4.38\frac{1}{3}$; 8,001.00 $\frac{1}{2}$; $10.071\frac{1}{11}$.

9. 1.0019 mi.; 224.000084 T.

10. $26.0100\frac{1}{2}$ cu. ft.; $38.150\frac{2}{11}$ sq. mi.

CASE II. A fraction to a decimal.

200. 7, 7.0, 7.00, 7.000 all express the same value; also $\frac{7}{1}$, $\frac{70}{10}$, $\frac{700}{100}$, $\frac{7000}{1000}$ all express the same value. Hence,

Annexing decimal ciphers to the numerator of a fraction does not change its value (189).

Ex. Reduce $\frac{7}{8}$ to the decimal form.

PROCESS.

EXPLANATION.—Since $\frac{7}{8}$ expresses the quotient of 7 divided by 8, I reduce the numerator 7 to thousandths by annexing three decimal ciphers, and then reduce $\frac{7.000}{8}$, the fraction thus formed, to a decimal, by dividing its numerator 7.000 by its denominator 8.

The result, .875, is the required decimal.

PROBLEMS.

Reduce to decimals

11. $\frac{2}{3}$, $\frac{3}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$.
 12. $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, and $\frac{1}{10000}$.
 13. $\frac{1}{10000}$ mi., $\frac{1}{100}$ T., $\frac{1}{100}$, and $\frac{1}{1000}$.
 14. $.25\frac{1}{100}$, $.125\frac{1}{100}$, and $.0004\frac{1}{100}$.

Change to mixed decimal numbers

15. $15\frac{1}{10}$, $518\frac{1}{10}$, and $100\frac{1}{10}$.
 16. $19\frac{1}{10}$, $5\frac{1}{10}$, and $7\frac{1}{10}$.
 17. $6\frac{1}{10}$, $2\frac{1}{10}$ yd., and $1\frac{1}{10}$ gal.
 18. $6\frac{1}{10}$, $72\frac{1}{10}$, $154\frac{1}{10}$, and $24\frac{1}{10}$.

201. RULES FOR THE CONVERSE REDUCTIONS OF DECIMALS AND FRACTIONS.

I. A decimal to a fraction:—Write the given number without the decimal point for a numerator, and under it write its denominator.

II. A fraction to a decimal:—Annex a decimal cipher or ciphers to the numerator, and divide the number thus formed by the denominator.

Sometimes the decimal is *interminable*. In such cases

Write a fraction after the decimal figures; or

Carry the quotient to any desired number of decimal places, and annex the sign + to it.

The sign + shows that the division is incomplete, or that there was a remainder after the last decimal figure of the quotient was obtained. Thus, $\frac{2}{3} = .666 +$;
 $\frac{1}{4} = .25 +$. (See Repetends; Supplement, page 391.)

PROBLEMS.

19. Reduce .784, .003175, and .69032 to fractions.
 20. Change $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, and $\frac{1}{10000}$ to decimals.
 21. Change $7.56\frac{1}{10}$, 21.0888, and 100.08 to mixed fractional numbers.
 22. $.2\frac{1}{10}$ da. and $.66\frac{2}{3}$ da. are what fractions of a day?
 23. Express the value of $.80045\frac{1}{10}$ ft., and $114.44\frac{4}{10}$ mi. in fractional form.
 24. What mixed decimal number equals $8\frac{1}{10}\frac{1}{10}\frac{1}{10}$?
 25. Reduce $\frac{1}{10}$, $\frac{1}{100}$, and $\frac{1}{1000}$ to decimals.

SECTION IV.

REDUCTIONS OF DENOMINATE NUMBERS,—

SIMPLE, COMPOUND, DECIMALS, AND FRACTIONS.

In computations in compound numbers it is often necessary to change given units to those of other denominations—as gallons to quarts and pints; feet to yards, rods, and miles; etc.

1. Reduction descending.

202. Reduction descending is performed by multiplication.

CASE I. A compound number to a denominate integer.

203. Ex. How many pints are 17 gal. 3 qt. 1 pt. ?

ANALYTICAL PROCESS.

$$17 \times 4 \text{ qt.} + 3 \text{ qt.} = 71 \text{ qt.}$$

$$71 \times 2 \text{ pt.} + 1 \text{ pt.} = 143 \text{ pt.}$$

$$\text{Hence, } 17 \text{ gal. } 3 \text{ qt. } 1 \text{ pt.} = 143 \text{ pt.}$$

EXPLANATION.—1 gal. is 4 qt., 17 gal. are 17 times 4 qt., and 17 gal. 3 qt. are 17 times 4 qt. + 3 qt. or 71 qt.

1 qt. is 2 pt., 71 qt. are 71 times 2 pt., and 71 qt. 1 pt. are 71 times 2 pt. + 1 pt. or 143 pt., which is the required result.

The custom of using the true multiplicand as the multiplier is shown in the Common Process.

COMMON PROCESS.

$$17 \text{ gal. } 3 \text{ qt. } 1 \text{ pt.}$$

$$\quad 4$$

$$\hline 71 \text{ qt.}$$

$$\quad 2$$

$$\hline 143 \text{ pt.}$$

PROBLEMS.

Reduce

1. 18 gal. 3 qt. 1 pt. 2 gi. to gills.

2. 8 mi. 173 rd. 3 yd. 2 ft. to inches.

3. £403 10s. 8d. to pence.

4. 15 bu. 6 qt. to pints.

5. 100 cd. 48 cu. ft. to cubic feet.

6. \$13 3s. 19 cts. Canada money to cts.

7. 18 bu. 3 pk. 5 qt. to quarts.

8. 42° 37' to seconds.

9. 7 B. 1 bun. 8 rn. to sheets.

10. 25 Tp. of land to acres.

11. 450 sq. rd. 95 sq. in. to square inches.

12. 3 mi. 25 ch. 42 l. to links.

13. 17 T. 5 cwt. 4 lb. 9 oz. to ounces.

14. 21 lb. 9 oz. Troy to grains.

15. 11 oz. 11 pwt. 11 gr. to grains.

16. 2 quad. 77° 38' to minutes.

Reduce to lowest denomination named

- | | |
|--|--|
| 17. 5 yr. 7 mo. 23 da. 21 h. 59 min. 49 sec. | 20. 3 A. 51 sq. rd. 11 sq. yd. 7 sq. ft. |
| 18. 12 cu. yd. 20 cu. ft. 88 cu. in. | 21. 1 Sec. 50 A. 7 sq. ch. 10 P. 600 sq. l. |
| 19. 21 sq. rd. 8 sq. yd. 5 sq. ft. 72 sq. in. | 22. 13 wk. 4 da. 12 h. 30 min. 15 sec. |

CASE II. A denominate decimal or a denominate fraction to units of a lower denomination.

204. Ex. 1. Reduce .05 gal. to the decimal of a pint.

PROCESS

EXPLANATION.—The successive multipliers for reducing gallons to pints are 4 and 2. I reduce the .05 of a gallon to the decimal of a quart by multiplying it by 4; and the result, .2 of a quart, to the decimal of a pint by multiplying it by 2.

.05 gal.
 $\frac{4}{}$
 .20 qt.
 $\frac{2}{}$
 .4 pt.

The last result, .4 pt., is the required denominate decimal.

Ex. 2. Change $\frac{1}{128}$ rd. to the fraction of an inch.

PROCESS.

EXPLANATION.—The successive multipliers for reducing rods to inches are $16\frac{1}{2}$ ($= \frac{32}{2}$) and 12. $\frac{1}{128} \times \frac{32}{2} \times 12 \text{ in.} = \frac{3}{8} \text{ in.}$ I therefore multiply the given denominate fraction $\frac{1}{128}$ rd. by the factors $\frac{32}{2}$ and 12 ($= 12$), and obtain $\frac{3}{8}$ in., the required denominate fraction.

PROBLEMS.

Reduce	to the decimal	Reduce	to the fraction
23. .009 yd.	of an inch.	27. $\frac{3}{128}$ bu.	of a quart.
24. .0015 mi.	of a rod.	28. $\frac{15}{128000}$ da.	of a minute.
25. .000125 lb. Troy	of a grain.	29. $\frac{7}{12800}$ mi.	of a foot.
26. .07 gal.	of a pint.	30. $\frac{5}{128000}$ T.	of an ounce.

Reduce

- | | |
|----------------------------------|--|
| 31. .3945 da. to seconds. | 36. $\frac{3}{8}$ gal. to pints. |
| 32. .02875 bu. to pints. | 37. $\frac{7}{128}$ lb. Troy to pennyweights. |
| 33. .021 T. to ounces. | 38. $\frac{441}{128000}$ A. to square yards. |
| 34. £25.375 to farthings. | 39. $7\frac{1}{2}$ hhd. to gills. |
| 35. 83.29 ch. to links. | 40. $5\frac{1}{8}\frac{1}{2}$ wk. to seconds. |

CASE III. A denominate decimal or a denominate fraction to a compound number.

205. Ex. 1. Reduce .78 of a common year to a compound number.

EXPLANATION.—The successive multipliers for reducing years to lower denominations are those in the descending scale of the Table of Time. To reduce .78 yr. to days I multiply by 365, and obtain 284.7 da. To reduce the .7 day to hours, I multiply it by 24, and obtain 16.8 h. To reduce the .8 h. to minutes, I multiply it by 60, and obtain 48 minutes.

Writing the integral parts of the several results in order, I have 284 da. 16 h. 48 min., the required compound number.

PROCESS.

$$\begin{array}{r} .78 \text{ yr.} \\ \underline{365} \\ 284.70 \text{ da.} \\ \underline{24} \\ 16.8 \text{ h.} \\ \underline{60} \\ 48.0 \text{ min.} \end{array}$$

Hence, .78 yr. = 284 da. 16 h. 48 min.

Ex. 2. Reduce $\frac{1}{11}$ of a square mile to a compound number.

ANALYTICAL PROCESS.

$$\begin{aligned} \frac{1}{11} \times \frac{5280}{1} \text{ A.} &= \frac{5280}{11} \text{ A.} = 465 \frac{5}{11} \text{ A.} \\ \frac{1}{11} \times \frac{160}{1} \text{ sq. rd.} &= \frac{160}{11} \text{ sq. rd.} = 72 \frac{8}{11} \text{ sq. rd.} \\ \frac{1}{11} \times \frac{160}{1} \text{ sq. yd.} &= \frac{160}{11} \text{ sq. yd.} = 22 \frac{8}{11} \text{ sq. yd.} \end{aligned}$$

$$\text{Hence, } \frac{1}{11} \text{ sq. mi.} = 465 \text{ A. } 72 \text{ sq. rd. } 22 \text{ sq. yd.}$$

EXPLANATION.—1 sq. mi. is 640 A., and $\frac{1}{11}$ sq. mi. is $\frac{1}{11}$ of 640 A. or $465 \frac{5}{11}$ A.

1 A. is 160 sq. rd., and $\frac{5}{11}$ A. is $\frac{5}{11}$ of 160 sq. rd. or $72 \frac{8}{11}$ sq. rd.

1 sq. rd. is $(30\frac{1}{2})$ $\frac{160}{11}$ sq. yd., and $\frac{8}{11}$ sq. rd. is $\frac{8}{11}$ of $\frac{160}{11}$ sq. yd. or $22 \frac{8}{11}$ sq. yd.

Writing the integral parts of the several results in order, I have 465 A. 72 sq. rd. 22 sq. yd., the required compound number.

PROBLEMS.

Reduce to compound numbers

41. .86 bu.	44. .8941 wk.	47. .1245°.	50. .005 gr.
42. .6875 gal.	45. .45 cd.	48. .485 lb. Troy.	51. 31.015 A.
43. .625 bar.	46. .86425 T.	49. .67 T.	52. 185.0075 mi.
53. Cong. 14.3074.	56. $8\frac{1}{11}$.	59. $\frac{3}{4}$ da.	62. $7\frac{1}{11}$ A.
54. £6.42.	57. $£\frac{1}{11}$.	60. $\frac{3}{4}$ sq. mi.	63. $\frac{1}{11}$ T.
55. 7.3125 a.	58. $\frac{7}{11}$ wk.	61. $\frac{3}{11}$ cu. yd.	64. $\frac{3}{11}$ rm.
65. $17\frac{1}{2}$ gt. gro.	66. $100\frac{1}{11}$ rd.	67. $2\frac{1}{11}$ lb.	

2. Reduction ascending.

206. *Reduction ascending is performed by division.*

CASE IV. A denominate integer to a compound number.

207. Ex. Reduce 817 quarts to units of higher denominations.

ANALYTICAL PROCESS.

$$\begin{array}{r}
 8 \text{ qt.}) \overline{817 \text{ qt.}} \\
 \underline{102 \text{ times and } 1 \text{ qt. rem.}} \\
 817 \text{ qt.} = 102 \text{ pk. } 1 \text{ qt.} \\
 4 \text{ pk.}) \overline{102 \text{ pk.}} \\
 \underline{25 \text{ times and } 2 \text{ pk. rem.}} \\
 102 \text{ pk.} = 25 \text{ bu. } 2 \text{ pk.} \\
 \text{Hence, } 817 \text{ qt.} = 25 \text{ bu. } 2 \text{ pk. } 1 \text{ qt.}
 \end{array}$$

EXPLANATIONS. 1. *Analytical Process.*—Since 8 qt. are 1 pk. 1 qt. is as many pk. as the times 8 qt. are contained in 817 qt., which is 102 times and 1 qt. remainder; i. e. 817 qt. are 102 pk. 1 qt.

4 pk. are 1 bu., and 102 pk. are as many bu. as the times 4 pk. are contained in 102 pk., which is 25 times and 2 pk. remainder; i. e. 102 pk. are 25 bu. 2 pk.

Writing the last quotient and the several remainders in order, I have 25 bu. 2 pk. 1 qt., the required compound number.

2. *Common Process.*—Since 8 qt. are 1 pk. and 4 pk. are 1 bu., the successive divisors for reducing quarts to bushels are 8 and 4. To reduce 817 qt. to pk., I divide by 8 and obtain 102 pk. 1 qt. To reduce 102 pk. to bu., I divide by 4 and obtain 25 bu. 2 pk.

Writing the last quotient and the several remainders in order, I have 25 bu. 2 pk. 1 qt., the required compound number.

COMMON PROCESS.

$$\begin{array}{r}
 8 \overline{) 817 \text{ qt.}} \\
 \underline{4 \overline{) 102 \text{ pk. } 1 \text{ qt.}}} \\
 25 \text{ bu. } 2 \text{ pk. } 1 \text{ qt.}
 \end{array}$$

PROBLEMS.

Reduce the following denominate numbers to compound numbers :

68. 9,120 gi. to gallons.	72. 219,302 oz. to T.	76. 58,600 sheets to rm.
69. 4,021 pt. to bushels.	73. 872,413 cu.in. to cu.yd.	77. 40,320 far. to £.
70. 14,442.6" to degrees.	74. 1,000,000 sq. ft. to A.	78. 125,780 l. to mi.
71. 472 $\frac{1}{2}$ to pounds.	75. 54,175,640 sec. to wk.	79. 14,005 gr. to lb. Troy.
80. 38,416 lb. to gross tons.	81. 125,205 buttons to great-gross.	

CASE V. A denominate decimal or a denominate fraction to units of a higher denomination.

208. Ex. 1. Reduce 65.52 pounds to the decimal of a gross ton.

PROCESS.

$$\begin{array}{r}
 28 \overline{) 65.52 \text{ lb.}} \\
 \underline{4 \overline{) 2.34 \text{ qr.}}} \\
 20 \overline{) .585 \text{ cwt.}} \\
 .02925 \text{ T.}
 \end{array}$$

EXPLANATION.—The successive divisors for reducing pounds to gross tons are 28, 4, and 20. To reduce 65.52 lb. to tons, I therefore divide, successively, by 28, 4, and 20, and obtain .02925 T., the denominate decimal required.

Ex. 2. Reduce $7\frac{1}{2}$ d. to the fraction of a £.

EXPLANATION.—To reduce $7\frac{1}{2}$ d. (or $\frac{15}{2}$ d.) to the fraction of a £, I divide successively by 12 and 20, and obtain $\frac{1}{16}$, the required denominate fraction.

Or, since in any given sum of sterling money there are $\frac{1}{12}$ as many shillings as pence, and $\frac{1}{20}$ as many pounds as shillings, I multiply $\frac{15}{2}$ d. successively by $\frac{1}{12}$ and $\frac{1}{20}$, and obtain $\frac{1}{16}$, as before.

ANALYTICAL PROCESS.

$$\begin{aligned} 7\frac{1}{2}d. &= \frac{15}{2}d. \\ \frac{15}{2}d. \div 12 &= \frac{5}{4}s. \\ \frac{5}{4}s. \div 20 &= \frac{1}{16}\text{£} \end{aligned}$$

Hence, $7\frac{1}{2}d. = \frac{1}{16}\text{£}$

COMMON PROCESS.

$$\frac{15}{2}d. \times \frac{1}{12} \times \frac{1}{20} = \frac{1}{16}\text{£}$$

PROBLEMS.

Reduce	to the decimal	Reduce	to the fraction
82. .24 oz.	of a ton.	87. $\frac{3}{4}$ yd.	of a mile.
83. 1.44"	of a degree.	88. $\frac{1}{4}$ gi.	of a gallon.
84. .9 in.	of a yard.	89. $\frac{3}{4}$ far.	of a pound.
85. 40.725 pt.	of a bushel.	90. $\frac{1}{4}$ sq. ft.	of a square rod.
86. 715.48 min.	of a day.	91. $\frac{1}{16}$ sq. l.	of an acre.

CASE VI. A compound number to a denominate decimal or to a denominate fraction.

209. Ex. 1. Reduce 2 pk. 3 qt. 1 pt. to the decimal of a bushel.

EXPLANATION.—I write the numbers expressing the several given denominations in order, in a column, the lowest denomination at the top.

The successive divisors for reducing pints to bushels are 2, 8, and 4. To reduce the 1 pt. to the decimal of a quart, I annex a decimal cipher and divide by 2, and write the result after 3 qt., making 8.5 quarts.

To reduce the 8.5 qt. to the decimal of a peck, I divide by 8, and annex the result to the pecks, making 2.4375 pecks.

To reduce this result to the decimal of a bushel, I divide by 4, and obtain .609375 of a bushel, the required decimal.

PROCESS.

$$\begin{array}{r|l} 2 & 1.0 \text{ pt.} \\ 8 & 3.5 \text{ qt.} \\ 4 & 2.4375 \text{ pk.} \\ \hline & .609375 \text{ bu.} \end{array}$$

Ex. 2. Reduce 22 h. 13 min. 20 sec. to the fraction of a day.

EXPLANATION.—The successive divisors for reducing seconds to days are 60, 60, and 24. To reduce the 20 sec. to the fraction of a minute, I divide by 60, and obtain $\frac{1}{3}$ min., which I annex or add to the 13 minutes, making 13 $\frac{1}{3}$ min., or $13\frac{1}{3}$ min.

FIRST PROCESS.

$$\begin{aligned} 20 \text{ sec.} \div 60 &= \frac{1}{3} \text{ min.} = \frac{1}{3} \text{ min.} \\ 13 \text{ min.} + \frac{1}{3} \text{ min.} &= 13\frac{1}{3} \text{ min.} = \frac{40}{3} \text{ min.} \\ \frac{40}{3} \text{ min.} \div 60 &= \frac{2}{9} \text{ h.} = \frac{2}{9} \text{ h.} \\ 22 \text{ h.} + \frac{2}{9} \text{ h.} &= 22\frac{2}{9} \text{ h.} = \frac{200}{9} \text{ h.} \\ \frac{200}{9} \text{ h.} \div 24 &= \frac{25}{27} \text{ da.} = \frac{25}{27} \text{ da.} \\ \text{Hence, } 22 \text{ h. } 13 \text{ min. } 20 \text{ sec.} &= \frac{25}{27} \text{ da.} \end{aligned}$$

To reduce $4^{\frac{1}{2}}$ min. to the fraction of an hour, I divide by 60, and obtain $\frac{1}{12}$ h., which I add to the 22 hours, making $22\frac{1}{12}$ h., or $22\frac{1}{4}$ h.

To reduce $2\frac{1}{2}$ h. to the fraction of a day, I divide by 24 and obtain $\frac{5}{24}$ da., which is the denominator fraction required.

SECOND PROCESS.

$$\begin{array}{l|l}
 \text{sec. } 20 & 60 \left| \frac{20}{60} \text{ min.} = \frac{1}{3} \text{ min.} \right. \\
 \text{min. } 13 + \frac{1}{3} = \frac{40}{3} & 60 \left| \frac{40}{3} \text{ h.} = \frac{13}{3} \text{ h.} \right. \\
 \text{h. } 22 + \frac{13}{3} = 22\frac{13}{3} & 24 \left| 22\frac{13}{3} \text{ da.} = 22\frac{1}{3} \text{ da.} \right.
 \end{array}$$

Hence, $22 \text{ h. } 13 \text{ min. } 20 \text{ sec.} = 22\frac{1}{3} \text{ da.}$

PROBLEMS.

Reduce to the decimal	Reduce to the fraction
92. 9 oz. 12 pwt. of a pound.	98. 31 cents $2\frac{1}{2}$ mills of a dollar.
93. 2 yd. 2 ft. 3 in. of a rod.	99. 10 quires 16 sheets of a ream.
94. 4 da. 4 h. 48 min. of a week.	100. 2 pk. $\frac{1}{2}$ pt. of a bushel.
95. 9 cwt. 85 lb. 5 oz. of a ton.	101. 60 gal. 3 qt. 1 pt. of a hogshead.
96. 9d. 3 far. of a pound.	102. 219 da. 14 h. 24 min. of a leap-year.
97. 21 cu. ft. 430 cu. in. of a cu. yd.	103. $8\frac{1}{3}$ doz. of a gross.

104. 5 ii 3 v 9 i gr. xv is what decimal of a pound?

105. 245 rd. 4 yd. 2 ft. 8 in. is what decimal of a mile?

106, 107. 273 bu. 3 pk. is what mixed decimal number? What mixed fractional number?

(See Supplement, page 397.)

210. RULES FOR REDUCTIONS OF DENOMINATE NUMBERS.

1. For reduction descending.

I. A compound number to a denominate integer:—

1. Multiply the units of the highest denomination given, by that number of the next lower denomination that equals 1 of this higher, and to the product add the units given of the lower denomination.

2. In like manner, reduce this result to units of the next lower denomination; and so continue until the given number is reduced to units of the required denomination.

II. A denominate decimal or a denominate fraction to units of a lower denomination:—

Multiply the given decimal, or the given fraction successively by the units of the scale between the given and the required denomination.

III. A denominate decimal or a denominate fraction to a compound number :—

1. *Multiply the decimal or the fraction by that number of the next lower denomination which equals 1 of the given denomination.*

2. *In like manner, reduce the decimal or the fractional part of this and of each succeeding result, to the next lower denomination, until the result is an integer, or until the lowest denomination is reached.*

3. *Write the integral parts of the several results and the decimal or fractional part of the final result in order, for the required compound number.*

2. For reduction ascending.

I. A denominate integer to a compound number :—

1. *Divide the given integer by that number of the same denomination which equals 1 of the next higher; writing the quotient as units of the higher denomination, and the remainder as units of the denomination divided.*

2. *In like manner, reduce this quotient to units of the next higher denomination; and so continue until the given number is reduced to units of the required denomination.*

3. *Write the last quotient and the several remainders in their order, for the required compound number.*

II. A denominate decimal or a denominate fraction to units of a higher denomination :—

Divide the given decimal, or the given fraction, successively by the units of the scale between the given and the required denomination.

III. A compound number to a denominate decimal or to a denominate fraction :—

1. *Write in a column the successive denominate numbers of the given compound number,—placing the lowest denomination at the top, and supplying denominations omitted in the given number.*

Reduce to the decimal of	Reduce to the fraction of
157. 3 vi 3 viij gr. xix a pound.	162. 374 A. 10½ sq. ft. a square mile.
158. 43¢ 7½ mills a dollar.	163. 2 quires 19 sheets a ream.
159. 10a. 7½d. a pound sterling.	164. 5 h. 15 min. 24 sec. a day.
160. 11 cu. ft. 917 cu. in. a cubic yard.	165. O 3 m 17 a Congius.
161. 2 cwt. 15 oz. a gross ton.	166. 11 oz. 19 pwt. 23 gr. a pound.

167. How many degrees of latitude are 12,498 statute miles on a meridian?

168. What is the length of the equator, in feet?

169. How many rails each 30 feet long will be required for 1 mile of a single-track railway?

170. How many revolutions will a carriage-wheel 14 ft. 8 in. in circumference make, in going 5 mi. 5 rd. 5 ft. 6 in.?

171. What is the distance in miles from the Tropic of Cancer (23° 28' 7" N. latitude) to the equator on the opposite meridian, *via* the north pole?

172. A Texas ranch 1.5 miles long and .45 mile wide is inclosed by a wire fence 3 wires high, the wire weighing .1195 lb. to the yard. How much will the wire cost at \$77.50 per net ton?

173. 2 yd. 9 in. is what fraction of 3 rods?

174. If a man digs 79½ rd. of ditch in a month, how many rods of ditch will he dig in .205 mo.?

175. Of a farm of 160 A., ½ is under improvement. What compound number expresses the part of the farm unimproved?

176. A farmer planted three hills of broom-corn on every 2 square yards of a 10-acre field. How many hills did he plant?

177. A cubic foot of distilled water at 62° Fahrenheit weighs 62.418 lb. What is the weight of 3 cu. yd. 13 cu. ft. 864 cu. in.?

178. 18 cu. yd. 25.2 cu. ft. of stone is what fraction of 8 cords?

179. 22 bu. 3 pk. 6¼ qt. is what decimal of 87½ bushels?

180. How many times can a lamp that holds 3½ pt. be filled from a barrel of head-light oil?

181, 182. What is the difference between 100 lb. of iron and the U. S. Mint weight of 100 lb. of gold—in avoirdupois weight? In Troy weight?

- 183.** What fraction of a ream is 87 half-sheets of paper?
- 184.** At the rate of 1 mile in 6 minutes, how many miles could a steam-ship sail in the year 1884?
- 185.** How many days are there in the present century?
- 186.** A pharmacist put up ̄s ix 3 vi of morphine in 1-grain powders. How many powders were there?
- 187.** How much time would be required to count \$1,000,000,000, allowing 10 h. for a day's work, and counting at the rate of \$1 per second?
- 188.** Find the number of seconds in the first three months of this year.
- 189.** When the moon has passed through 3 S. 37.75° of her orbit, over how many seconds of her orbit has she moved?
- 190.** How many half-pint bottles can be filled from a hogshead of wine?
- 191.** A coal dealer bought 2,625 gross tons of coal, and sold it by the net ton. How many tons did he sell?
- 192.** What decimal of a mile is $\frac{1}{4}$ of a yard?
- 193.** How many degrees of longitude are 1,561.328 miles on the equator?
- 194.** What decimal of an acre is $\frac{1}{4}$ of a square yard?
- 195.** What fractional part of a bushel is 3 pk. 3 qt. 1 pt.?
- 196.** What decimal part of a cord of stone is 85 cu. ft. 976 cu. in.?
- 197.** Bought 6 quires 18 sheets of paper. What decimal of a ream did I buy?
- 198.** 4 gt. gro. 3 gro. 5 doz. clothes-pins are how many dozen?
- 199.** What is the value in U. S. currency of 13s. 10 $\frac{1}{2}$ d.?
- 200.** Express $14^\circ 41' 33''$ of latitude in the denominations of linear measures.
- 201.** Change \$17.625 to the denominations of English money.
- 202.** The boundaries of a farm measure 217 ch. 93 l. What is the distance in miles?
- 203.** My farm contains 82.1265 acres. Express the contents by a compound number.

SECTION V.

INTERCHANGING UNITS OF LONGITUDE AND UNITS OF TIME.

211. *Longitude* is distance east or west of any given meridian.

212. The *prime meridian* is the meridian from which astronomers, navigators, and surveyors reckon longitude.

a. The *international prime meridian*, adopted in 1884, passes through the Astronomical Observatory, Greenwich, England.

b. Some nations also reckon longitude from the capital of their own country.

c. Longitude is measured on the equator or on a parallel of latitude, and is expressed in units of the table of circular measure.

213. The earth revolves east on its axis once in 24 hours, and the sun appears to pass west around the earth in the same time; that is, the sun appears to pass west over 15° of longitude in 1 hour of time, over $15'$ of longitude in 1 minute of time, and over $15''$ of longitude in 1 second of time. Hence,

The relative position to the sun, of any place on the earth—except at the poles—determines the time at that place.

214. A difference of 15° in longitude makes a difference of 1 h. in time.

"	$15'$	"	"	"	1 min.	"
"	$15''$	"	"	"	1 sec.	"

On these facts are based the following

PRINCIPLES.

I. *Fifteen times any difference in time equals the difference in longitude; h., min., and sec. being changed respectively to $^{\circ}$, ', ''.* And, conversely,

II. *One fifteenth of any difference in longitude equals the difference in time; $^{\circ}$, ', and '' being changed respectively to h., min., sec.* Hence,

215. RULES.

1. Difference in time to difference in longitude:—

Multiply the difference in time by 15.

2. Difference in longitude to difference in time:—

Divide the difference in longitude by 15.

a. Meridian time, or true time, is time that coincides with the meridian of a place; *i. e.*, when the sun is on the meridian of the place, it is 12 o'clock, noon, at that place.

b. Standard time is time determined from the meridians 60°, 75°, 90°, 105°, and 120° west longitude from Greenwich. The difference in time between two adjacent standard meridians is 1 hour.

c. The five meridians of standard time are designated as follows:

<i>Inter-colonial,</i>	based on 60° meridian;
<i>Eastern,</i>	“ 75° “
<i>Central,</i>	“ 90° “
<i>Mountain,</i>	“ 105° “
<i>Pacific,</i>	“ 120° “

d. Standard time was adopted by the principal railroad companies in the United States in 1888. It has been legalized by legislative and official enactments, in several of the States and in numerous cities of this country.

PROBLEMS.

The difference in time between two places	{	1. Is 25 min. 18 sec.;	}	what is the difference in longitude?
		2. Is 1 h. 15 min. 24 sec.;		
		3. Is 4 min. 15 sec.;		
		4. Is 7 h. 13 min. 5 sec.;		

5. When it is 9 o'clock A.M. at the State-House, Boston, it is 53 min. 43 sec. past 7 A.M. at the City Hall, Chicago. What is the difference in longitude between the two places?

Find the difference in longitude between two places, the difference in time being

6. 18 min.	8. 2 h. 30 sec.	10. 19 min. 36 sec.	12. 8 h. 4 min. 9 sec.
7. 27 min. 20 sec.	9. 7 h. 14 min.	11. 3 h. 5 min.	13. 3½ h.

14. The difference in time between Philadelphia, Pa., and Rome, Italy, is 5 h. 50 min. 28 sec., and the longitude of Philadelphia is 75° 10' west. What is the longitude of Rome?

The difference in time between

15. Washington and San Francisco	} is {	3 h. 1 min. 46 sec.	} What is the difference in longitude?
16. London and New Orleans		5 h. 59 min. 47 $\frac{1}{8}$ sec.	
17. Paris and St. Louis		6 h. 10 min. 21 sec.	
18. Cincinnati and New York		42 min. 25 $\frac{1}{2}$ sec.	
19. Greenwich and Washington		5 h. 8 min. 1 sec.	

The difference in longitude between

20. Philadelphia and San Francisco is $47^{\circ} 16' 45''$;	} What is the difference in time?
21. London and Paris " $2^{\circ} 25' 38''$;	

Find the difference in time between two places, the difference in longitude being

22. $4^{\circ} 30' 45''$.	25. $97^{\circ} 54' 26''$.	28. 131° .
23. $17^{\circ} 3'$.	26. $13^{\circ} 17''$.	29. $4^{\circ} 17' 55\frac{1}{2}''$.
24. $9^{\circ} 35' 40''$.	27. $47^{\circ} 18' 19''$.	30. $13^{\circ} 27''$.

31. Canton, China, is in $113^{\circ} 14'$ E. longitude, and San Francisco is in $122^{\circ} 26' 45''$ W. longitude. What is the difference in time between the two places?

When the difference in longitude between two places exceeds 180° , subtract the difference from 360° before computing the difference in time.

32. If the difference in longitude between two places is $197^{\circ} 13' 24''$, what is the difference in time?

33. Peking, China, is in $116^{\circ} 28'$ E. longitude, and Philadelphia is in $75^{\circ} 10'$ W. longitude. What is the difference in time between the two places?

34. A is in Canton, China, $113^{\circ} 14'$ E. longitude, and B is in New York City, $74^{\circ} 3'$ W. longitude. When it is 9 o'clock 30 min. P.M. by A's watch, what time is it by B's?

35. A man left Pittsburgh, and travelled until the local time was 2 h. 13 min. 20 sec. slow by Pittsburgh time. Through how many degrees of longitude, and in what direction, did he travel?

36. The sun passes the meridian at noon. When the moon passes the meridian at 9 o'clock 41 min. P.M., is the moon east or west of the sun? How many degrees east or west?

SECTION VI.

REVIEW.

216. TEST QUESTIONS IN THE THEORY OF REDUCTIONS.

What change is made

- I. In the fractional unit, in reducing a fraction to lower terms ?
 - II. In the fractional units, in reducing dissimilar to similar fractions ?
 - III. Give reasons for the process in reducing dissimilar to least similar fractions.
 - IV. State wherein *pound* and *sovereign* in English money are not synonymous terms.
 - V. State wherein the word *reduce* differs in its mathematical sense, from its common meaning.
 - VI. Show that the common denominator obtained by reducing dissimilar to similar fractions, is a common multiple of all the denominators of the dissimilar fractions.
- Prove that one of two fractions can be changed so as to have the same denominator as the other fraction
- VII. When the denominator of the first fraction is a factor of the denominator of the second ;
 - VIII. When the denominator of the first fraction is a multiple of the denominator of the second, and the numerator of the first is a multiple of the quotient of the denominator of the first divided by the denominator of the second.
 - IX. Prove that when the terms of a fraction are prime to each other, no fraction having smaller terms can have the same value.
 - X. Prove that either multiplying or dividing by any number produces the same result as performing the converse operation by the reciprocal of that number.

State when the several kinds of reductions included in the next four questions should be used :

- XI. Each of the six cases of the reductions of fractions.
- XII. Each of the two cases of converse reductions of decimals and fractions.
- XIII. Each of the six cases in reductions of compound numbers.
- XIV. The reductions of decimal numbers and of decimal currency.

XV. Show that 6 quarts dry measure are equal to 7 quarts liquid measure (nearly).

XVI. Show by illustration the difference between a purely concrete number and a denominate number.

XVII. Show that 6 quarts dry measure equal (very nearly) 7 quarts liquid measure.

XVIII. Show that 5 cubic feet of space equal (very nearly) 4 statute bushels.

XIX. Show why h., min., and sec. are interchangeable with °, ', and ''.

XX. Suppose a person to start from the intersection of the Eastern Standard Meridian with the parallel of 40° N., and travel constantly north, south, east, or west; how many geographical degrees must he travel in each instance to have a difference of 12 hours between his local time and the time at the place from which he starts?

XXI. Show that a bushel heaped measure equals 5 pecks even measure.

XXII. Name the seven denominate units derived from the quarter of 28 pounds, and state which of these units are English and which American.

XXIII. Show that 28 heaped bushels of coal equal a gross ton.

XXIV. Show that the next centennial year will not be a leap-year.

XXV. Prove that 175 pounds Troy weight equal 144 pounds avoirdupois weight.

XXVI. The length of a meter, in inches, and the distance between any two places, in miles, being known, how can the distance in metric denominations be found?

XXVII. The weight of a kilogram, in ounces, being known, how can the weight of a cargo of iron, in tons, be found in kilograms?

XXVIII. The proportionate length of a meter to a quadrant of the earth's meridian being known, how can the earth's polar circumference be found (1) in meters? (2) In miles?

XXIX. How can the capacity of a cistern, tank, or reservoir, known or given in gallons or hogsheads, be found in hektoliters?

XXX. Show that George Washington was born on the 11th day of February, 1732, instead of the 22nd of that month.

217. OUTLINES OF REDUCTIONS.

I. TERMS.	{	1. _____		5. _____
		2. _____		6. _____
		3. _____		7. _____
		4. _____		_____

A. REDUCTIONS OF DECIMAL NUMBERS.	{	I. PRINCIPLE.	{	1. _____
		II. RULES.		2. _____
				3. _____

B. OF FRACTIONS.	{	I. FOR WHAT USED.	{	1. _____
		II. PRINCIPLES.		2. _____
				3. _____
		III. CASES AND RULES.		4. _____
				5. _____
6. _____				
		7. _____		
		8. _____		

C. OF DENOMI- NATE NUMBERS.	{	I. KINDS, AND HOW PERFORMED.	{	1. _____ ; _____	
		II. CASES AND RULES.		2. _____ ; _____	
				1. <i>Descend- ing.</i>	a. _____
					b. _____
				2. <i>Ascend- ing.</i>	c. _____
		a. _____			
		b. _____			
		c. _____			

D. OF LONGITUDE AND TIME.	{	I. KINDS OF TIME.	{	1. _____
		II. PRINCIPLES.		2. _____
				3. _____
				4. _____
				III. RULES.
2. _____				
3. _____				

CHAPTER IV.

THE FOUR FUNDAMENTAL PROCESSES.

218. The *fundamental processes of computation* are addition, subtraction, multiplication, and division.

Each of these processes, in its application to the four classes of numbers, is based upon the same principles, but is necessarily varied to meet the requirements of the uniform and varying scales of notation.

219. A number can be changed only by increasing or diminishing it. It can be increased only by uniting parts to it; these parts may be unequal or equal to the number increased; hence, a number can be increased only by addition, or by multiplication derived from addition. It can be diminished only by taking parts from it; these parts may be unequal or equal; hence, it can be diminished only by subtraction, or by division derived from subtraction.

From the foregoing it appears that, in arithmetical computations,

220. The *two primitive processes* are addition and subtraction;

221. The *two derivative processes* are multiplication and division; and

222. All arithmetical computations are performed by one or more of these four fundamental processes.

These four processes are often called *the fundamental rules of arithmetic*.

In this chapter these processes are presented in their relations to the four general classes of numbers—integers, decimals, fractions, and compound numbers.

223. The various fundamental processes are based upon axioms, principles, and important facts.

a. Pupils should illustrate the application of the axioms, and demonstrate the several principles and important facts that form the basis of each fundamental process.

b. They should also show the special application of the axioms, principles, and important facts, in the solutions of the illustrative examples.

SECTION I.

ADDITION.

224. Addition is the process of finding a number whose value is the same as that of two or more given like numbers.

a. The **parts** are the numbers added.

b. The **sum** or **amount** is the number obtained by addition.

225. Basis for the processes of addition.

I. The fact stated in the axiom 1, page 45, Art. 158, is included in the following

AXIOM.—*The number of units in the sum is the same as the number of units in all the parts.*

II. From the definition of addition, given above, is deduced the following **PRINCIPLE.**—*Only like orders of units of like numbers can be added.*

III. From 64, V, is taken the following

IMPORTANT FACT.—*Each ten of the sum of the units of any order, integral or decimal, is a unit of the next higher order.*

I. ADDITION OF NUMBERS IN THE DECIMAL SCALE.

226. Ex. The parts are 547; 4,628; 83.059; 2,326.47; and 473.856. What is their sum?

EXPLANATION.—I write the parts so that units of the same order stand in the same column.

Beginning with the units of the lowest order, I add, successively, the units of each order, writing the ones of the sum as units of the same order in the result, adding the tens of the sum with the given units of the next higher order, and writing the entire sum of the units of the highest order.

PROCESS.

$$\begin{array}{r}
 547 \\
 4,628 \\
 83.059 \\
 2,326.47 \\
 473.856 \\
 \hline
 8,058.386
 \end{array}$$

PROBLEMS.

	1	2	3	4
5.	1,375	61,754	980	1,369,452
6.	45	9,803	45,698	64,896
7.	98,769	685	7,693	99
8.	15	37	955	8,045
9.	<u>9,389</u>	<u>96,456</u>	<u>4</u>	<u>967</u>

10-13. In each of the following sets of numbers find the sum of all the numbers above *o*.

19,365— <i>a</i>	2,796,824— <i>a</i>	\$247,320.50— <i>a</i>	14.637867— <i>a</i>
42,321— <i>b</i>	49,380,715— <i>b</i>	745.— <i>b</i>	527.008— <i>b</i>
72,516— <i>c</i>	4,135,296— <i>c</i>	.75— <i>c</i>	9.0009— <i>c</i>
89,347— <i>d</i>	273,416— <i>d</i>	12.19— <i>d</i>	75.1— <i>d</i>
28,916— <i>e</i>	5,237— <i>e</i>	4,000.— <i>e</i>	8,236.1573— <i>e</i>
73,698— <i>f</i>	29,628— <i>f</i>	92,624.25— <i>f</i>	95.315— <i>f</i>
90,875— <i>g</i>	479,536,800— <i>g</i>	62,009.09— <i>g</i>	526.708547— <i>g</i>
42,907— <i>h</i>	57— <i>h</i>	436.375— <i>h</i>	1,900— <i>h</i>
23,560— <i>i</i>	96,006— <i>i</i>	9,379.05— <i>i</i>	.15— <i>i</i>
81,654— <i>j</i>	543,729— <i>j</i>	11.1875— <i>j</i>	.0095— <i>j</i>
32,976— <i>k</i>	2,068,305— <i>k</i>	.50— <i>k</i>	1.1— <i>k</i>
43,568— <i>l</i>	526,372,958— <i>l</i>	497,606.98— <i>l</i>	28.36— <i>l</i>
72,355— <i>m</i>	40,382— <i>m</i>	7,240.01— <i>m</i>	500.005— <i>m</i>
89,372— <i>n</i>	9— <i>n</i>	600,729.57— <i>n</i>	726.925647— <i>n</i>
70,008— <i>o</i>	15— <i>o</i>	98,024.32— <i>o</i>	9,825.37— <i>o</i>
15,000— <i>p</i>	2,078— <i>p</i>	1.375— <i>p</i>	42.016— <i>p</i>
50,236— <i>q</i>	50,000,927— <i>q</i>	696.625— <i>q</i>	78.0087— <i>q</i>
37,295— <i>r</i>	3,286,932— <i>r</i>	10,526.19— <i>r</i>	5,396.5— <i>r</i>
12,637— <i>s</i>	756,819— <i>s</i>	415,096.785— <i>s</i>	408— <i>s</i>
94,378— <i>t</i>	7,485— <i>t</i>	144.125— <i>t</i>	7,296.001— <i>t</i>
54,867— <i>u</i>	96— <i>u</i>	11,973.20— <i>u</i>	417.273469— <i>u</i>
91,643— <i>v</i>	708— <i>v</i>	628,700.— <i>v</i>	526.19685— <i>v</i>
25,536— <i>w</i>	23,456— <i>w</i>	37,925.— <i>w</i>	75.75— <i>w</i>
19,687— <i>x</i>	78,901— <i>x</i>	.04— <i>x</i>	197.0375— <i>x</i>
30,790— <i>y</i>	325,158— <i>y</i>	295.57— <i>y</i>	29.15— <i>y</i>
24,715— <i>z</i>	824,106,375— <i>z</i>	.75— <i>z</i>	1,436.1735— <i>z</i>

What is the sum of { **14-16.** Below *n*? | **20-22.** From *k* to *x*?
all the abstract numbers { **17-19.** From *h* to *u*? | **23-25.** From *c* to *y*?
26. Of all the concrete numbers? | **27-30.** Of each column of numbers?

What is the { **31.** In the first and fourth columns?
sum of all { **32.** In the second and fourth columns?
the numbers { **33.** In the first, second, and fourth columns?

34. Of the numbers from *e* to *t* in the third column, added to the sum of all the numbers below *c* in the same column?

35. What is the sum of ninety-seven quadrillion one hundred eight trillion fifty-nine billion thirty-nine thousand forty-five, seven million five hundred eighteen thousand six hundred eighty-four, thirty-one million ninety-two, seven hundred twenty-seven decillion five quintillion one hundred fifteen thousand nine hundred forty-two, and three septillion forty-nine million one thousand eighty-one?

36. Add five and five hundredths, fourteen million seventy-nine thousand three hundred forty-nine and five hundred sixteen ten-millionths, three hundred nine hundred-thousandths, seventy hundred eighty-seven trillion five hundred billion fourteen million three hundred eighteen thousand and nineteen million seven hundred four thousand eighty-one hundred-millionths, five tenths, and one and ninety-nine ten-thousandths.

II. ADDITION OF NUMBERS IN VARYING SCALES.

227. Ex. 1. Add 40 gal. 3 qt. 1 pt., 46 gal. 1 pt., 25 gal. 1 qt. 1 pt., and 54 gal. 2 qt.

EXPLANATION.—I write the parts so that the units of each denomination stand in a separate column.

I add, successively, the units of each denomination, beginning with the lowest, and reduce the sum to units of the next higher denomination, writing the excess in the result as units of the denomination added.

I add the units of the higher denomination obtained by this reduction, with the given units of that denomination; and I write the entire sum of the units of the highest denomination.

PROCESS.		
40 gal. 3 qt. 1 pt.		
46	0	1
25	1	1
54	2	
166 gal. 3 qt. 1 pt.		

Ex. 2. Find the sum of 243 A. 120 P. 12 sq. yd. 6 sq. ft., 88 A. 80 P. 16 sq. yd. 4 sq. ft., and 124 A. 90 P. 20 sq. yd. 5 sq. ft.

EXPLANATION.—I first perform the work as in example 1, and obtain 456 A. 131 P. 18½ sq. yd. 6 sq. ft.

I then reduce the ½ sq. yd. to integers of lower denominations, and obtain 6 sq. ft. 108 sq. in.; this I add to 456 A. 131 P. 18 sq. yd. 6 sq. ft., and obtain 456 A. 131 P. 19 sq. yd. 8 sq. ft. 108 sq. in., the sum required.

- a.** In adding yards, consider every 11 yards as 2 rods, and thus avoid fractions.
b. In adding tons and pounds, add the pounds as in integers, and add or "carry" 1 for every 2 from the column of thousands of pounds to the columns of tons.

PROCESS.			
243 A. 120 P. 12 sq. yd. 6 sq. ft.			
88	80	16	4
124	90	20	5
456 A. 131 P. 18½ sq. yd. 6 sq. ft.			
		½	= 6
456 A. 131 P. 19 sq. yd. 8 sq. ft. 108 sq. in.			

PROBLEMS.

Add { 37. 17 bu. 3 pk. 2 qt. 1 pt., 58 bu. 1 pk., 3 pk. 2 qt. 1 pt., 14 bu. 5 qt.
 38. 7 yr. 196 da. 13 h. 50 min. 36 sec., 5 yr. 17 min., 37 da. 9 h. 44 sec.
 39. O 5 f3 3 f3 7 u 47, Cong. 1 f3 11 u 19, O 2 f3 3, Cong. 2 f3 5 u 24.

40. What is the sum of 27 rd. 9 ft. 7 in., 119 rd. 3 yd. 2 ft. 11 in., 3 mi. 227 rd. 9 in., 97 rd. 5 yd. 2 ft. 7 in., and 2.25 mi.?

41. Find the amount of 6 lb. 7 oz. 12 pwt., 5 lb. 13 pwt. 14 gr., 127 lb. 9 oz., 9 lb. 16 pwt., and 11 oz. 19 pwt. 23 gr.

42. A man owned four farms, the first containing 150 A., the second 197 A. 120 sq. rd., the third 218 A. 36 sq. rd., and the fourth 476 A. 198 sq. rd. How many acres were there in all his farms?

43. What is the sum of £1,024 8s. 5d., £34 7½d., £955 19s., 5s. 7d. 3 far., and £99 9s. 9d. 1 far.?

44. My cash receipts in four months were \$2,147.50, \$3,207.37½, \$1,312, and \$7,298.87½. What was the sum of my receipts?

45. Find the amount of 1 circumference 2 quad. 37° 35' 20", 3 quad. 89° 42', 67° 13' 27", 2.5 quad., and .5 circumference.

46. I sold to one man 13 T. 15 cwt. of hay, to another 17 T. 7 cwt. 75 lb., to another 9 T. 97 lb., and to another 13 cwt. 50 lb. How much hay did I sell?

47. Find the amount of lb. ij 3 iij 3 iv, 3 ix 3 ij gr. xvij, lb. i 3 v 3 vij gr. ix, lb. iij 3 xi 3 i, lb. ij gr. xiv, and lb. i.

48. How many great gross are there in 11 gro. 7 doz., 3 grt. gro. 9 gro. 5½ doz., 2 grt. gro. 10½ gro., and 3 grt. gro. 3 doz.?

49. One week a grocer sold 84 gal. 2 qt. 1 pt., 55 gal., 23 gal. 3 qt. 1 pt., 73 gal. 2 qt., 96 gal. 1 qt. 3 gi., and 134 gal. 3 qt. 1 pt. of molasses. How much molasses did he sell in the week?

50. The cubical contents of four boxes are, respectively, 26 cu. ft. 137 cu. in., 1 cu. yd. 8 cu. ft. 1,038 cu. in., 20 cu. ft. 108 cu. in., and 25.15 cu. ft. What are the total contents?

51. What is the sum of 14 yr. 109 da., 7 yr. 357 da. 14 h. 38 min., 314 da. 9 h. 45 sec., 5 yr. 210 da. 9 min. 5 sec., and 11 yr. 35 da. 13 min. 24 sec.?

III. ADDITION OF FRACTIONS.

228. In adding fractions observe the following

IMPORTANT FACTS.

I. *Similar fractions are added, by adding the number of fractional units expressed by the several fractions.*

II. *Dissimilar fractions can be added only after they have been reduced to similar fractions (225, II).*

Give three examples illustrating

1. The first fact. | 2. The second fact.

Ex. 1. What is the sum of $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$, and $\frac{1}{4}$?

EXPLANATION. — Since the fractions are similar, I add the number of fractional units expressed by the several given fractions. The result is an improper fraction, which I reduce to a mixed number.

PROCESS.

$$\frac{1}{8} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{1+2+2+2+2}{8} = \frac{9}{4} = 2\frac{1}{4}$$

Ex. 2. Find the sum of $\frac{2}{3}$, $\frac{2}{3}$, $\frac{7}{12}$, and $\frac{1}{12}$.

EXPLANATION. — Since the fractions are dissimilar, I reduce them to similar fractions, and add the

PROCESS.

number of fractional units expressed by the equivalent similar fractions. The result is an improper fraction, which I reduce to a mixed number.

$$\frac{2}{3} + \frac{2}{3} + \frac{7}{12} + \frac{1}{12} = \frac{8+8+7+1}{12} = \frac{24}{12} = 2$$

Ex. 3. Add $18\frac{1}{2}$, $42\frac{1}{2}$, and $34\frac{1}{2}$.

PROCESS.

EXPLANATION. — Since the fractional parts of the given numbers are dissimilar fractions, I reduce them to similar fractions, after which I add, first, these similar fractions, and, next, the integral parts. The sum obtained is the required result.

$$\begin{aligned} 18\frac{1}{2} &= 18\frac{1}{2} \\ 42\frac{1}{2} &= 42\frac{1}{2} \\ 34\frac{1}{2} &= 34\frac{1}{2} \\ \hline 96\frac{3}{2} \end{aligned}$$

PROBLEMS.

What is the sum

52. Of $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$?

56. Of $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$?

53. Of $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$?

57. Of $378\frac{1}{2} + 96\frac{1}{2} + 105\frac{1}{2}$?

54. Of $1\frac{1}{2} + 3\frac{1}{2} + 7\frac{1}{2} + 13\frac{1}{2}$?

58. Of $\frac{1}{3} + 2\frac{1}{3} + 1\frac{1}{3} + 17\frac{1}{3}$?

55. Of $24\frac{1}{2} + 36\frac{1}{2} + 9\frac{1}{2} + 4\frac{1}{2}$?

59. Of $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 91\frac{1}{2} + \frac{1}{2}$?

60. Add $\frac{1}{2}$ mi., $\frac{1}{3}$ mi., $\frac{1}{4}$ mi., $\frac{1}{5}$ mi., $\frac{1}{6}$ mi., and $\frac{1}{10}$ mi.
61. What is the sum of \$175 $\frac{1}{2}$, \$6 $\frac{1}{2}$, \$7 $\frac{1}{2}$, \$ $\frac{1}{2}$, and \$23 $\frac{1}{2}$?
62. Find the amount of 60 $\frac{1}{2}$ A., 13 $\frac{1}{2}$ A., 59 $\frac{1}{2}$ A., 126 $\frac{1}{2}$ A., and $\frac{1}{2}$ A.
63. How many gallons are 45 $\frac{1}{2}$ gal., 177 $\frac{1}{2}$ gal., 213 $\frac{1}{2}$ gal., 71 gal., 59 $\frac{1}{2}$ gal., and 144 $\frac{1}{2}$ gal.?
64. Add 117 $\frac{1}{2}$, 90 $\frac{1}{2}$, $\frac{1}{2}$, 1,277 $\frac{1}{2}$, and 15 $\frac{1}{2}$.
65. Add $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, 18 $\frac{1}{2}$, $\frac{1}{5}$, 17, and 16,974 $\frac{1}{2}$.
66. What is the sum of 26 $\frac{1}{2}$, 915 $\frac{1}{2}$, and 4,688 $\frac{1}{2}$?
67. Add 4,175 $\frac{1}{2}$, 8,015 $\frac{1}{2}$, 971 $\frac{1}{2}$, 430,058 $\frac{1}{2}$, 190,868 $\frac{1}{2}$, and 430 $\frac{1}{2}$.

229. RULES FOR ADDITION.

I. To add integers, decimals, and compound numbers:—

1. Add the units of the lowest order or denomination, reduce the sum to units of the next higher order or denomination, and write the excess in the result as units of the order or denomination added.
2. Add the units of the higher order or denomination obtained by this reduction, with the given units of that order or denomination, and reduce the sum to units of the next higher order or denomination.
3. So proceed with the units of each successive order or denomination, and write the entire sum of the units of the highest order or denomination.

II. To add fractions:—

1. Reduce dissimilar to similar fractions.
2. Add the numerators, and under the sum write the common denominator.
 - a. Reduce fractions to lowest terms before reducing to similar fractions.
 - b. In all final results, reduce proper fractions to lowest terms, and improper fractions to integers or mixed numbers.

PROOFS.—The sums obtained by adding the parts in reverse orders should be equal.

Or,

The parts subtracted successively from the sum and several remainders should leave no remainder.

PROBLEMS.

68. A producer has four casks of oil containing 9,375.5 bar., 6,008.32 bar., 917.975 bar., and 25,979.405 bar. How many barrels of oil has he?

69, 70. I bought $17\frac{3}{4}$ yd. of cloth for \$1.44 $\frac{1}{4}$, $13\frac{1}{2}$ yd. for \$9.75, $26\frac{2}{3}$ yd. for \$2.86, $16\frac{1}{2}$ yd. for \$25.60, and 9 yd. for \$16.875. How many yards of cloth did I buy? What was the amount of the bill?

71. Monday I traveled $245\frac{1}{2}$ mi.; Tuesday, 289 mi.; Wednesday, $308\frac{1}{2}$ mi.; Thursday, $117\frac{1}{4}$ mi.; Friday, $297\frac{1}{4}$ mi.; and Saturday, 311 mi. How many miles did I travel in the week?

72. In A's farm are $77\frac{3}{10}$ A. of meadow, $197\frac{1}{4}$ A. of woodland, 57 A. of pasture, $98\frac{1}{4}$ A. of tillable land, and 37 A. of swamp. How many acres are there in the farm?

73. From a barrel which held 42 gal. of kerosene a grocer sold 7 qt. 1 pt., 3 gal. 1.5 pt., 4.75 gal., 2.5 qt., 6 gal., 5 gal. 3 qt. 1 pt. 2 gi., and 2.25 gal. How much kerosene did he sell?

74. A man owns five farms, containing respectively 144 A. 7 sq. ch. 13 P. 327 sq. l., 98 A. 13.4 P., 108 A. 2 sq. ch. 568 sq. l., 196 A. 8 sq. ch. 13.875 P., and 513 A. $97\frac{1}{4}$ sq. rd. How many acres are there in the five farms?

75. In five successive days a stone-mason laid $5\frac{1}{4}$ cu. yd., $6\frac{3}{4}$ cu. yd., $4\frac{1}{2}$ cu. yd., $3\frac{1}{2}$ cu. yd., and $4\frac{1}{2}$ cu. yd. of stone. How many cubic yards of stone did he lay?

76. A produce-dealer sold 4,650 bu. of wheat to one party, 7,809 bu. to another, 19,409 bu. to another, and 3,725 bu. to another. How many bushels of wheat did he sell?

77, 78. A farmer sold 97 bu. 3 pk. of oats for \$48.87 $\frac{1}{2}$, 136 bu. $1\frac{1}{2}$ pk. of wheat for \$149.94, $275\frac{1}{2}$ bu. of corn for \$171.87 $\frac{1}{2}$, 39 bu. 2 pk. 1 qt. of rye for \$38.97 $\frac{1}{2}$, and 17 bu. 1 qt. 1 pt. of barley for \$18.65. How many bushels of grain did he sell? How much did he receive for it?

79. One day a stationer sold $\frac{1}{2}$ of a ream, $\frac{3}{4}$ of a quire, $\frac{2}{3}$ of a ream, $\frac{1}{2}$ of a ream, and $\frac{1}{4}$ of a quire of note-paper. How much paper did he sell?

80. In a wood-pulp mill $216\frac{1}{2}$ cd. of wood were used in Nov., $173\frac{3}{4}$ ca. in Dec., $188\frac{3}{4}$ cd. in Jan., $216\frac{3}{4}$ cd. in Feb., and $315\frac{1}{2}$ cd. in March. How many cords of wood were used in the five months?

81. I filled orders for 182.3 T., 96.5 T., 37.95 T., and 317.625 T. of coal. How many tons of coal did I sell?

82-85. C, D, and E were engaged in peeling bark. The first day C peeled $1\frac{1}{4}$ cd., D $1\frac{1}{4}$ cd., and E $1\frac{1}{4}$ cd.; the second day C peeled $1\frac{1}{4}$ cd., D $1\frac{1}{4}$ cd., and E $1\frac{1}{4}$ cd.; the third day C peeled $2\frac{1}{4}$ cd., D 2 cd., and E $1\frac{1}{4}$ cd.; the fourth day C peeled the same amount as on the third day, D the same as C peeled the second day, and E the same as D peeled the third day. How much bark did each of the men peel in the four days? How much did all of them peel in the four days?

86. A farmer sold to five grocers 87.5 bu., 96.375 bu., 47 bu. 3 pk., 39 bu. $\frac{1}{2}$ pk., and 76.625 bu. of potatoes. How many bushels did he sell?

87. A grocer sold 5 lb. of coffee for \$.87 $\frac{1}{2}$, 17 $\frac{1}{4}$ lb. of sugar for \$1.62 $\frac{1}{2}$, 29 lb. of rice for \$1.81 $\frac{1}{2}$, 3 $\frac{1}{2}$ lb. of tea for \$3.93 $\frac{1}{2}$, and 4 $\frac{1}{2}$ lb. of cheese for \$.76. How much did his sales amount to?

88. I sold, at different times, 13.375 gal., 11 $\frac{1}{4}$ gal., 15 $\frac{1}{4}$ gal., 3 $\frac{1}{2}$ gal., and 11.75 gal. of maple sirup. How many gallons did I sell?

89. A coal-dealer sold to one customer 9 T. 13 cwt. 2 qr. of coal, to another 7 T. 1 qr. 15 lb., to another 14 T. 3.7 cwt., to another 20 T. 19 cwt. 3 qr. 13 lb., and to another 1,875 lb. How much coal did he sell?

90. The railroad distance from A to B is $11\frac{3}{8}$ miles, from B to C $4\frac{1}{4}$ miles, from C to D $12\frac{1}{2}$ miles, and from D to E $9\frac{1}{4}$ miles. What is the distance from A to E?

IV. COUNTING-ROOM DEVICES AND SHORT METHODS.

Sufficient practice should be given under each of the following Devices and Short Methods, to fix them so firmly in the mind of the learner that he can make ready application of them at any time.

CASE I. To add ledger columns.

230. The methods shown on page 104 are used principally by accountants.

Ex. 1. The unit figure of the sum of a column is written in the result, and a small figure expressing the tens (or the "carry") is written below.

By this method the computation may be suspended on writing the sum of any column, and resumed, at any time, beginning with the "carry" figure of the last column previously added.

Ex. 2. See 158, AXIOM 1.

Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.	Ex. 5.
\$ 328.43	\$823.34	41	54	(61)
679.54	706.52	76	65	(85)
582.17	59.32	25	21	(94)
836.94	408.64	92	18	(32)
270.—	367.48	30	97	(53)
86.85	541.82	47	83	(71)
93.67	215.70	83	56	(10)
5,219.—	632.—	64	41	(28)
8,043.75	278.36	53	32	(69)
3,269.25	419.50	62	77	(33)
395.52	87.05	59	19	(47)
\$19,805.12	1,632.61	632	563	583
876 54	\$4,539.73			

Ex. 3. Beginning at the foot of the right-hand column, add to as near 20 as possible, and write a small figure expressing the ones of the partial sum thus obtained, at the right of the last figure added; begin with the next figure in the column, and add as before; and so on, to the top of the column.

Next, add the digits (if there be any in the column above the last small figure), the numbers expressed by the small figures, and as many tens as there are small figures at the right of the column, and write the ones of this sum for the ones of the final result.

Then, beginning with the tens of the sum of the first column, add the second column in the same manner; and so continue to add, till all the columns are added.

Ex. 4. Beginning at the foot of the right-hand column, add till the sum exceeds 9, place a dot at the right of the last digit added, and drop the ten of the partial sum; begin with the ones of the partial sum, and add as before; and so on, to the top of the column.

Next, to the sum of the digits above the last dot, add as many tens as there are dots, write the ones of this sum for the ones of the final result, and reserve the tens to add to the next column.

Then, beginning with the tens of the sum of the first column, add the second column in the same manner; and so continue to add, till all the columns are added.

The additions may begin at either the foot or the top of the columns.

Ex. 5. In adding a column, combine any two or more digits whose sum is 10, then two or more others, and so on till all the digits in the column are added. Thus, 1st column: (7 + 3) 10, (9 + 1) 20, (8 + 2) 30, (4 + 5 + 1) 40, (3) 43; write 3. 2d column: 4(4 + 2) 10, (8 + 6 + 1) 20, (7 + 3) 30, (5 + 9 + 6) 50, (8) 58; write 58. Sum, 583.

The curves are here used to show the digits combined. In practice omit them.

Of what special value is the operation shown 1. In example 1?

2. In example 2? 3. In example 3? 4. In example 4? 5. In example 5?

CASE II. To add two or more columns at once.

231. Ex. 1. Beginning with the number 81 at the foot of the column, add first the tens and then the ones of the next number, as follows :

81 + 20 (= 2 tens) = 51, and 51 + 5 (ones) = 56; then 56 + 10 (= 1 ten) = 66, and 66 + 7 (ones) = 73; and so on, until all the numbers are added.

	Ex. 1.	Ex. 2.
	32	659
	41	304
	53	127
	17	512
	25	230
	31	465
	199	2,297

Ex. 2. Beginning with the number 465 at the foot of the column, add the hundreds, tens, and ones of each number successively.

MAKING THE COMPUTATIONS.—665, 695; 1,195, 1,205, 1,207; 1,807, 1,827, 1,834; 1,634, 1,638; 2,238, 2,288, 2,297.

The additions are commonly carried from the highest to the lowest unit; but this is not necessarily the order. The addition may begin at any order of units, and proceed from order to order as combinations may be more easily made.

CASE III. Horizontal extensions.

232. Ex. $324 + 281 + 2,312 + 467 + 5,128 + 7,560 + 735 = 16,807$

Beginning with the number at the left, add the ones, then the tens, then the hundreds, and then the thousands of all the numbers. Or, as in Case II, add the entire number in each instance.

MAKING THE COMPUTATIONS.—524, 604, 605; 2,605, 2,905, 2,915, 2,917; 3,817, 3,877, 3,884; 8,384, 8,484, 8,504, 8,512; 15,512, 16,012, 16,072; 16,772, 16,802, 16,807.

Writing numbers with units of the same orders in the same columns for addition, is not a necessity, but a convenience,—as shown by this method.

What are the advantages over the ordinary method of computation in using the method

1. Of adding the parts of a number in two or more sets?
2. Of adding two or more columns at once?
3. Of adding numbers that stand in a horizontal line?

EXERCISES.

Apply the methods now given, by adding the numbers in a part or all of the problems on pages 97, 98.

SECTION II.

SUBTRACTION.

233. Subtraction is the process of finding a number which is the difference between two given like numbers.

a. The **minuend** is that one of the two numbers from which the other is to be subtracted.

b. The **subtrahend** is the number to be subtracted from the minuend.

c. The **remainder** or **difference** is the number obtained by subtraction.

If you have \$17 and spend \$8 of it, you have \$9 remaining; *i. e.*, you have a **remainder** of \$9. But, if you have \$17 and I have \$8, the **difference** between your money and mine is \$9. Hence,

When a part of any number is subtracted from the whole of it, the result is a **remainder**; and,

When one of two like numbers is subtracted from the other, the result is a **difference**.

The terms **remainder** and **difference** are commonly used as synonyms.

234. BASIS FOR THE PROCESSES OF SUBTRACTION.

I. On page 45, Art. 158, is the following

AXIOM.—*If the same number or equal numbers be added to each of two numbers, the difference between the sums will be the same as the difference between the numbers.*

II. From the definition of subtraction (233) is deduced the following

PRINCIPLE.—*Units of any order can be subtracted from units of the same order only.*

III. **IMPORTANT FACTS.**—1. *A unit of any order of any number in the decimal scale is ten units of the next lower order (64, V).*

2. *The remainder or difference comprises the excess of the number of units in the minuend over the number of units in the subtrahend.*

I. SUBTRACTION OF NUMBERS IN THE DECIMAL SCALE.

235. Ex. From 38,069.409 subtract 9,683.0728.

PROCESS.

EXPLANATION.—I write the units of the several orders in the subtrahend under the units of like orders in the minuend.

$$\begin{array}{r} 38,069.409 \\ 9,683.0728 \\ \hline 28,386.3362 \end{array}$$

Beginning with the lowest order, I subtract, successively, the units of each order in the subtrahend from the units of the same order in the minuend, and write the difference as units of that order in the result.

When the units of any order in the subtrahend exceed those of the same order in the minuend, I add 10 to the units of the minuend before subtracting. I then consider the units of the next higher order in the minuend 1 less (64, V), or the units of the next higher order in the subtrahend 1 more (Ax.).

Give the reasons

1. For writing the right-hand figure of the subtrahend, in this process, one place further to the right than the right-hand figure of the minuend.

2. For considering the units of the next higher order in the minuend 1 less.

3. For considering the units of the next higher order in the subtrahend 1 more.

4. Which of the two methods (2 or 3) is preferable, and why?

PROBLEMS.

1	2	3	4	5
1,408,695	9,236,000	100,000	.68725	.85
968,297	5,815,234	99	.14329	.00315
6	7	8	9	10
.5	.01	204.0075	9.12	100.5375
.1875	.00011	54.19	8.40625	.09382
11	12	13	14	15
10,000	\$87.50	\$13,756.—	\$.205	\$450.—
333.333	15.625	897.25	.08	.05565

16. Find the difference between 917 ft. and 649.55 ft.

17. \$100 less \$.375 are how much?

18. Subtract 9,867 from 5,634,278.

19. The minuend is forty-nine thousand and eleven hundred-thousandths, and the subtrahend is seventy-six and eight hundred forty ten-millionths. What is the remainder?

20. Subtract seventeen and nine million nine hundred-millionths from three hundred twenty and one thousand seven hundred eight ten-thousandths.

II. SUBTRACTION OF NUMBERS IN VARYING SCALES.

236. Ex. 1. From 82 bu. 7 qt. subtract 35 bu. 3 pk. 4 qt. 1 pt.

EXPLANATION.—I write the units of the several denominations in the subtrahend under units of like denominations in the minuend.

Beginning with the lowest denomination, I subtract, successively, the units of each denomination in the subtrahend from the units of the same denomination in the minuend, and write the difference as units of that denomination in the result.

When the number of units of any denomination in the subtrahend exceeds the number of the same denomination in the minuend, I add to the units of that denomination of the minuend as many units as equal a unit of the next higher denomination, before subtracting. I then consider the units of the next higher denomination of the minuend 1 less (235), or the units of the next higher denomination of the subtrahend 1 more (Ax.).

Ex. 2. From 80 rd. 3 yd. 1 ft. subtract
60 rd. 5 yd. 2 ft.

EXPLANATION.—I first subtract as in ex. 1, and obtain 19 rd. $2\frac{1}{2}$ yd. 2 ft. $\frac{1}{2}$ yd. = 1 ft. 6 in., which I add to 19 rd. 2 yd. 2 ft., and obtain 19 rd. 3 yd. 8 in., the required remainder.

PROCESS.

$$\begin{array}{r}
 80 \text{ rd. } 3 \text{ yd. } 1 \text{ ft.} \\
 60 \quad 5 \quad 2 \\
 \hline
 19 \text{ rd. } 2\frac{1}{2} \text{ yd. } 2 \text{ ft.} \\
 \quad \frac{1}{2} = 1 \quad 6 \text{ in.} \\
 \hline
 19 \text{ rd. } 3 \text{ yd. } 0 \text{ ft. } 6 \text{ in.}
 \end{array}$$

Apply questions 1, 2, 3, Art. 235, to this case in subtraction.

PROBLEMS.

21. From 2 mi. 87 rd. 2 yd., take 1 mi. 297 rd. 3 yd. 2 ft.

22. From 16 gal. 1 pt., take 14 gal. 2 qt. 1 pt. $1\frac{1}{2}$ gi.

23. From O 5 f3 7 f3 3 m 2, take f3 14 f3 7 m 35.

24. From 2 lb. 7 oz. 20 gr., take 11 oz. 17 pwt. 22 gr.

25. From lb. xi 3 v 3 ii, take 3 ix 3 ii gr. xij.

26. From 3 quad. 7° take 2 quad. $18^{\circ} 37' 24''$.
 27. From 114 bu. 3 qt. 1 pt. take 72 bu. 3 pk. 7 qt. $1\frac{1}{2}$ pt.
 28. From 98 T. 13 cwt. 17 lb. take 69 T. $17\frac{1}{2}$ cwt.
 29. From 12 A. subtract 7 A. 40 sq. rd. 29 sq. yd. 7 sq. ft.
 30. Subtract 17 cd. 97 cu. ft. from 50 cd. 8 cu. ft.
 31. Find the difference between 1,417 mi. 37 rd. 2 yd. and 1,389 mi. 298 rd. 1.5 ft.
 32. Subtract 468 A. 7 sq. ch. 13 P. 459 l. from 1 Tp.
 33. Find the difference between 9 s. 28° and 4 s. $54' 38''$.
 34. Take 2 yr. 308 da. 17 h. 45 min. 12 sec. from 5 yr. 9 h. 8 min. 5 sec.
 35. From 37 cu. yd. 9 cu. ft. 927 cu. in. take 19 cu. yd. 1,309 cu. in.

Ex. Find the difference in time between Oct. 17, 1882, and May 5, 1886.

EXPLANATION.—I write the later of the two dates for the minuend and the earlier for the subtrahend, writing the number of the year, month, and day in order.

I then subtract as in other compound numbers, calling 30 days a month—as the number of days in the subtrahend is greater than that in the minuend.

PROCESS.

1886	yr.	5	mo.	5	da.
1882		10		17	
		3	yr.	6	mo. 18 da.

36. Find the difference of time between Jan. 15, 1877, and July 4, 1883.
 37. What is the length of time from May 7, 1880, to June 1, 1884?
 38. Find the time from Oct. 19, 1870, to Dec. 1, 1885.
 39. From 37 rm. 5 quires 12 sheets take 28 rm. 17 quires 18 sheets.

III. SUBTRACTION OF FRACTIONS.

237. In subtracting fractions observe the following

IMPORTANT FACTS.

I. *Similar fractions are subtracted, by subtracting the number of fractional units in the subtrahend from the number of fractional units in the minuend.*

II. *Dissimilar fractions can be subtracted only after they have been reduced to similar fractions.*

Ex. 1. From $\frac{7}{12}$ subtract $\frac{1}{3}$.

PROCESS.

EXPLANATION.—Since the given fractions are dissimilar, I reduce them to similar fractions, and then subtract the number of fractional units in the subtrahend from the number in the minuend, and write the result over the common denominator.

$$\frac{7}{12} - \frac{1}{3} = \frac{4\frac{1}{12}}{12} = \frac{1}{3}$$

Ex. 2. Subtract $32\frac{1}{2}$ from $59\frac{3}{4}$.

PROCESS.

EXPLANATION.—Since the fractional parts of the given numbers are dissimilar, I reduce them to similar fractions, and then subtract first the fraction and then the integer of the subtrahend from the like parts of the minuend, and obtain the required result.

$$\begin{array}{r} 59\frac{3}{4} = 59\frac{6}{8} \\ 32\frac{1}{2} = 32\frac{4}{8} \\ \hline 27\frac{2}{8} \end{array}$$

Ex. 3. Subtract $49\frac{3}{4}$ from $86\frac{1}{4}$.

PROCESS.

EXPLANATION.—Since the fractional parts of the given numbers are dissimilar, I reduce them to similar fractions.

Since the fractional part of the subtrahend is greater than the fractional part of the minuend, I add $\frac{1}{4}$ ($=1$) to the minuend and 1 to the subtrahend, and then subtract first the fraction and next the integer of the subtrahend from the like parts of the minuend.

$$\begin{array}{r} 86\frac{1}{4} = 86\frac{2}{4} \\ 49\frac{3}{4} = 49\frac{3}{4} \\ 86\frac{2}{4} + \frac{1}{4} = 86\frac{3}{4} \\ 49\frac{3}{4} + 1 = 50\frac{3}{4} \\ \hline 36\frac{3}{4} \end{array}$$

PROCESS.

Ex. 4. From 379 subtract $26\frac{7}{8}$.

$$\begin{array}{r} 379 = 378\frac{8}{8} \\ 26\frac{7}{8} = 26\frac{7}{8} \\ \hline 352\frac{1}{8} \end{array}$$

PROBLEMS.

From	take	From	subtract	Subtract	from
40. $\frac{1}{4}\frac{1}{4}$	$\frac{1}{8}\frac{1}{8}$	43. $\frac{3}{8}\frac{1}{4}$	$\frac{1}{4}$	46. $\frac{1}{8}$ T.	$\frac{3}{8}$ T.
41. $\frac{3}{8}\frac{1}{4}$	$\frac{1}{8}$	44. $19\frac{7}{8}\frac{1}{4}$	$13\frac{1}{4}$	47. $16\frac{7}{8}\frac{0}{8}$ mi.	$205\frac{1}{4}$ mi.
42. $\frac{1}{10}\frac{2}{10}$	$\frac{1}{20}\frac{2}{10}$	45. $40\frac{6}{8}\frac{3}{8}$	$9\frac{6}{8}\frac{3}{8}$	48. $29\frac{1}{8}\frac{0}{8}$ bu.	$50\frac{3}{8}$ bu.
49. Take $\frac{1}{11}$ hhd. from $\frac{5}{11}$ hhd.		52. From 1,000 mi. take $52\frac{2}{5}$ mi.			
50. Take $\frac{3}{4}$ T. from $\frac{10}{10}$ T.		53. From $96\frac{1}{4}$ lb. take 85 lb.			
51. Take $\frac{2}{3}\frac{2}{3}$ gal. from $4\frac{2}{3}$ gal.		54. From 17 cwt. take $5\frac{2}{10}$ cwt.			
55. What is the difference between $298\frac{2}{11}$ rd. and $97\frac{1}{11}$ rd.?					
56. The minuend is $400\frac{1}{8}$, and the subtrahend is $297\frac{1}{8}$. What is the difference?					

238. RULES FOR SUBTRACTION.

I. To subtract integers, decimals, and compound numbers:—

1. *Beginning with the lowest order or denomination, subtract, successively, the units of each order or denomination in the subtrahend from the*

units of the corresponding order or denomination in the minuend, and write the difference as units of that order or denomination in the result.

2. When the number of units of any order or denomination in the subtrahend exceeds the number of units of the corresponding order or denomination in the minuend, add to the units of that order or denomination in the minuend as many units as equal 1 of the next higher order or denomination, before subtracting.

Then, consider the units of the next higher order or denomination in the subtrahend 1 more, or the units of the next higher order or denomination in the minuend 1 less.

II. To subtract fractions:—

1. Reduce dissimilar to similar fractions.

2. Subtract the numerator of the subtrahend from the numerator of the minuend, and under the difference write the common denominator.

See remarks *a* and *b* under Rules for Addition, Art. 229.

PROOFS.—The sum of the subtrahend and remainder should equal the minuend;
or, The difference between the minuend and remainder should equal the subtrahend.
(See Supplement, pages 416, 417.)

PROBLEMS.

57. What number added to 400,375 makes 70,000,001?

58. From 9106 ten-thousandths subtract 24319 millionths.

59. Subtract 2,137,408.032785 from 17,400,916.05.

60. From $\$ \frac{7}{8}$ take \$.625.

61. The United States contains 3,026,504 square miles, and Canada contains 686,353 square miles. How much larger is the United States than Canada?

62. From a piece of land containing .925 A., 65 sq. rd. 118 sq. ft. was sold. How much of the land remained unsold?

63. A note dated Sept. 19, 1880, was paid July 1, 1884. How long did it run?

64. Find the difference between .785 of a year and 67 da. 13 h. 45 sec.

65. From $1,008\frac{1}{8}$ ft. take the difference between $928\frac{1}{4}$ ft. and $579\frac{3}{8}$ ft.

66. Find the difference between $\frac{7}{8}$ hhd. and $28\frac{3}{8}$ gal.

67. If I pay \$4,375 for a farm and sell it at a loss of \$173.75, how much do I receive for it?

68. Subtract one million three hundred twenty-eight thousand two hundred sixteen billionths from ten.

69. $413,701,568,714\frac{2122}{10000000000}$ is the minuend, and $865,314,907\frac{2222}{10000000000}$ is the subtrahend. What is the remainder?

70. What is the difference between $211\frac{345}{1000000}$ and $197\frac{1}{8}$ less $13\frac{14}{11}$?

71. A man raised $197\frac{3}{11}$ bu. of potatoes, from which he sold 97 bu. $3\frac{1}{4}$ pk. How many had he left?

72. Find the difference of time between 10:30 o'clock A.M., May 9, 1848, and 6:45 o'clock P.M., June 3, 1884.

73. I owned $\frac{1}{4}$ of a Tp. of Western land. After selling $1\frac{1}{4}$ sec. to one man, and 5 quarter-sections and 3 forties to another, how many acres had I remaining?

74. Find the difference between three hundred seventy-four trillion ninety-five million one hundred eight thousand sixty-nine and three thousandths, and nine billion fourteen million thirty-six and five million six hundred eighty-seven thousand two hundred four hundred-billionths.

Find the difference in time between	{	75. March 7, 1823, and May 1, 1885.
		76. June 30, 1879, and July 29, 1883.
		77. Feb. 29, 1856, and Jan. 1, 1870.
		78. Sept. 10, 1880, and April 9, 1881.
		79. Oct. 29, 1708, and Dec. 18, 1900.
		80. 7th mo. 15th da., 1886, and 4th mo. 27th da., 1888.

IV. SHORT METHODS.

CASE I. To subtract from a minuend that is one or more units of any order higher than the highest order in the subtrahend.

239. The *complement* of a number is the difference between the number and a unit of the order next higher than that expressed in the number.

The complement of 7 is 3; of 40 is 60; of 46 is 54; of 800 is 700; of 820 is 680; of 828 is 672; of .7 is .3; of .0081 is .0069; etc.

By inspection (*i. e.*, at sight) find the complement of

20	500	7,300	60,000	830,000	9,016	.608
17	240	4,620	37,000	773,600	251,832	.0002
92	632	5,179	300,000	867	.04	.000056

Perform the subtraction in each of the six examples following, beginning at the left in each example; and show

1. That the number of units of the highest order in the remainder is 1 less than the units of the same order in the minuend; and

2. That, in subtracting, each cipher of the minuend may be regarded as 9, except the last, which must be regarded as 10.

Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.	Ex. 5.	Ex. 6.
700	6,000	50,000	20,000,000	3,000	800
<u>8</u>	<u>32</u>	<u>2,017</u>	<u>312,745</u>	<u>9.81</u>	<u>24.073</u>

Under what conditions is this method better than the common method?

CASE II. To subtract at once two or more numbers from another number.

240. Ex. 1. From 4,356 subtract the sum of 827, 534, and 316. PROCESS.

Add the ones of the subtrahends, subtract the sum, 17, from 26 (= 2 tens + 6), and write the remainder, 9, for the ones of the required result.	4,356
Then, to 2 (the number of tens added to the minuend in the previous subtraction) add the tens of the subtrahends, subtract the sum, 8, from 15, and write the remainder, 7, for the tens of the required result.	827
	534
	316
	<u>2,679</u>

In the same manner add the remaining columns, and subtract their sum in each instance from the units of the corresponding order in the minuend.

Ex. 2. From 92,817	COMPUTATIONS.	PROCESS.
subtract the sum of	4, 11, 12 from 17, write 5;	92,817
13,921, 24,017, 8,934,	1, 7, 10, 11, 13 from 21, write 8;	13,921
and 30,860.	2, 10, 19, 28 from 28, write 0;	24,017
	2, 10, 14, 17 from 22, write 5;	8,934
	2, 5, 7, 8 from 9, write 1.	30,860
		<u>15,085</u>

CASE III. Horizontal Extensions. (See 232.)

241. Ex. \$21,319.50 — \$8,174.87½ = \$13,144.62½.

Subtract the units of the several orders of the subtrahend from the units of the corresponding orders of the minuend, as in the common method.

SECTION III. MULTIPLICATION.

242. Multiplication is a short process of finding a number that is the sum of as many times one of two numbers as there are ones in the other ; or,

It is the process of finding such a part of one of two numbers as the other number is part of 1.

a. The **product** is the number obtained by multiplication.

b. The **factors** are the numbers used to obtain the product.

c. The **multiplicand** is the factor that is to be taken any required number of times ; or,

It is the factor a required part of which is to be taken.

d. The **multiplier** is the factor used to obtain any required number of times the multiplicand ; or,

It is the factor used to obtain any required part of the multiplicand.

e. **Continued multiplication** is the process of finding the product of more than two factors.

f. In the process, either factor may be used as a multiplicand, but

g. The **true or logical multiplicand** is the factor that would be used as one of the equal parts in addition ; or,

It is the factor an equal part of which is to be obtained.

Illustrate the application of the terms multiplication, multiplicand, multiplier,

1. When the multiplier is an integer.
2. When the multiplier is a decimal.
3. When the multiplier is a fraction.
4. When the multiplier is a mixed decimal number.
5. When the multiplier is a mixed fractional number.
6. Give three examples in continued multiplication.
7. Give four concrete examples, and name the true multiplicand in each.

What are the relative values of multiplicand and product,—

8. When the multiplier is less than 1?

9. When the multiplier is 1?

10. When the multiplier is more than 1?

243. PROPOSITION I.—*Prove that to multiply by any number is to perform the same operation upon the multiplicand as must be performed upon unity to produce the multiplier.*

244. Basis for the processes of multiplication. (See 223, a, b.)

I. AXIOM.—*The repetition of a number does not change its unit.*

II. GENERAL PRINCIPLES.—1. *The multiplier is an abstract number.*

2. *The multiplicand may be either an abstract or a concrete number.*

3. *The multiplicand and product are like numbers.*

4. *When the multiplier is ones, each partial product is of the same order of units as the order of the units multiplied to obtain it.*

III. PRINCIPLES OF INTEGERS AND DECIMALS.—1. *Each removal of a number one place to the left multiplies the number by 10.*

2. *The sum of the partial products obtained by multiplying a number successively by the units of each order in another number, is the product of the two numbers.*

3. *The product of ones by ones is ones.*

4. *When one factor is an integer, the number of decimal places in the product equals the number of decimal places in the other factor.*

5. *There must be as many decimal places in the product as there are in the factors.*

IV. PRINCIPLES OF FRACTIONS.—1. *Multiplying the numerator or dividing the denominator of a fraction by an integer, multiplies the fraction by that integer.*

2. *Dividing the numerator or multiplying the denominator of a fraction by an integer, divides the fraction by that integer.*

V. IMPORTANT FACTS.—1. *An integer may be moved one place to the left, by annexing to it either a cipher or a digit.*

2. *A decimal may be moved one place to the left, by moving the decimal point one place to the right.*

I. MULTIPLICATION OF NUMBERS IN THE DECIMAL SCALE.

CASE I. To multiply by any integer less than 10.

245. EXPLANATION.—I write the multiplier under the right-hand figure of the multiplicand.

Beginning at the right, I multiply the units of each order in the multiplicand by the multiplier, and in the product I write the ones of each result and add the tens to the next result.

Since, in Ex. 2, the lowest order of units in the multiplicand is hundredths, I point off two decimal places in the product.

ILLUSTRATIVE EXAMPLES

Ex. 1.	Ex. 2.
30,852	308.52
<u>7</u>	<u>7</u>
215,964	2,159.64

PROBLEMS.

	1	2	3	4	5
Multiply	90,086	52,800	.4075	.29785	37.08
by	<u>5</u>	<u>9</u>	<u>6</u>	<u>4</u>	<u>8</u>
6	7	8	9	10	
3,264.1	9.909	\$375	\$.565	\$3,900.75	
<u>3</u>	<u>5</u>	<u>2</u>	<u>8</u>	<u>9</u>	

CASE II. To multiply by one or more integral units of any order.

246. ILLUSTRATIVE EXAMPLES.

Ex. 1.	Ex. 3.	Ex. 5.	
$2,768 \times 10 = 27,680$	$2,768 \times 100 = 276,800$	$2,768 \times 10,000 = 27,680,000$	
Ex. 2.	Ex. 4.	Ex. 6.	
$2.768 \times 10 = 27.68$	$2.768 \times 100 = 276.8$	$2.768 \times 10,000 = 27,680$	
Ex. 7.	Ex. 8.	Ex. 9.	Ex. 10.
2,768	2,768	2,768	2,768
<u>30</u>	<u>30</u>	<u>3,000</u>	<u>3,000</u>
83,040	83,040	8,304,000	8,304,000

How many ciphers must there be on the right of the product of two integers? When may there be more?

PROBLEMS.

- | | | | |
|-----------|------------|-----------|-----------|
| 11 | 12 | 13 | 14 |
| 2.768 | 2.768 | 936,758 | 5,309,000 |
| 300 | 30,000 | 900 | 5,000 |
| 830.400 | 83,040.000 | | |
- Multiply {
- | | |
|--|--|
| <p>15. 43,007 by 100.</p> <p>16. 8,059 by 1,000.</p> <p>17. 609,304 × 10,000 = ?</p> <p>18. 12,057,890 × 100,000 = ?</p> | <p>19. 320.07 × 100 = ?</p> <p>20. 71.3192 × 1,000,000 = ?</p> <p>21. .860071 × 10,000 = ?</p> <p>22. .0007208 × 100,000 = ?</p> |
|--|--|
23. What is the cost of a farm of 160 A., at \$100 per acre?
24. A manufacturer sold 170 corn-shellers at \$20 each. How much did he receive for them?
25. In 1,000 barrels of flour of 196 pounds each are how many pounds?
26. In 1 mile are 5,280 feet. How many feet are there in 3,000 miles?
27. In a pail factory 150 workmen receive \$40 a month each. How much are their total wages for a month?
28. Multiply 29,004,320 by 90,000.

CASE III. To multiply by integral units of different orders.

247. EXPLANATION.—I write the factors—the right-hand digit of the multiplier under the right-hand digit of the multiplicand.

I multiply the multiplicand by the ones, tens, hundreds, etc., of the multiplier; place the right-hand figure of each partial product under the figure of the multiplier used to obtain it; and add the partial products (**158**).

Since in Ex. 2 there are four decimal places in the multiplicand, I point off four decimal places in the product.

ILLUSTRATIVE EXAMPLES.

Ex. 1.	Ex. 2.
24,857	24,857
3,062	3,062
49714	49714
149142	149142
74571	74571
76,112,134	7,611,2134

EXPLANATION.—
I write the factors
—the right-hand
digit of the multiplier under the right-hand
digit of the multiplicand.

Ex. 3.
Multiply 83,400
by 600
50,040,000

Ex. 4.	Ex. 5.
216,000	5,470
790	2,800
1944	4376
1512	1094
170,640,000	15,316,000

Omitting the ciphers on the right of the factors, I multiply the remaining part of the multiplicand by the remaining part of the multiplier, and to the sum of the partial products thus obtained I annex the ciphers omitted.

PROBLEMS.

	29	30	31	32
Multiply	61,908	95,674	60,899	6,875
by	<u>57</u>	<u>5,378</u>	<u>35,068</u>	<u>7,900</u>
33		34	35	36
360,297		90,710	7,380,000	58,370
<u>36,500,000</u>		<u>4,600</u>	<u>6,090,000</u>	<u>698,000</u>

37-40. Multiply 7,083 by 75,000; 31,507 by 29,700; 698 by 50,008; 90,736 by 70,708.

Multiply	41. 9,360.5 by 739.	47. \$500.505 by 6,825.
	42. 30.875 by 1,903.	48. \$10,000.625 by 97.
	43. .368752 by 37,000.	49. 3.21378 by 84,000.
	44. .000378 by 9,999.	50. .1586304 by 707.
	45. \$198.75 by 6,400.	51. \$35,693.01 by 536.
	46. \$.375 by 358,000.	52. \$110,378.25 by 960.

53. What is the product of $17,360 \times 540,000 \times 7,000$?

54. Find the product of $1,030,600 \times 17,000,580$.

55. Multiply 219,000,000 by 714,000.

56. Multiply 608,900,300 by 100 times 4,087,090.

57. Find the product of $5,090 \times 60,800$ multiplied by $299,000 \times 890,700$.

58. Multiply ninety-three quintillion eight hundred seventy-five quadrillion five million by seven hundred thirty-four decillion.

CASE IV. To multiply by decimal units.

248. This process is based on Axiom, and Principles III, V, Art. 244.

ILLUSTRATIVE EXAMPLES.

Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.	Ex. 5.	Ex. 6.
246	246	24.6	.5761	\$.325	.0004
.3	4.3	4.3	.8	.3	.002
<u>73.8</u>	<u>1,057.8</u>	<u>105.78</u>	<u>.40088</u>	<u>\$.0975</u>	<u>\$.0000008</u>

EXPLANATION.—I write the numbers and multiply as in integers, and point off as many figures for decimals in the product as there are decimal places in both factors.

When the number of places in the result is less than the number of decimal places in both factors, I prefix decimal ciphers.

PROBLEMS.

59	60	61	62	63	64
74.687	872,306	\$75.75	\$.25	\$.625	\$.10635
<u>42.345</u>	<u>.342009</u>	<u>1.375</u>	<u>2.5</u>	<u>.596</u>	<u>9.09</u>

	65	66	67
Multiply	73 times 2.904	2,094,007	953.3 times .0826
by	.67	36.15 times 734	4.00005 times 28.509

68. How much will 240.5 bu. of oats cost, at \$.37½ per bushel?

69. Multiply three and six hundred ninety-nine millionths by five and seventy-six hundred-thousandths.

70. What is the product of the factors 27.3, 9.08, 7.065, and 13.0091?

71. 709 thousand multiplied by 36098 millionths = what number?

72. What is the cost of 1,000 barrels of petroleum, at \$1.08½ per barrel?

73. If 2.625 bushels of wheat will seed one acre of land, how many bushels will seed 217.16 acres?

74. I bought 17.5 yd. of silk at \$2.25 per yd., 2.75 yd. of velvet at \$3.87½ per yd., 14.375 yd. of cashmere at \$.42 per yd., and 11.625 yd. of satin at \$1.62½ per yd. What was the total cost?

75. What is the product of .00075 × .0000909?

249. PROPOSITION II.—*Prove that decimal ciphers must be prefixed to the product, when the number of figures in the product is less than the number of decimal figures in the factors.*

II. MULTIPLICATION OF NUMBERS IN VARYING SCALES.

250. The processes in this case are based on the Axiom, and Principle II, Art. 244.

Ex. Multiply 2 da. 9 h. 15 min. 35 sec. by 9.

EXPLANATION.—I write the factors—the multiplier under the lowest denomination of the multiplicand.

PROCESS.

$$\begin{array}{r}
 2 \text{ da. } 9 \text{ h. } 15 \text{ min. } 35 \text{ sec.} \\
 \phantom{2 \text{ da. }} 9 \\
 \hline
 21 \text{ da. } 11 \text{ h. } 20 \text{ min. } 15 \text{ sec.}
 \end{array}$$

I multiply, successively, the units of each denomination, and reduce each product to units of the next higher denomination. I write the excess in each product as units of the denomination multiplied, and add the units of the higher denomination obtained by this reduction, to the product in that higher denomination.

PROBLEMS.

76	77	78
7 cu. yd. 13 cu. ft. 1,309 cu. in.	3 lb. 23 gr.	39 T. 17 cwt. 90 lb.
8	5	7
79	80	81
15 mi. 48 rd. $2\frac{1}{4}$ ft.	19 A. 15 P. 327 sq. l.	£45 10s. $4\frac{1}{2}$ d.
56	109	200

Multiply

82. 15 bu. 3 pk. 2 qt. by 26. 85. 8 lb. 5 oz. 13 pwt. 18 gr. by 224.
 83. 12 mi. 218 rd. 5 yd. 5.5 in. by 19. 86. 19 cwt. 2 qr. 5 lb. by 290.
 84. 3 gal. 1 qt. 3 gi. by 75. 87. 150 A. 29 sq. rd. 15 sq. yd. by 3.7.
88. Find the product of 13 cu. yd. 9 cu. ft. 27 cu. in. multiplied by 8.
 89. Find the product of £205 3.25s. multiplied by 84.
 90. Find the product of £5 9 s 4 m 20 multiplied by 312.
91. If a teamster can draw 1 T. 875 lb. of hay at a load, how much can he draw in 10 loads?
92. How many bushels are there in 25 sacks of potatoes, each sack containing 3 bu. .5 pk.?
93. A silversmith used 1 oz. 13 pwt. 9 gr. of silver, and 7 pwt. 5 gr. of gold for each of two dozen napkin-rings. How much of each kind of metal did he use?
94. How much will 19 T. 12 cwt. 75 lb. of hay cost, at \$11.50 per ton?
95. 16 cd. 36 cu. ft. of wood costs how much, at \$1.75 per cord?
96. 31 gal. 1 qt. 2 gi. of molasses costs how much, at \$.42 per gal.?
97. Multiply 17 rm. 18 sheets by 2.
98. Find the product of 14 yr. 3 mo. 17 h. 38 sec. multiplied by 100.
99. Multiply 13 grt. gr. 7 gr. 8 doz. by 1,000.

III. MULTIPLICATION OF FRACTIONS.

The processes in this subject are based on the Axiom, and Principles II, 4, and IV, 1, Art. 244.

CASE I. To multiply a fraction by an integer.

251. ILLUSTRATIVE EXAMPLES.

Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.
$\frac{7}{11} \times 9 = \frac{63}{11} = 5\frac{8}{11}$	$\frac{11}{11} \times 4 = \frac{44}{11}$	$\frac{8}{11} \times 6 = \frac{48}{11}$	$36\frac{7}{10} \times 5 = \left\{ \begin{array}{l} 36\frac{7}{10} \\ 5 \end{array} \right.$
			$183\frac{1}{2}$

How is the multiplication performed

- | | |
|------------------------|------------------------|
| 11. In example 1? Why? | 13. In example 3? Why? |
| 12. In example 2? Why? | 14. In example 4? Why? |

In multiplying a fraction by an integer

15. When may the denominator be divided? Why?
 16. When must the numerator be multiplied? Why?

PROBLEMS.

Multiply	Find the product	How much is
100. $\frac{11}{11}$ by 8.	103. Of $413 \times \frac{11}{11}$.	106. 75 times $\frac{11}{11}$ bu.?
101. $\frac{11}{11}$ by 55.	104. Of $916 \times \frac{11}{11}$.	107. 108 times $\frac{11}{11}$ A.?
102. $\frac{11}{11}$ by 168.	105. Of $50 \times \frac{11}{11}$.	108. 907 times $\frac{11}{11}$ mi.?

CASE II. To multiply an integer by a fraction.

252. Ex. 1. Multiply 16 by $\frac{7}{13}$.

ANALYTICAL PROCESS.

ANALYSIS.—To multiply by a fraction is to find such a part of the multiplicand as the multiplier is part of 1 (Art. 242).

$$16 \div 13 = \frac{16}{13}$$

$$\frac{16}{13} \times 7 = \frac{112}{13} = 8\frac{8}{13}$$

Since $\frac{7}{13}$ is $\frac{7}{13}$ of 1, to multiply 16 by $\frac{7}{13}$ is to find $\frac{7}{13}$ of 16.

Dividing 16 by 13 I obtain $\frac{16}{13}$, or $\frac{1}{13}$ of 16; and multiplying $\frac{16}{13}$ by 7 I obtain $8\frac{8}{13}$, or $\frac{7}{13}$ of 16, the required product.

SECOND PROCESS.

The work is commonly written as shown in the

$$16 \times 7 = 112$$

COMMON PROCESS.

$$112 \div 16 = \frac{112}{16} = 8 \frac{0}{16}$$

$$16 \times \frac{7}{1} = \frac{112}{1} = 8 \frac{0}{1}$$

17. Wherein does the second process differ from the first?

18. Analyze the second process.

Ex. 2. Multiply $\frac{4}{11}$ by 75.Ex. 3. Multiply 75 by $\frac{4}{11}$.

19. Finding the product of a fraction by an integer consists of the multiplication of what number? Or of a division of what number?

20. Wherein does the process of multiplying a fraction differ from that of multiplying by a fraction?

21. When is the process shortened by performing the division first?

PROBLEMS.

Multiply	How much is	
109. 17 by $\frac{1}{11}$.	113. $\frac{4}{101} \times 27$?	117. $\frac{1}{4} \times 196$ da. = ?
110. 104 by $\frac{3}{11}$.	114. $\frac{1}{100} \times 5,000$?	118. $\frac{1}{11} \times 1,365$ ft. = ?
111. 609 by $\frac{4}{1005}$.	115. $\frac{1}{11} \times 68$ lb.?	119. $\frac{1}{100} \times 103$ rd. = ?
112. 913 by $\frac{3}{11}$.	116. $\frac{1}{100} \times 31$ mi.?	120. $110 \times \frac{1}{11}$ h. = ?

CASE III. To multiply a fraction by a fraction.

253. Ex. 1. Multiply $\frac{1}{4}$ by $\frac{1}{8}$.

ANALYTICAL PROCESS.

ANALYSIS.—Since $\frac{1}{4}$ is $\frac{1}{4}$ of 1, to multiply $\frac{1}{4}$ by $\frac{1}{8}$ is to find $\frac{1}{8}$ of $\frac{1}{4}$, or 7 times $\frac{1}{4}$ of $\frac{1}{8}$.

$$\frac{1}{4} \div 8 = \frac{1}{32}$$

$$\frac{1}{4} \times 7 = \frac{7}{4}$$

A fraction is divided by dividing its numerator or multiplying its denominator (Art. 244, Principle IV, 2).

COMMON PROCESS.

Dividing $\frac{1}{4}$ by 8 I obtain $\frac{1}{32}$, or $\frac{1}{8}$ of $\frac{1}{4}$; and multiplying $\frac{1}{32}$ by 7 I obtain $\frac{7}{32}$, or $\frac{7}{8}$ of $\frac{1}{4}$, the required product.

$$\frac{1}{4} \times \frac{7}{8} = \frac{7}{32}$$

Ex. 2. Multiply $\frac{1}{11}$ by $\frac{1}{11}$.

PROCESS.

Since the numerators and denominators have common factors, I cancel all these common factors, and find the product of the remaining factors of each term for the required numerator and denominator, respectively.

$$\frac{1}{11} \times \frac{1}{11} = \frac{1}{121}$$

Ex. 3. Multiply $38\frac{1}{2}$ by $7\frac{1}{2}$.

FIRST PROCESS.

Reduce mixed numbers to improper fractions.

$$38\frac{1}{2} \times 7\frac{1}{2} = \frac{76}{2} \times \frac{15}{2} = 284\frac{1}{2}$$

For remainder of analysis see Ex. 1.

SECOND PROCESS.

The product of two mixed numbers equals the product of their integral parts plus the products of the integral part of each factor by the fraction of the other factor, plus the product of the fractional parts.

$$38\frac{1}{2} \times 7\frac{1}{2} = \begin{cases} 38 \times 7 = 266 \\ 38 \times \frac{1}{2} = 19 \\ 7 \times \frac{1}{2} = 3\frac{1}{2} \\ \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \end{cases}$$

$$\underline{284\frac{1}{2}}$$

What terms of the given fractions are used as factors to obtain

22. The numerator of the product?

23. The denominator of the product?

24. Then, how is a fraction multiplied by a fraction?

25. Under what conditions may cancellation be applied to multiplication of fractions?

26. When either factor is a mixed number, what preparatory step is necessary?

27. When both factors are mixed numbers, in what two ways may the product be found?

PROBLEMS.

Find the product in each of the following problems :

121. $\frac{1}{2} \times \frac{1}{3}$

125. $\frac{201}{1000} \times \frac{800}{1000} \times \frac{1}{2}$

129. $\frac{1}{2}$ of $\frac{2}{3} \times \frac{1}{4}$

122. $\frac{3}{4} \times \frac{5}{6}$

126. $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5}$

130. $\frac{1}{100} \times \frac{1}{10}$ of 1,554

123. $\frac{1}{2} \times \frac{1}{3}$

127. $\frac{1}{10} \times 5\frac{1}{2} \times \frac{1}{2} \times 7$

131. $\frac{1}{2}$ of $7 \times 197\frac{1}{2}$

124. $\frac{1}{1000} \times \frac{1}{1000}$

128. $\frac{1}{1000} \times 177\frac{1}{2} \times \frac{1}{10}$

132. $\frac{1}{2}$ of $\frac{1}{2} \times \frac{1}{2}$ of 50

133. $14\frac{1}{2} \times 18\frac{1}{2}$ | 134. $48\frac{1}{2} \times 12\frac{1}{2}$ | 135. $117\frac{1}{2} \times 92\frac{1}{2}$ | 136. $107\frac{1}{2} \times 88\frac{1}{2}$

137. How much is $\frac{1}{2} \times \frac{1}{3}$ of $\frac{1}{4} \times 1,000$?

138. Find the product of $\frac{1}{2} \times 9$ multiplied by $4\frac{1}{2} \times 19$.

139. How much is $\frac{1}{2}$ times $\frac{1}{3}$ multiplied by $2\frac{1}{2} \times 3\frac{1}{2}$?

140. What is the value of $\frac{3}{4}$ of 27 barrels of flour, at $\frac{1}{2}$ of \$12 $\frac{1}{2}$ per barrel?

141. How many acres are $\frac{1}{4}$ of $\frac{1}{2}$ of $\frac{3}{10}$ of 160 A.?

254. RULES FOR MULTIPLICATION.

I. Numbers in the decimal scale:—

Beginning with the lowest order of units in both factors, multiply, successively, the units of each order in the multiplicand by the units of each order of the multiplier; write the ones of each result in the product, and add the tens to the next result; place the right-hand figure of each partial product under the figure of the multiplier used to obtain it; add the partial products; and point off as many places for decimals in the final product as there are decimal places in the factors.

II. Numbers in varying scales:—

Multiply, successively, the given units of each denomination, from lowest to highest, and reduce each product to units of the next higher denomination, writing the excess in each product as units of the denomination multiplied, and adding the units of the higher denomination obtained by this reduction to the product in that denomination.

III. Fractions:—

Reduce mixed numbers to improper fractions and integers to the form of fractions; multiply all the numerators together for the numerator, and all the denominators together for the denominator, of the product.

PROOFS.—*The products of the factors used in reverse orders should be the same.*

Or

The quotient of the product divided by either factor should equal the other factor.

(See Supplement, pp. 416, 418.)

PROBLEMS.

142. Multiply 7 decillion 107 quadrillion 327 trillion 9 by 108 million 14 thousand 20.

Find the product

143. Of $\frac{3}{4}$ of $\frac{1}{2}$ of $\frac{1}{11}$ of $\frac{1}{4}$ of 360 and $\frac{1}{16}$ of $\frac{3}{11}$ of 1,000.

144. Of .000376 and 5.103005.

Find the product

145. Of sixty-eight hundred-thousandths and one hundred seventy-five ten-millionths.

146. Of $17^{\circ} 37' 52''$ multiplied by 48.

147. Of 93 and 713 hundred-millionths multiplied by 1,738 ten-thousandths.

148. If a railroad train runs at the rate of 38 mi. 20 rd. per hour, how many miles will it run in $2\frac{1}{2}$ days?

149. G paid $\$1,357\frac{1}{2}$ for a farm, and sold it for $\frac{1}{11}$ of the cost. For how much did he sell it?

150. If a man plant $\frac{7}{8}$ of an acre of corn in 1 day, how many acres will 7 men plant in $3\frac{2}{5}$ days?

151. If a druggist's average daily sales of laudanum are $57\ 35\ 9\ 1$ gr. 13, how much will he sell in 23 days?

152. In cultivating cotton a laborer ploughs one bout to each row of plants. How many miles does he plough in cultivating a field of 76 rows, each row being 1 mi. 136 rd. 15 ft. long?

153. Find the cost of one dozen silver table-knives, each weighing 1 oz. 17 pwt. 23 gr., at $\$.62\frac{1}{2}$ per oz.

What is that number the factors of which are

154. 507, 638, 96, and 1,007? | **155.** 3.00875, .5, .6483, and 14?

156. How much must be paid for the rent of a store $2\frac{1}{2}$ years at $\$62\frac{1}{2}$ per month?

Multiply $\left\{ \begin{array}{l} \textbf{157.} \text{ .0001073586 by 10,000,000.} \\ \textbf{158.} \text{ 9 million 13 by 5 hundred and 396 ten-millionths.} \\ \textbf{159.} \text{ 18 da. 19 h. 17 min. 38 sec. by } 14\frac{1}{2}. \end{array} \right.$

Find the value $\left\{ \begin{array}{l} \textbf{160.} \text{ Of } \frac{211}{11} \times \frac{22}{100} \times \frac{50}{100} \times \frac{22}{11}. \\ \textbf{161.} \text{ Of } 3\frac{1}{2} \times \frac{1}{11} \times 7 \times \frac{11}{11} \times 1\frac{1}{2} \times 100. \end{array} \right.$

162. The factors of a number are three ten-thousandths, nine and seven hundredths, five and one tenth, and nine hundred eight hundred-thousandths. What is the number?

163. A chemist bought 18 lb. $3\frac{1}{2}$ oz. of chemicals at \$9.37 $\frac{1}{2}$ per lb. How much did they cost him?

164. If an oil well flows 2 bar. 28 gal. $2\frac{1}{2}$ qt. per minute, how much will it flow in 7 da. 5 h. 30 min.?

165. Find the cost of 13 T. 19 cwt. 67 lb. of hay at \$10.50 per T.

166. I bought 50 pounds of sugar at 6 $\frac{1}{4}$ ¢, and $\frac{2}{3}$ as much coffee at 4 times the price I paid for the sugar. What did I pay for the coffee?

167. A field of $5\frac{1}{4}$ acres of potatoes yielded 148 bushels per acre. What was the total yield?

IV. SHORT METHODS.

CASE I. The multiplier a composite number.

PROCESS.

255. Ex. Multiply 2,589 by 35. $5 \times 7 = 35$

7 times 5 times a number is 35 times the number.

$$\begin{array}{r} 2,589 \\ 5 \\ \hline 12,945 \\ 7 \\ \hline 90,615 \end{array}$$

PROBLEMS.

Write a different multiplicand for each, and multiply

168. By 36 ($= 6 \times 6$)

172. By 125

176. By 72

169. By 48 ($= 6 \times 8$)

173. By 54

177. By 168

170. By 63 ($= 7 \times 9$)

174. By 88

178. By 280

171. By 648 ($= 8 \times 9 \times 9$)

175. By 49

179. By 729

CASE II. The multiplier 11.

256. Ex. 1. Multiply 25 by 11.

Ex. 2. Multiply 13,584 by 11.

COMPUTATIONS.

Ex. 1. Write 5; ($5 + 2 = 7$) write 7; write 2.

Ex. 2. Write 4; ($4 + 8 = 12$) write 2; ($1 + 8 + 5 = 14$) write 4; ($1 + 5 + 3 = 9$) write 9; ($3 + 1 = 4$) write 4; write 1.

PROCESSES.

(1)	(2)
25	13,584
11	11
275	149,424

CASE III. The multiplier containing only two orders of units expressed by digits, either digit being 1.

257. Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.
$2,135 \times 21$	$1,754 \times 701$	$6,249 \times 12$	$31,625 \times 107$
$\begin{array}{r} 4270 \\ 44835 \\ \hline \end{array}$	$\begin{array}{r} 12278 \\ 1,229,554 \\ \hline \end{array}$	$\begin{array}{r} 12498 \\ 74,988 \\ \hline \end{array}$	$\begin{array}{r} 221375 \\ 3,383,875 \\ \hline \end{array}$

PROBLEMS.

Write a different multiplicand for each, and multiply

180. By 19.	182. By 1004.	184. By 81.	186. By 71.
181. By 106.	183. By 4001.	185. By 501.	187. By 107.

CASE IV. One part of the multiplier a factor of another part.

258. Ex. 1. Ex. 2. Multiply 807 by 735.

$$\begin{array}{r} 519 \\ 84 \quad 8|4 \\ \hline 2076 \quad (\times 2) \\ 4152 \quad (\text{product by 8 tens}) \\ \hline 43,596 \end{array}$$

PROCESS.

$$\begin{array}{r} 807 \\ 735 \quad 7|35 \\ \hline 5649 \quad (\text{product by 7 hundreds}) \\ 28245 \quad (\text{product by 35 ones}) \\ \hline 593,145 \end{array}$$

Ex. 3. Multiply 21,584 by 6,315. | Ex. 4. Multiply 24,217 by 54,618.

3	PROCESSES.	4
$\begin{array}{r} 21,584 \\ 6,315 \quad 6 3 15 \\ \hline 64602 \quad (\times 2) \quad (\text{product by 3 hundreds}) \\ 129204 \quad (\text{product by 6 thous.}) \\ 323010 \quad (\text{product by 15 ones}) \\ \hline 135,987,210 \end{array}$		$\begin{array}{r} 24,217 \\ 54,618 \quad 54 6 18 \\ \hline 145202 \quad (\times 2) \quad (\text{prod. by 6 hund.}) \\ 435606 \quad (\times 3) \quad (\text{product by 18 ones}) \\ 1206818 \quad (\text{prod. by 54 thous.}) \\ \hline 1,221,773,806 \end{array}$

PROBLEMS.

Write a different multiplicand for each, and multiply

188. By 427.	191. By 8,412.	194. By 84,426.	197. By 63,009.
189. By 742.	192. By 9,684.	195. By 96,432.	198. By 54,060.
190. By 639.	193. By 1,188.	196. By 144,848.	199. By 803,216.

CASE V. The multiplier expressed by any number of 9's.

259. Ex. 1. Multiply 756 by 9. 10 - 1 = 9

Ex. 2. Multiply 3,852 by 99. 100 - 1 = 99

Ex. 3. Multiply 487 by 999. 1000 - 1 = 999

COMPUTATIONS.**PROCESSES.**

	1	2	3
Ex. 1. $756 \times 10 - 756 \times 1 = 756 \times 9$			
Ex. 2. $3,852 \times 100 - 3,852 \times 1 = 3,852 \times 99$	7,560	385,200	487,000
	756	3,852	487
Ex. 3. $487 \times 1,000 - 487 \times 1 = 487 \times 999$	6,804	381,348	486,513

This method may also be used
where the multiplier ends with any number of 9's.

Ex. 4. Multiply 4,137 by 5,999.

$$5,999 = 6,000 - 1$$

COMPUTATION.

$$4,137 \times 6000 - 4,137 \times 1$$

PROCESS.

$$\begin{array}{r} 4,137 \\ 6,000 \\ \hline 24,822,000 \\ 4,137 \\ \hline 24,817,863 \end{array}$$

PROBLEMS.

Write different multiplicands, each expressed by three or more figures, and multiply

- | | | | |
|----------------|----------------|----------------|-----------------|
| 200. By 999. | 202. By 8,999. | 204. By 249. | 206. By 40.99. |
| 201. By 9,999. | 203. By 599. | 205. By 3,299. | 207. By 70,099. |

CASE VI. The multiplier an aliquot part.

260. An *aliquot part* of a number is one of the equal parts of the number.

The *unit of an aliquot part* is the number divided to obtain the part.

50, 83 $\frac{1}{3}$, 25, 20, etc., are aliquot parts of 100; and

100 is the unit of each of these aliquot parts.

Name 5 aliquot parts and the unit of each

- | | | | |
|-------------|-----------------|------------------|--------------------|
| 28. Of 12. | 31. Of 1 yard. | 34. Of 1 bushel. | 37. Of \$1. |
| 29. Of 40. | 32. Of 1 pound. | 35. Of 1 mile. | 38. Of 1 gross. |
| 30. Of 300. | 33. Of 1 day. | 36. Of £1. | 39. Of 1 shilling. |

261. Complete the following

TABLE OF ALIQUOT PARTS.

Aliquot parts of	1	10	100	1000	1 Ton or 2000 lb.	1 ft. or 12 in.	1 lb. or 16 oz.	1 doz. or 12	1 yd.	1 A.
One half is	$\frac{1}{2}$	5	50	500	1000 lb.	6 in.	8 oz.	6	$\frac{1}{3}$ yd.	80 sq. rd.
One third is	$\frac{1}{3}$									
One fourth is	$\frac{1}{4}$									
One fifth is	$\frac{1}{5}$									
One sixth is	$\frac{1}{6}$									
One eighth is	$\frac{1}{8}$									
One tenth is	$\frac{1}{10}$									
One twelfth is	$\frac{1}{12}$									
One sixteenth is	$\frac{1}{16}$									

a. In prices in retail business, 8ths, 16ths, and 32ds of 1 cent are seldom used; but in wholesale transactions not only 8ths, 16ths, and 32ds, but also 64ths of 1 cent are extensively used.

b. A difference of $\frac{1}{100}$ of a pound makes a difference of \$31.25 in the cost of 100,000 pounds of cotton. Or,

A difference of $\frac{1}{100}$ per barrel makes a difference of \$39.06 $\frac{1}{4}$ on 250,000 barrels of petroleum.

262. Since 25 is $\frac{1}{4}$ of 100, 25 times any number is $\frac{1}{4}$ of 100 times the same number. Hence,

To multiply by 25, *annex two ciphers to the multiplicand and divide the result by 4.*

In similar manner, deduce a short method for multiplying

1. By 5, 50, or 500.
2. By $3\frac{1}{2}$, $33\frac{1}{2}$, or $333\frac{1}{2}$.
3. By $2\frac{1}{2}$ or 250.
4. By 2, 20, or 200.
5. By $1\frac{1}{2}$, $12\frac{1}{2}$, or 125.
6. By $1\frac{2}{3}$, $16\frac{2}{3}$, or $166\frac{2}{3}$.
7. By any aliquot part of a higher unit.

263. 75 is $\frac{3}{4}$ of 100, and 75 times any number is 3 times $\frac{1}{4}$ of 100 times the same number. Hence,

To multiply by 75, *multiply $\frac{1}{4}$ of 100 times the multiplicand by 3.*

In similar manner deduce a short rule for multiplying

1. By $62\frac{1}{2}$. | 3. By 875. | 5. By $66\frac{2}{3}$. | 7. By $18\frac{2}{3}$.
 2. By 375. | 4. By 6.25. | 6. By $83\frac{1}{3}$. | 8. By $57\frac{1}{2}$.

The principle upon which this method is based may be used when the right-hand figure or figures of the multiplier express an aliquot part of some higher unit.

		PROCESSES.	
Ex. 1.	Ex. 2.	1	2
COMPUTATIONS.			
35 = $8\frac{1}{2} \times 10$	3,125 = $8\frac{1}{2} \times 1000$	470	$3,428,000$
$8\frac{1}{2} \times 10 \times 47$	$8\frac{1}{2} \times 1000 \times 3,428$	$3\frac{1}{2}$	$3\frac{1}{2}$
		235	428500
		1410	10284000
		$1,645$	$10,712,500$

PROBLEMS.

For each of the following multipliers write a different multiplicand of not less than three figures, and multiply

208. By 225. | 210. By 2,025. | 212. By $433\frac{1}{3}$. | 214. By $1,242\frac{2}{3}$.
 209. By 375. | 211. By 5,875. | 213. By $1,016\frac{2}{3}$. | 215. By 3,625.

CASE VII. To find a product correct to any required decimal place.

264. Ex. 1. What is the product, correct to thousandths, of 34.86754 multiplied by 27.3067?

EXPLANATION.—I first write the multiplier under the multiplicand, in such order that each figure of the multiplicand multiplied by the figure of the multiplier directly under it will produce *thousandths*.

I then multiply, by each digit of the multiplier, that part of the multiplicand to the left of and including the figure directly over the digit used, carrying in each instance, from the figure or figures to the right in the multiplicand, the nearest integral number of units of the order expressed by the figure of the multiplicand directly over the figure of the multiplier used.

PROCESS.
3486754
760372
697341
244073
10460
209
24
$952.107\pm$

In carrying from rejected figures on the right,

Add 1 for products from 5 to 14 inclusive;

Add 2 for products from 15 to 24 inclusive;

Add 3 from 25 to 34; 4 from 35 to 44; etc. Why?

Ex. 2. Multiply 865.0754 by .4785 correct to three decimal places.

Ex. 3. Multiply .86759 by .048732 correct to four decimal places.

Ex. 4. Multiply 8,706.2 by 275.86451 correct to two decimal places.

PROCESSES.

2	3	4
865.0754	.86759	8,706.2000
5874	237840	15468572
346030	347	174124000
60555	69	60943400
6920	6	4353100
433	.0422	696496
413938		52237
		3482
		435
		8

From an examination of the preceding examples and the processes employed, show

1. That when the multiplier is in the order required, the figures are in reversed order.

2. That when the multiplier is in the order required, the figure in the ones place will stand under the order of units in the multiplicand that corresponds,—by the terms of the problem,—to the final order of the product.

3. That the several partial products are parts of the products obtained from the same example in ordinary multiplication, but that they are in reverse order.

PROBLEMS.

Find the product

216. Of 48.176 by 32.046 correct to the 4th decimal place.

217. Of 317.9542 by .01752 correct to the 3d decimal place.

218. Of 8450.1 by 10.7231 correct to the 2d decimal place.

219. Of .052346 by .043251 correct to the 5th decimal place.

220. Of 1968.42 by 1.62815 correct to the 2d decimal place.

Retaining only 3 decimal places in the product, multiply

221. 876.945 by 24.122.

222. 921.00475 by 8.9765.

223. 71.08064 by .009271.

224. .781305 by .2357126.

225. .09543 by .08547.

226. 10.0045 by 10.7325.

(For other Short Methods, see Supplement, page 397.)

SECTION IV.

DIVISION.

265. Division is the process of separating a number into equal parts; or,

It is the process of comparing two like numbers, by measuring one of them by the other.

The first definition given above is *general*, and includes all possible arithmetical applications of the process. The second definition is *special*, and is limited to computations in which the numbers to be compared have like units.

(See Supplement, page 403.)

- a.** The *dividend* is the number to be divided.
- b.** The *divisor* is the number by which the dividend is to be divided.
- c.** The *quotient* is the number obtained by division.
- d.** *Long division* is the method in which the result of each step of the process is written.
- e.** *Short division* is the method in which only the final result is written.
- f.** A *partial dividend* is the part of the dividend used to obtain the units of any order or denomination in the quotient, when the quotient expresses units of different orders or denominations.
- g.** Division is *complete* and *exact*, when all the dividend is divided and the quotient is an integer (143).
- h.** A *remainder* is that part of the dividend left undivided at any stage of the process before the division is completed.
- i.** The *reciprocal of a number* is 1 divided by the number.
- j.** A fraction is *inverted* by interchanging its terms.

Illustrate the application of the terms division, dividend, divisor,—when the divisor is

- 1. An integer. | 2. A decimal. | 3. A fraction.
- 4. A mixed decimal number. | 5. A mixed fractional number.

What are the relative values of dividend and quotient

- 6. When the divisor is less than 1?
- 7. When the divisor is 1?
- 8. When the divisor is more than 1?

266. PROPOSITIONS.

Prove by illustrative examples

- I. *That the reciprocal of a fraction is the fraction inverted.*
- II. *That the process of division is a short method of subtraction*
 - 1st. *Only when the dividend and divisor are like numbers.*
 - 2d. *Only when the dividend is greater than the divisor.*
- III. *That when the divisor is more than 1, such a part of the dividend is to be obtained as 1 is part of the divisor.*
- IV. *That when the divisor is less than 1, as many times the dividend are to be obtained as 1 is times the divisor.*
- V. *That to divide by any number is to perform the same operation upon the dividend as must be performed upon the divisor to obtain 1.*
- VI. *That dividing by any number is the same in effect as multiplying by the reciprocal of that number.*

267. BASIS FOR THE PROCESSES OF DIVISION.

- I. AXIOM.—*When all the parts of a number are divided, the entire number is divided.*
- II. GENERAL PRINCIPLES.—1. *When the divisor and dividend are like numbers, the quotient is an abstract number.*
2. *When the divisor is an abstract number, the dividend and quotient are like numbers.*
3. *When the divisor is a denominate number, it must be of the same denomination as the dividend.*
4. *When the dividend and divisor are of the same order of units, the quotient is ones.*
5. *The value of the quotient is not changed by dividing the divisor and dividend by any common factor. (See Cancellation, 180.)*
6. *When the divisor is an integer, any quotient figure expresses units of the same order or denomination as the last figure or denomination of the dividend used to obtain it.*
7. *Any partial remainder is of the same order of units, or of the same denomination, as the last figure or last denomination of the dividend used to obtain it.*

III. PRINCIPLES OF INTEGERS AND DECIMALS.—1. *Pointing off one decimal place in an integer, or removing the decimal point one place to the left in a decimal or a mixed decimal number, divides the number by 10.*

2. *When the divisor is an integer, the quotient has the same number of decimal places as the dividend.*

3. *There must be as many decimal places in the quotient as the number of decimal places in the dividend exceeds the number in the divisor.*

IV. IMPORTANT FACT.—*When the dividend is less than the divisor the quotient is less than 1, and may be expressed by either a proper fraction or a decimal.*

V. PRINCIPLES OF FRACTIONS.—*The principles, 244, IV, apply also to division of fractions.*

I. DIVISION OF NUMBERS IN THE DECIMAL SCALE.

CASE I. To divide by an integer expressed by a digit or digits.

268. ILLUSTRATIVE EXAMPLES.

Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.
32) 3,658,432 (113,076	15) 872.355 (58.157	6) 12,719	5) 138.28
32	75	2,119½	27.656
45	122	85	
36	120	75	
98	23	105	
96	15	105	
248			
224			
192			
192			

ANALYSIS.—

Since in each of these examples the entire dividend can not

Ex. 5.

8) 59,726
7,465.75

readily be divided at one time, I divide all its parts from the highest order of units to the lowest (267, Axiom); and Since the divisor is an integer, each partial quotient, and also each partial remainder, is of the same order of units, respectively, as the partial dividend used to obtain it (267, II, 6, 7); and

Since each divisor is an abstract number, the unit of each quotient is the same as the unit of the dividend (267, II, 2).

Therefore, I divide the first partial dividend and place the result as units of the highest order in the quotient; I then multiply the divisor by this quotient, and subtract the product from the partial dividend. To the remainder I annex the units expressed by the next figure of the dividend, and divide this partial dividend as before.

I proceed in the same manner with each partial dividend, until all the parts of the given dividend are divided.

In examples 1, 2, the process is *long division*; and

In examples 3, 4, 5, the process is *short division*.

9. In the process of division what steps are taken, and in what order?
10. Give the reason for each step.
11. What is obtained by multiplying the divisor by a quotient figure?
12. What numbers are trial partial dividends?
13. What numbers are true partial dividends?
14. When there is a remainder after the last trial dividend has been divided, how is the division completed?
15. In example 1, what part of the quotient would be affected by omitting the 0, and why?

PROBLEMS.

- | | | |
|--|--|---|
| 1. $75 \overline{) 785,625}$ | 2. $16 \overline{) 6,304.8}$ | 3. $9 \overline{) 43.9275}$ |
| 4. $\begin{array}{r} .639056 \\ 8 \end{array}$ | 5. $\begin{array}{r} 23.00625 \\ 25 \end{array}$ | 6. $\begin{array}{r} 16,098,800 \\ 1,685 \end{array}$ |
| 7. $293,654 \div 6$ | 8. $9.327562 \div 775$ | 9. $100.009 \div 87$ |
- Divide $\left\{ \begin{array}{l} 10. 5,340,780 \text{ by } 5. \\ 11. 78,315,246 \text{ by } 78. \\ 12. 15,436.7 \text{ by } 29. \end{array} \right. \left\{ \begin{array}{l} 13. 91.72008 \text{ by } 22,534. \\ 14. 63,375,125 \text{ by } 375. \\ 15. .4697255 \text{ by } 1,362. \end{array} \right. \left\{ \begin{array}{l} 16. 158.329 \text{ by } 897. \\ 17. .0003865 \text{ by } 128. \\ 18. .22032 \text{ by } 54. \end{array} \right.$

CASE II. To divide by a number that expresses a unit or units of any order or orders higher than ones.

269. ILLUSTRATIVE EXAMPLES.

- | | | | |
|---|---|---|--|
| Ex. 6.
$1 \overline{) 0} 8,20 \overline{) 0}$ | Ex. 7.
$7 \overline{) 0} 13 \overline{) 6.29}$
$\begin{array}{r} 1.947 \end{array}$ | Ex. 8.
$8 \overline{) 00} 5 \overline{) 28}$
$\begin{array}{r} .6725 \end{array}$ | Ex. 9.
$300 \overline{) 2.6358}$
$3 \overline{) 00} .026358$
$\begin{array}{r} .008786 \end{array}$ |
| Ex. 10.
$3,6 \overline{) 00} 97,8 \overline{) 48}$ (27.18)
$\begin{array}{r} 72 \\ 258 \\ 252 \\ \hline 288 \\ 288 \end{array}$ | | Ex. 11.
$275, \overline{) 000} 2, \overline{) 925.75}$ (.010631)
$\begin{array}{r} 275 \\ 1757 \\ 1650 \\ \hline 1075 \\ 825 \\ \hline 250 \div 275 = \frac{1}{11} \end{array}$ | |

16. Show that the location of the vertical line used to cut off figures from the right of the integer of the dividend, in the illustrative examples, corresponds to the place for the decimal point of the quotient.

17. In each of the examples 7, 8, 10, what part of the given dividend is used to obtain the tenths of the quotient?

18. In examples 8, 9, 11, why are there no integers in the quotients?
 19. What peculiarity is shown in the process in examples 9, 11?
 20. When is this preparatory step necessary?
 21. Explain the process in each of these six examples.

CASE III. To divide by a decimal or by a mixed decimal number

270. ILLUSTRATIVE EXAMPLES.

Ex. 12.	Ex. 14.	Ex. 15.	Ex. 16.
$.7 \overline{) 9.8}$ 14	$.28 \overline{) 21}$ $.28 \overline{) 21.00} (75$	$2.56 \overline{) 7.296} (2.85$ 512 2176 2048 1280 1280	$156.8 \overline{) .9000} (.0057\frac{1}{2})$ 7840 11600 10976 $624 \div 1568 = \frac{3}{4}$
Ex. 13. $.6 \overline{) .972}$ 1.62	196 140 140		

ANALYSIS.—In solving each of these examples, to obtain the integral part of the quotient there must be as many decimal places in the dividend as there are in the divisor (267, II, 4); and to obtain the decimal part of the quotient, there must be as many decimal places in the quotient as the number of decimal places in the dividend exceeds the number in the divisor (267, III, 3). Therefore, I first annex decimal ciphers to the dividend when necessary, and divide as in integers, placing the decimal point in the quotient before writing the first decimal figure.

When there is a remainder after all the figures expressing the dividend have been used, I form a new partial dividend—by annexing a decimal cipher to the remainder—and so continue the division to any required degree of exactness.

In ordinary computations, the quotient will be sufficiently exact when carried to three or four decimal places.

Examples 15 and 17 are the same. The latter shows that the process is simplified if we

Omit the decimal point from the divisor, and remove the decimal point in the dividend as many places to the right as the original divisor contains decimal figures.

Ex. 17.
$2.56 \overline{) 7.296} ($ $256 \overline{) 7296} (28$ 512 2176 2048 1280 1280

PROBLEMS.

19. $.9 \overline{) 59,872}$	24. $3154 \overline{) 4.1} ($	29. $58.58 \text{ mi.} \div 89.89$
20. $.25 \overline{) 3,702.5} ($	25. $16 \div .0325$	30. $432.967 \text{ gal.} \div 158.2$
21. $.0231 \overline{) 3.268058} ($	26. $.0875 \div 240$	31. $17,326,950 \text{ bu.} \div 3.21$
22. $4.18 \overline{) .0932} ($	27. $.5 \div .000125$	32. $5.1365 \div 7,800 = ?$
23. $.0075 \overline{) .000565} ($	28. $104,377 \div .63926$	33. $51.365 \div .0078 = ?$

II. DIVISION OF NUMBERS IN VARYING SCALES.

271. Ex. 1. Divide 187 rd.
4 yd. 2 ft. by 8.

PROCESS.

$$\begin{array}{r} 8) 187 \text{ rd. } 4 \text{ yd. } 2 \text{ ft.} \\ \underline{23 \text{ rd. } 2 \text{ yd. } 1 \text{ ft. } 11\frac{1}{2} \text{ in.}} \end{array}$$

Analyze each of these processes.

Ex. 2. Divide 27 da. 15 h. 40 min. by
4 da. 6 h.

PROCESS.

$$27 \text{ da. } 15 \text{ h. } 40 \text{ min.} = 39,820 \text{ min.}$$

$$4 \text{ da. } 6 \text{ h.} = 6,120 \text{ min.}$$

$$39,820 \text{ min.} \div 6,120 \text{ min.} = 6\frac{1}{3} \text{ times.}$$

PROBLEMS.

34. 9) 54 sq. mi. 68 sq. rd. 96 sq. in. | **35.** 7) 82 gal. 1 qt. 1 pt.

Divide

36. 113 T. 7 cwt. 18 lb. by 18.

37. 258 bu. 5 qt. 1 pt. by 96.

38. 68 A. 13½ sq. yd. by 147.

39. 42 gal. 1 pt. by 2 gal. 1 qt.

40. 27 cu. yd. 18 cu. ft. by 5.4.

Divide

41. 7 rm. 8 quires by 7 quires.

42. 196° 11' by 2° 31'.

43. 2 yr. 7 mo. 17 da. by 3 wk. 18 h.

44. 17 pwt. 1.5 gr. by 19.

45. 7 cd. 17 cu. ft. by 4.25 cd.

III. DIVISION OF FRACTIONS.

CASE I. To divide a fraction by an integer.

272. The processes in this case are based on **267**, Axiom; and **244**, IV.

ILLUSTRATIVE EXAMPLES.

Ex. 1.

$$\frac{2}{3} \div 5 = \frac{2}{15}$$

Ex. 2.

$$\frac{3}{4} \div 15 = \frac{3}{60}$$

Ex. 3.

$$\frac{1}{2} \div 12 = \frac{1}{24}$$

Ex. 4.

$$11\frac{1}{2} \div 13 = \frac{23}{2} \div 13 = \frac{23}{26}$$

Ex. 5.

$$24) 219\frac{3}{4} \left(9\frac{3}{4} \right. \\ \underline{216}$$

$$3\frac{3}{4} \div 24 = \frac{3}{8} \div 24 = \frac{3}{192}$$

How is the division performed

22. In example 1? Why?

23. In example 2? Why?

24. In example 3? Why?

25. In example 4? Why?

26. In example 5? Why?

27. Show that $\frac{1}{4} \div 4 = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$ of $\frac{1}{4}$.

In dividing a fraction by an integer

28. When *may* the numerator be divided? Why?

29. When *must* the denominator be multiplied? Why?

PROPOSITION VII.—*Prove that dividing a fraction by an integer is the same as multiplying the fraction by the reciprocal of the integer.*

PROBLEMS.

46	47	48	49	50
$\frac{1}{11} \div 7$	$\frac{1}{11} \div 18$	$24\frac{1}{2} \div 9$	$25\frac{1}{2} \div 50$	$\frac{2}{3} \div 6$

What is the quotient

51. Of $\frac{1}{100} \div 68$?	54. Of $\frac{1}{1000} \div 5,100$?	57. Of $\frac{1}{1000} \div 100$?
52. Of $39\frac{1}{11} \div 15$?	55. Of $1,561\frac{1}{11} \div 87$?	58. Of $3\frac{1}{2} \div 195$?
53. Of $7\frac{2}{11} \div 154$?	56. Of $\frac{3}{100} \div 28$?	59. Of $\frac{1}{10} \div 250$?

60. What is the quotient of $1,736,900\frac{2}{5} \div 2,734$?

CASE II. To divide an integer by a fraction.

273. Ex. 1. Divide 28 by $\frac{1}{4}$.

ANALYTICAL PROCESS.

ANALYSIS.—To divide by a fraction is to measure the dividend by the divisor.

$$28 \times \frac{1}{4} = 7$$

To measure one number by another, the unit of the two numbers must be the same (**265**). The unit of 28 is 1, and the unit of $\frac{1}{4}$ is $\frac{1}{4}$. The 4 sevenths can not be reduced to ones, but the 28 ones can be reduced to sevenths. Therefore,

$$7 \div \frac{1}{4} = 28$$

I first reduce the 28 to sevenths; then dividing the result—196 sevenths—by 4 sevenths, I obtain 49, the required quotient.

COMMON PROCESS.

The work is commonly written as shown at the right.

$$28 \div \frac{1}{4} = 28 \times \frac{4}{1} = 112$$

30. Wherein does the common process differ from the analytical process?

Ex. 2. Divide 57 by $\frac{2}{3}$.

Ex. 3. Divide 121 by $13\frac{2}{7}$.

PROCESS.

PROCESS.

$$57 \div \frac{2}{3} = 57 \times \frac{3}{2} = 85\frac{1}{2} \quad | \quad 121 \div 13\frac{2}{7} = 121 \div \frac{94}{7} = 121 \times \frac{7}{94} = 8\frac{1}{2}$$

31. Dividing an integer by a fraction consists of a multiplication by what number? Of a division by what number?

32. When is the process shortened by performing the division first?

33. Wherein does the process of dividing a fraction differ from that of dividing by a fraction?

PROPOSITION VIII.—*Prove that the process of dividing by a fraction and that of multiplying by the reciprocal of the fraction are the same.*

PROBLEMS.

61	62	63	64
$6 \div \frac{3}{15}$	$1 \div \frac{1}{11}$	$4,081 \div 27\frac{9}{10}$	$755 \div 890\frac{1}{2}$

What is the quotient

- | | | |
|------------------------------------|--|--|
| 65. Of $13 \div \frac{4}{7}$! | 68. Of $3 \div 1\frac{1}{11}$! | 71. Of $572 \text{ bu.} \div \frac{3}{4}$! |
| 66. Of $5 \div \frac{1}{11}$! | 69. Of $60 \div 56\frac{1}{2}$! | 72. Of $177 \text{ gal.} \div 8\frac{1}{2}$! |
| 67. Of $1,000 \div \frac{1}{11}$! | 70. Of $60,075 \div 7\frac{1}{2}\frac{1}{2}$! | 73. Of $948 \text{ T.} \div 1,008\frac{1}{2}$ T. |
74. What is the quotient of $1,205 \div 649\frac{1}{2}$?
75. What is the quotient of $903,725 \div 99\frac{1}{10}\frac{1}{10}$?

CASE III. To divide a fraction by a fraction.

274. Ex. 1. Divide $\frac{1}{11}$ by $\frac{1}{7}$.

ANALYTICAL PROCESS.

ANALYSIS.—Any number is divided by a fraction, by multiplying the number by the denominator of the fraction and dividing the resulting product by the numerator (244, IV).

$$\frac{1}{11} \times 9 = \frac{9}{11}$$

$$\frac{9}{11} \div 7 = \frac{9}{77}$$

Therefore, I multiply $\frac{1}{11}$ the dividend by 9 the denominator of the divisor, and then divide $\frac{9}{11}$ the resulting product by 7 the numerator of the divisor, and obtain $\frac{9}{77}$ the required quotient.

COMMON PROCESS.

The work is commonly written as here shown.

$$\frac{1}{11} \div \frac{1}{7} = \frac{1}{11} \times \frac{7}{1} = \frac{7}{11}$$

Ex. 2. Divide $17\frac{1}{2}$ by $\frac{1}{2}$.

PROCESS.

$$17\frac{1}{2} \div \frac{1}{2} = \frac{35}{2} \times \frac{2}{1} = 35 = 18\frac{1}{2}$$

Ex. 3. Divide $\frac{1}{11}$ by $7\frac{1}{11}$.

PROCESS.

$$\frac{1}{11} \div 7\frac{1}{11} = \frac{1}{11} \div \frac{78}{11} = \frac{1}{11} \times \frac{11}{78} = \frac{1}{78}$$

Ex. 4. Divide $9\frac{2}{5}$ by $35\frac{7}{8}$.

PROCESS.

$$9\frac{2}{5} \div 35\frac{7}{8} = \frac{462}{50} \div \frac{287}{8} = \frac{462}{50} \times \frac{8}{287} = 1\frac{24}{125}$$

What terms of the given fractions are multiplied, to obtain

34. The numerator of the quotient? | 35. The denominator?

36. Then, how is a fraction divided by a fraction?

PROBLEMS.

76. $1\frac{1}{2} \div \frac{2}{3}$ | 77. $\frac{217}{100} \div \frac{7}{8}$ | 78. $\frac{1985}{1100} \div \frac{1}{818}$ | 79. $6\frac{2}{17} \div 10\frac{1}{10}$

What is the quotient

80. Of $\frac{7}{10} \div \frac{1}{4}$?

83. Of $413\frac{1}{2} \div \frac{1}{4}$?

86. Of $\frac{9}{10} \div \frac{220}{1000}$?

81. Of $1\frac{2}{3} \div \frac{2}{3}$?

84. Of $5\frac{22}{100} \div 13\frac{2}{3}$?

87. Of $\frac{88}{185} \div \frac{35}{1157}$?

82. Of $17\frac{1}{2} \div 20\frac{1}{10}$?

85. Of $9,075\frac{1}{4} \div 1\frac{1}{2}$?

88. Of $\frac{44}{4} \div \frac{4}{8}$?

89. What is the quotient of $2\frac{1}{2} \div 15\frac{1}{2}$?

275. RULES FOR DIVISION.

I. Numbers in the decimal scale:—

Annex decimal ciphers to the dividend when necessary, divide the first partial dividend, and place the result in the quotient; multiply the divisor by this quotient figure, and subtract the product from the partial dividend. Annex to the remainder the units of the next lower order of the dividend, and divide the partial dividend thus formed, as before.

Proceed in the same manner with each partial dividend, placing the decimal point before writing the tenths of the quotient.

II. Numbers in varying scales:—

1. The divisor an abstract number;—*Divide the units of each denomination separately, from highest to lowest, and in the quotient write the several results for units of the same denominations as those divided.*

Reduce each partial remainder to units of the next lower denomination, and add to it the given units of that denomination, for the next partial dividend.

2. The divisor a compound or denominate number;—*Reduce both divisor and dividend to units of the lowest denomination contained in either, and divide as in integers and decimals.*

III. Fractions :—

Reduce mixed numbers to improper fractions and integers to the form of fractions, and multiply the dividend by the reciprocal of the divisor.

PROOFS.—*The product of divisor and quotient should equal the dividend.*

Or,

The quotient of the dividend divided by the quotient should equal the divisor.

(See Supplement, pp. 416, 418.)

PROBLEMS.

Dividend	Divisor	Dividend	Divisor	Dividend	Divisor
90. 9,376,856,740	1,800,569	94. $\frac{3}{10}$	$\frac{1}{2}$	98. .00005	4.25
91. 328.00875	495.193625	95. 42	$\frac{1}{11}$	99. $73\frac{1}{2}$	$80\frac{1}{2}$
92. .417	.2018653	96. 43.75	1,000,000	100. 913.75	$248\frac{1}{10}$
93. .732684	1,000	97. $\frac{1}{2}$	55	101. $\frac{1}{10}$ of $\frac{1}{2}$	$\frac{1}{2}$ of $\frac{1}{2}$
102. .7 of .324	.002 of .90009	103. 136,800,572	3.62567452		

What is the quotient

104. Of four and five ten-thousandths divided by nine hundred twelve hundred-thousandths ?

105. Of 19 A. 17 sq. yd. 136 sq. ft. 139 sq. in. divided by 29 ?

106. Of 3 lb. 15 pwt. $9\frac{1}{2}$ gr. divided by 12 ?

107. Of 196 mi. 18 rd. 13.5 ft. divided by 57 ?

108. Of 42 cu. yd. 16 cu. ft. 1,080 cu. in. divided by 326 ?

109. The dividend is 7,801,006, the quotient is 502, and the remainder is 428. What is the divisor ?

110. The dividend is $\frac{1}{4}$ and the quotient is $\frac{1}{11}$. What is the divisor ?

111. $3\frac{1}{2} \times 4\frac{1}{2} \times$ by what number = $26\frac{1}{2}$?

112. A steam-ship runs 120 mi. 200 rd. in 7 h. 20 min. What is her rate per hour ?

How many cords are

113. 25,390 pounds of green hemlock bark, at 2,800 pounds to the cord ?

114. 25,390 pounds of partially dry hemlock bark, at 2,500 pounds to the cord ?

115. 25,390 pounds of dry hemlock bark, at 2,200 pounds to the cord ?

- 116.** If \$1,456 $\frac{1}{2}$ is $\frac{1}{3}$ of the cost of a boat, what is the entire cost?
- 117.** By what number must 31 $\frac{1}{2}$ be multiplied to obtain 112 $\frac{3}{4}$?
- 118.** Express in the form of a complex fraction $24\frac{2}{3} \times 5\frac{2}{3} \div 17\frac{1}{3} \times \frac{4}{5}$ of $6\frac{1}{10}$, and find the quotient.
- 119.** A car-wheel 8 ft. 3 in. in circumference will make how many revolutions in running 19 mi. 287 rd.?
- 120.** \$100 for 12 T. 17 cwt. 96 lb. of hay, is how much per T.?
- 121.** What is the quotient of $\frac{18\frac{1}{2} \times \frac{1}{17} \times \frac{1}{11}}{14\frac{1}{11} \times \frac{1}{11}}$?
- 122, 123.** I sold 49 $\frac{1}{2}$ cords of wood for $\frac{2}{3}$ of its cost, and received \$197.50. How much per cord did I receive for the wood? How much per cord did I pay for it?
- 124.** A wholesale carpet dealer paid \$2,514.30 for 49 $\frac{1}{10}$ pieces of carpeting of 76 $\frac{1}{2}$ yards each. How much was the price per yard?
- 125.** If 26 factory boys earn \$507 in a month of 26 working days, what are the daily wages of each boy?
- 126.** At 17 $\frac{1}{2}$ bricks to the course and 5 courses to the foot, what is the height of a chimney that contains 3,171 bricks?
- 127.** Express in the form of a complex fraction, and find the quotient of, 34 $\frac{1}{2}$ times 14 $\frac{1}{2}$ divided by $\frac{4}{5}$ of $\frac{7}{8}$ of 24 $\frac{1}{2}$.
- 128.** \$650 equals how many pounds sterling?
- 129.** The product of a number multiplied by 3 is how many times the product of the same number multiplied by $\frac{3}{8}$?

IV. SHORT METHODS.

CASE I. The divisor a composite number.

PROCESS.

276. Ex. Divide 90,615 by 35. $5 \times 7 = 35$

5) 90,615

One seventh of one fifth of a number is
one thirty-fifth of that number.

7) 18,123
2,589

PROBLEMS.

Using the foregoing method, divide

- | | | |
|---------------------------|-----------------------------|---------------------------------|
| 130. 69,624 by 72. | 132. 104,400 by 144. | 134. 1,562,325 by 1.625. |
| 131. 22,848 by 96. | 133. 568.75 by 125. | 135. 1,905,792 by 2,688. |
- (See Supplement, page 408.)

CASE II. Omitting the partial products from the written work.

277. Ex. Divide 3,606,900 by 4,372.

COMPUTATION.

- (8 × 2) 16 from 19, 8;
- (8 × 7 + 1) 57 from 66, 9;
- (8 × 3 + 6) 30 from 30, 0;
- (8 × 4 + 3) 35 from 36, 1.

4,372 is contained in 10,990, 2 times.

- (2 × 2) 4 from 10, 6;
- (2 × 7 + 1) 15 from 23, 8;
- (2 × 3 + 2) 8 from 9, 1;
- (2 × 4) 8 from 10, 2;

PROCESS.

$$\begin{array}{r} 4,372 \overline{) 3,606,900} \quad 825 \\ \underline{1093} \\ 2186 \\ \underline{0000} \end{array}$$

- (5 × 2) 10 from 10, 0;
 - (5 × 7 + 1) 36 from 36, 0;
 - (5 × 3 + 3) 18 from 18, 0;
 - (5 × 4 + 1) 21 from 21, 0.
- 4,372 in 21,860, 5 times.

By this process solve 10 problems on pages 185, 186.

CASE III. The divisor 15, 25, 35, 45, 55, or 65.

PROCESS.

278. Ex. Divide 8,685 by 45.

$$\begin{array}{r} 45 \overline{) 8,685} \\ \underline{2} \\ 90 \overline{) 17,370} \\ \underline{193} \end{array}$$

Multiply both dividend and divisor by 2 (~~244~~, IV). Cut off the cipher of the divisor (~~269~~), and divide by short division.

PROBLEMS.

Using the foregoing method, divide

136. 4,820 by 35. | **137.** 8,246 by 25. | **138.** 495 by 65. | **139.** 343.475 by 55.

CASE IV. The divisor an aliquot part or parts.

279. Ex. 1. Divide 3,125 by 25.

$$100 \div 4 = 25$$

Divide the dividend by 100, i. e., point off 2 decimal places, and multiply the quotient thus obtained by 4.

PROCESS.

$$\begin{array}{r} 3125 \\ \underline{4} \\ 12500 \end{array}$$

Ex. 2. Divide 3,648 by 12½.

$$100 \div 8 = 12\frac{1}{2}$$

Divide the dividend by 100 (i. e., by 8 times the divisor), and multiply the result by 8.

PROCESS.

$$\begin{array}{r} 36.48 \\ \underline{8} \\ 291.84 \end{array}$$

PROBLEMS.

Using the foregoing method, find the quotient

- | | | |
|------------------------------|------------------------------|----------------------------|
| 140. Of 75.5 ÷ 16½ | 143. Of 123,456 ÷ 83½ | 146. Of 2,618½ ÷ 20 |
| 141. Of 687,530 ÷ 500 | 144. Of 3,263 ÷ 133½ | 147. Of 8,124 ÷ 375 |
| 142. Of 673.25 ÷ 1.25 | 145. Of 7,396 ÷ 6½ | 148. Of 476 ÷ 875 |

CASE V. Multiplication and division combined in one operation.

280. Ex. Divide $8 \times 162 \times 35 \times 13$ by $72 \times 20 \times 42$.

FIRST PROCESS.

$$\begin{array}{r} 1\ 8\ 3\ 5 \\ 8 \times 162 \times 35 \times 13 \\ \hline 72 \times 20 \times 42 \\ 9\ 4\ 6 \end{array} = 2\frac{1}{2} = 9\frac{1}{2}$$

SECOND PROCESS.

$$\begin{array}{r|l} 9\ 72 & 8 \\ 4\ 20 & 162\ 13\ 3 \\ 6\ 42 & 35\ 5 \\ & 13 \\ \hline 4 & 39 \\ & 9\frac{1}{2} \end{array}$$

For problems, see Cancellation and those review problems in which the only processes employed are multiplication and division.

CASE VI. To find a quotient correct to any required decimal place.

EXERCISES.

281. How many integral places will there be in the quotient

37. Of $244.617 \div 81.342$? 40. Of $217,462.55 \div 9,317.8756$?

38. Of $765.9214 \div 341.9241$? 41. Of $.0036754 \div .0179632$?

39. Of $8.643215 \div .08623$? 42. Of $1.00098 \div .0070605$?

What order of units will be expressed by the first digit of the quotient

43. Of $92.875 \div 48.63118$? 45. Of $34.9721 \div 72,951.82$?

44. Of $.00387564 \div .0875643$? 46. Of $.627859 \div .016343$?

Ex. 1. Divide 1,643.8753 by 22.37425 correct to four decimal places.

EXPLANATION. — I ascertain the number of integral places required in the result,—remembering that the figures of the quotient will express integers when such figures are obtained by dividing any partial dividend down to, and including the order of units expressed by the left-hand digit of the divisor (267, II, 4). (In this example, the number of integral places is 2.)

Then, for a divisor I take as many figures—beginning with the left-hand digit of the given divisor—as equal

PROCESS.

$$\begin{array}{r|l} 2|2.37425| & 1633.8753 \\ \hline & 1566198 \\ & 67677 \\ & 67123 \\ & 554 \\ & 447 \\ & 107 \\ & 89 \\ & 18 \\ & 18 \end{array} \quad \begin{array}{l} 73.0248 \pm \end{array}$$

the sum of the integral places and the required decimal places, and find how many times it is contained in the first partial dividend, as in the common method of division. I multiply the figures used of the divisor by the quotient figure, carrying from the rejected figures on the right the nearest integral number of units of the order expressed by the right-hand figure of the divisor used, and subtract the product from the partial dividend.

Then, rejecting the right-hand figure of the divisor previously used, the remaining figures form the next divisor, with which I divide the remainder from the preceding division, multiplying and carrying as before :—and so on, until all the figures of the divisor have been rejected.

Ex. 2. Divide 86.274 by .74325 correct to two decimal places.

Ex. 3. Divide .0054326 by .000745 correct to three decimal places.

2	PROCESSES.	3
$ \begin{array}{r} .7\overline{4325} \overline{) 86.274} \quad \quad 116.07^+ \\ \underline{74325} \\ 11949 \\ \underline{7433} \\ 4516 \\ \underline{4459} \\ 57 \\ \underline{52} \\ 5 \end{array} $	$ \begin{array}{r} .0007\overline{450} \overline{) .0054326} \quad \quad 7.292^+ \\ \underline{52150} \\ 2176 \\ \underline{1490} \\ 686 \\ \underline{671} \\ 15 \\ \underline{15} \end{array} $	

If the number of figures in the divisor does not equal the entire number of integral and decimal places in the quotient, annex decimal ciphers to the divisor.

PROBLEMS.

Find the quotient

149-151. Of each of the exercises 37, 38, 39, page 144, correct to the second decimal place.

152-154. Of each of the exercises 40, 41, 42, correct to three decimal places.

155-157. Of exercises 43, 44, 45, correct to four decimal places.

158. Of exercise 46 correct to five decimal places.

SECTION V.

REVIEW.

282. THE TEN CASES IN THE FUNDAMENTAL PROCESSES.

Any arithmetical problem can be classed under one or more of these cases. Pupils should fill the blank in each case, and explain the process required.

CASE I. Given, the parts, to find ———.

CASE II. Given, the subtrahend and difference, to find ———.

CASE III. Given, the sum and all the parts but one, to find ———.

CASE IV. Given, the minuend and subtrahend, to find ———.

CASE V. Given, the factors, to find ———.

CASE VI. Given, the divisor and quotient, to find ———.

CASE VII. Given, the divisor and dividend, to find ———.

CASE VIII. Given, the dividend and quotient, to find ———.

CASE IX. Given, the product and one factor, to find ———.

CASE X. Given, the product of several factors and all the factors but one, to find ———.

283. PROPOSITIONS AND PROBLEMS.

Prove by illustrative examples involving successively integers, decimals, numbers in varying scales, and fractions :

PROPOSITION I. *Multiplying the dividend or dividing the divisor by any number multiplies the quotient by the same number.*

PROPOSITION II. *Dividing the dividend or multiplying the divisor by any number divides the quotient by the same number.*

PROPOSITION III. *Multiplying or dividing both dividend and divisor by any number does not change the quotient.*

PROBLEMS.

1. Add three hundred and seventy-nine ten-thousandths, eighty-five and one millionth, ninety-eight hundred-thousandths, and two hundred sixty-seven and twelve ten-thousandths.

2. Find the sum of $\frac{3}{16}$, $\frac{4}{5}$, $\frac{1}{11}$, and $\frac{488}{1000}$. | 3. From 6.5 subtract .097325.

4. What is the difference between $2\frac{11}{17}$ and $\frac{4}{11}$?

5. The product of four factors is 36; and three of the factors are $\frac{1}{3}$, $9\frac{1}{2}$, and $5\frac{1}{3}$. What is the other factor?

6. Subtract 342 times $\frac{1}{1000}$ from 3.7 times $1\frac{1}{10}$.

7. $\frac{1188}{1000} \div 4.625 =$ what number?

8. Find the time from Nov. 29, 1876 to July 4, 1887.

9. Divide $287\frac{1}{2}$ by the sum of $13\frac{1}{10}$ and $22\frac{1}{11}$.

10. The product of three numbers is .495, and two of the numbers are $1\frac{1}{11}$ and .35. What is the other number?

11. If 17 be added to both terms of $\frac{4}{5}$, how will the result compare in value with the original fraction?

12. Divide \$8 into 200 equal parts. | 13. Divide 2,000 by $\frac{7}{100}$.

14. From sixteen billion seven hundred eight thousand two hundred ninety-three hundred-quadrillionths take thirty-nine trillion four hundred seventy-two decillionths.

15. $7 \div .07$ minus $.07 \div 7 =$ what number?

16. Multiply thirteen and five tenths by one thousand five hundred-thousandths, divide the product by fifteen hundredths, and to the quotient add sixteen thousand eight hundred seventy-five hundred-millionths.

17. From 32 rd. 5 yd. 2 ft. 11 in. take 23 rd. 1 ft. 2 in.

18. Divide .7 of .324 by .02 of .90009.

19. Divide $8\frac{1}{2}$ by $3\frac{1}{2}$, and $5\frac{1}{2}$ by $10\frac{1}{10}$, and find the product of the quotients.

20. Find the time from May 10, 1816 to Jan. 1, 1887.

21. Reduce £23 9s. 2½d. sterling to U. S. money.

22. Divide 12 A. 108 sq. rd. 4.5 sq. yd. of land into 19 lots of equal area.

23. From 2 subtract .9999, multiply the remainder by .035, and divide the product by .25.

24. Find the sum of $426\frac{1}{2} - 6\frac{1}{2}$ and $426\frac{1}{2} \times \frac{1}{2}$.

25. To the sum of the two numbers $6\frac{1}{4}$ and $9\frac{1}{2}$ add their difference and their product.

26. Divide $6\frac{1}{2}$ by $\frac{1}{2}$, multiply the quotient by $2\frac{1}{2}$, from the product subtract $4\frac{1}{2}$, and to the remainder add $14\frac{1}{2}$.

27. A pharmacist used 33 2 gr. 10 of opium in 92 powders. How much opium was there in each powder?

28. The sum of four numbers is $897\frac{1}{2}$; the first of the numbers is $130\frac{1}{2}$; the second is $3\frac{1}{2}$ times the first; and the third is $\frac{1}{2}$ of the sum of the first and second. What is the fourth number?

29. Divide 17 by the quotient of $13\frac{1}{2} \div 16\frac{1}{2}$.

30. Divide $27\frac{1}{2}$ by $14\frac{1}{2} - 10\frac{1}{2}$.

31. $6\frac{1}{2} + 9\frac{1}{2} \times 5\frac{1}{2} - 4\frac{1}{2} \div 12\frac{1}{2} =$ what number?

32. $75 \times 38 - 1,918 + 679 \div 56 = (?)$ | 34. $93.3 \div 3\frac{1}{2} = (?)$

33. $75 \times 38 - [(1,918 + 679) \div 56] = (?)$ | 35. $\frac{31\frac{1}{2} \times \frac{1}{2} \times 8.375 \times \frac{1}{2}}{\frac{1}{2} \text{ of } 4\frac{1}{2} \times \frac{1}{2}} = (?)$

36. How many seconds are there in the circumference of a silver dollar?

37. Find the number to which, if $\frac{1}{2}$ of $\frac{1}{2}$ be added, the sum will be 1.

38. What number is that which being increased by $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of itself, the amount will be $3,516\frac{1}{2}$?

39. What is the difference of latitude between St. Augustine, Fla., 30° N., and Cape of Good Hope, Africa, 35° S.?

40. $37\frac{1}{2}$ cords of wood make how many piles of 7 cd. 96 cu. ft. each?

41. Divide 1 Tp. 20 Sec. 394.728 A. of government land into 240 farms of equal size.

42. 19 T. 13 cwt. 11 lb. of sugar will fill how many hogsheads averaging 19 cwt. 13 lb. 11 oz.?

43. The longitude of Cincinnati, O., is $84^\circ 39' 21''$ W.; that of Canton, China, is $113^\circ 14'$ E. What is the difference in longitude between the two places, reckoning west from Cincinnati?

Find the date

44. Of 1 yr. 2 mo. 15 da. from May 10, 1885. | 47. Of 9 yr. 21 da. from July 22, 1881.
 45. Of 8 mo. 12 da. from Sept. 19, 1884. | 48. Of 90 da. } from Jan. 8, 1883.
 46. Of 5 yr. 4 mo. from Apr. 20, 1886. | 49. Of 3 mo. }
 50. Of the 47th da. after Jan. 17, 1887. | 52. Of the 98th day before Dec. 2, 1881.
 51. Of the 213th da. after Aug. 6, 1886. | 53. Of the 171st da. before Apr. 29, 1888.

54. Add 81 days to May 16, and give the date.

55. Subtract 113 days from Nov. 1, and give the date.

What is the date

56. Of 90 da. + 3 da. after July 10? | 58. Of 60 da. + 3 da. after Feb. 28, 1892?
 57. Of 30 da. + 3 da. after Apr. 15? | 59. Of 120 da. + 3 da. after June 2?

How many days

60. From June 17 to Oct. 5? | 62. From Feb. 9, 1882, to Aug. 13, 1882?
 61. From Oct. 11 to Jan. 24? | 63. From Nov. 10, 1887, to Sept. 3, 1888?

64. What is the value of $\frac{\frac{1}{2}}{\frac{1}{3} \times \frac{1}{4}}$? | 65. Simplify the two complex fractions $\frac{\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}}{\frac{1}{5} \times \frac{1}{6}}$ and $\frac{2\frac{1}{2} \times 13\frac{1}{2}}{6\frac{1}{2} \times 4\frac{1}{2}}$, and find their sum.

66. Find their difference. | 67. Find their product.

68. Find their two quotients.

69. A man was born in 1824, his wife in 1829, his son in 1854, and his daughter in 1863. What is the sum of their ages this year?

70. From May 19 to July 18 are how many week-days?

71. At \$1 $\frac{1}{2}$ a yard what will be the cost of a carpet for a flight of stairs of 19 steps, each 7 $\frac{1}{2}$ inches high and 11 $\frac{1}{2}$ inches wide?

72. A merchant bought 975 $\frac{1}{2}$ yd. fancy prints @ 4 $\frac{1}{2}$ ¢, 1,037 yd. muslin @ 7 $\frac{1}{2}$ ¢, 726 $\frac{1}{2}$ yd. cashmere @ 37 $\frac{1}{2}$ ¢, 511 $\frac{1}{4}$ yd. silk @ \$1 $\frac{1}{2}$, 413 $\frac{1}{2}$ yd. satin @ 98¢, and 105 $\frac{1}{2}$ yd. broadcloth @ \$3.13. Find the amount of his purchases.

73. A miller in Minneapolis, Minn., bought 965,000 pounds of wheat and 1,006,400 pounds of corn. How many bushels of each kind of grain did he buy?

74. A farmer raised 5 bushels of clover seed, from which he sold the following amounts: .625 bu., $\frac{1}{8}$ bu., $\frac{1}{4}$ bu., .375 bu., .85 bu., and 3 pk. 3 qt. How much had he remaining?

75. \$7.20 for 1,280 lb. of fertilizer is how much per ton?

76. The difference in longitude between two places being $7^{\circ} 49' 28''$, what is the difference in time?

77. From a barrel of molasses was drawn at various times 6.25 gal., 4 gal. 2 qt. 1 pt., $5\frac{3}{4}$ gal., $7\frac{1}{8}$ gal., and .875 gal. How much molasses remained in the barrel?

78. The local time of St. Louis is 2 hr. 8 min. 46 sec. earlier than that of San Francisco. The longitude of St. Louis is $90^{\circ} 15' 15''$ W. What is the longitude of San Francisco?

79. A pharmacist put up $\frac{2}{3}$ of quinine, that cost him \$6.50 per lb., in pills of 2 grains each. He sold the pills at 20¢ per dozen. How much was his profit?

80. At 4s. 4d. per pair, how many dozen pairs of gloves can be bought for £26?

81. Of a section of Western land 117 A. 78 sq. rd. are woodland, and the remainder is in 8 equal fields. What is the area of each field?

82. A quarryman sold 218 cords of building stone at \$.75 a perch. How much did he receive for it?

83. A farmer sold to a merchant 2 tubs of butter, 50 lb. each, @ $22\frac{1}{2}$ ¢, and 30 doz. eggs @ 18¢. He bought from the merchant $5\frac{1}{2}$ lb. Java coffee @ 32¢; $2\frac{1}{4}$ lb. Japan tea @ 90¢; 2 sacks of flour @ \$1.62 $\frac{1}{2}$; 1 box bar soap, 60 lb., @ $7\frac{1}{2}$ ¢; 1 pr. boots, \$3.75; 2 pr. ladies' boots, \$5.25; 1 piece Lonsdale muslin, $42\frac{1}{4}$ yd., @ $9\frac{1}{2}$ ¢; 5 gal. head-light oil @ 15¢; 2 gal. sirup @ 65¢; and 50 lb. sugar @ 13 lb. for \$1. The balance was paid in money. What was the amount paid, and who paid it?

84. A company of 32 men own the 100 shares of stock of a Dakota land company, 10 of them owning 1 share each, 10 of them 3 shares each, and 12 of them 5 shares each. The company divides a tract of $30,376\frac{1}{4}$ acres of land among the shareholders, according to the number of their respective shares. How much land do the first 10 receive?

85. How much land do the second 10 receive? | 86. 7 of the second 10?

87. How much land do the 12 receive? | 88. 5 of the 12?

89. How much land does each one of the first 10 receive?

90. How many gross of clothes-pins can be bought for $\$7\frac{2}{5}$ at $2\frac{1}{4}$ ¢ per dozen?

91. A grocer paid \$4 for a barrel of kerosene, and retailed it at 15¢ per gallon. How much did he gain?

92. If you can do a piece of work in $27\frac{1}{4}$ days that you and I together can do in $19\frac{1}{4}$ days, in how many days can I do it alone?

93. 5,000 bushels of oats in St. Louis, Mo., equal how many bushels in Hartford, Conn.?

94. I have a piece of silk containing $42\frac{3}{4}$ yards. If I keep $\frac{3}{4}$ of it, and sell the remainder for $\$1\frac{1}{2}$ per yard, how much shall I receive for what I sell?

95. A clothing merchant took a lease of a store for 4 years at $\$81\frac{1}{2}$ per month. At the end of $2\frac{3}{4}$ years he sold the lease for $\$105\frac{1}{2}$ per month for the unexpired term. How much did his rent cost him per month for the time he occupied the store?

96. A fruit dealer bought $1,296\frac{1}{2}$ bu. of apples @ $37\frac{1}{2}$ ¢, and 768 bu. 3 pk. @ 45¢ per bu. He sold $\frac{1}{3}$ of the first lot at 50¢ per bu., and the remainder at 45¢. He sold $\frac{1}{3}$ of the second lot at cost, and the remainder at $62\frac{1}{2}$ ¢ per bu. How much did he gain?

97. How much sooner does the sun rise at New York than at Chicago, on the 21st day of March?

98. A grocer buys vinegar at 15¢ per gal., and retails it at $5\frac{1}{4}$ ¢ per qt. What are his profits on a cask that contains 37 gal. 2 qt. 1 pt.?

99. A person who owned $\frac{1}{2}$ of a Wisconsin lead mine, sold $\frac{1}{3}$ of his stock for \$5,390. What was the value of the entire mine at the same rate?

A market gardener bought 3 A. 90 sq. rd. of land at \$128.50 per A.; 7 A. 28 sq. rd. at \$235 per A.; and 10 A. $1\frac{1}{4}$ sq. yd. at \$68.75 per A.

100. How many acres did he buy? | **101.** How much did it cost him?

102. A, owning $\frac{1}{3}$ of a mill, sold $\frac{2}{7}$ of his share to B. What part of the mill did each man then own?

103. At \$140 for 27 T. 850 lb. of coal, how much will 97 T. 540 lb. cost?

104. When it is noon at Buffalo, N. Y., $78^{\circ} 52' W.$, what is the time at Berlin, Prussia, $13^{\circ} 23' 45'' E.$?

105. 234 saw-logs at \$.62 $\frac{1}{2}$ each will pay for sawing how many thousand feet of lumber at \$3.25 per M.?

106. If a man cuts 2 cords of wood in $\frac{1}{4}$ of a day, in what time will he cut .3 of a cord?

107. The latitude of Jacksonville, Fla., is $30^{\circ} 15' N.$, and of Cleveland, O.—on the same meridian—is $41^{\circ} 31' N.$ What is the air-line distance between the two cities?

108. The latitude of New York City is $40^{\circ} 43' 43'' N.$, and the latitude of Lima, S. A.—on the same meridian—is $11^{\circ} 36' S.$ How many miles is it in a direct line from New York City to Lima?

109. One year an orchard produced 888 $\frac{1}{4}$ bushels of apples, and $\frac{2}{7}$ of them were marketable. A fruit dealer bought $\frac{1}{3}$ of those marketable at \$1.12 $\frac{1}{2}$ per standard barrel. How much did he pay for them?

110. For digging potatoes a farmer pays 25¢ for every 4 bushels, and for picking them up 5¢ for every 3 bushels. On these terms one man dug 36 bu. 3 pk., a second man dug $1\frac{1}{2}$ times as many, a third man as many as the first two, and a fourth man $\frac{1}{2}$ as many as the second and third. One boy picked up 59 $\frac{1}{2}$ bu., a second boy $\frac{1}{2}$ as many as the first, and a third boy the remainder. How many bushels of potatoes were dug?

111. At what cost for the digging? | **112.** For the picking up?

113. How much did each person receive for his labor?

114. The difference in the local time of two places is 2 h. 18 min. 35 sec. What is the difference in longitude?

115. If 7.5 meters of French velvet cost \$29 $\frac{1}{2}$, what is the cost of 3 pieces of 31.35 meters each?

116. A farmer exchanged 18 bushels of apples @ 31 $\frac{1}{4}$ ¢ per bu., for coffee @ 28¢ per lb., and sugar @ 15 lb. for \$1. He received 4 times as many pounds of sugar as of coffee. How many pounds of each did he receive?

117. When the planet Mercury has moved 1,000,576" in the heavens, has it described more or less than one revolution in its orbit?

118. An express agent collected charges at the U. S. Mint in Philadelphia for transporting 347 pounds, commercial weight, of gold from California. How many mint pounds of gold were transported?

119. The heirs to an estate of \$180,500 are a widow, two sons, and three daughters. Each son receives three times as much as each daughter, and the widow as much as all the children. What is the share of each heir?

120. The longitude of Eastport, Maine, is 67° W., and that of Chicago, Ill., is 87° 37' 45" W. When it is noon at Chicago what is the time at Eastport?

121. A man having spent \$25 more than $\frac{2}{3}$ of his money, had \$70 more than $\frac{1}{3}$ of it left. How much money had he at first?

122. Suppose the planet Saturn to be visible at Greenwich, longitude 0, Philadelphia 75° 10' W., and New Orleans 90° 2' 30", at the same instant, and the time of its appearance at Philadelphia to be 9 h. 42 min. P.M., Dec. 31, 1892. What is the time of its appearance at Greenwich?

123. What is the time of its appearance at New Orleans?

124. A grocer bought 3 bu. of chestnuts at \$4.50 per bu., and retailed them @ 5¢ per half-pt., liquid measure. How much were his profits?

125, 126. 50,000 silver dollars weigh how many pounds avoirdupois? How many pounds Troy?

127. How much will an apothecary gain in buying 5 pounds of rhubarb at \$3.25 per pound avoirdupois weight, and selling it at the rate of 30¢ per half ounce apothecaries' weight?

284. TEST QUESTIONS.

I. When one of any set of factors is 0, what is the product?

II. When the divisor is a concrete number, what kind of a number must the dividend be? Why?

How is the quotient affected

III. By adding 1 to the divisor? By subtracting 1 from the divisor?

Show the effect upon the value of the fractional unit,

IV. If the number of fractional units in the unit of the fraction is increased.

V. If the number of fractional units is decreased.

VI. Show that the quotient of one fraction divided by another having an equal numerator, equals the denominator of the divisor divided by the denominator of the dividend.

VII. PROPOSITION.—Prove that multiplication, in each of the four general classes of numbers, may be performed by addition.

VIII. PROPOSITION.—Prove that in division of decimal numbers, for every figure of the dividend used after the first quotient figure is obtained, there must be a figure in the quotient.

IX. PROPOSITION.—Prove that the equation

(1) $\frac{1}{2}$ of 5 = $\frac{5}{2}$ of 1, is mathematically correct; but that

(2) $\frac{1}{2}$ of 15 = $\frac{15}{2}$ of 1, is not mathematically correct.

X. PROPOSITION.—Prove that "cutting off" or "pointing off" figures from the right of an integer, and moving the decimal point to the left in a decimal, are, in effect, like processes in division.

How is the value of a fraction affected

XI. By cancelling any factor from either term?

XII. By multiplying either term by any factor?

XIII. By multiplying or dividing both terms by the same factor?

XIV. Show how many of the factors of a concrete number can be concrete.

How is the product affected

XV. By adding 1 to either factor? By subtracting 1 from either factor?

XVI. In multiplying by a decimal number expressed by more than one figure, what parts of the several partial products are not written; and why?

XVII. PROPOSITION.—Prove that the sum of two fractions the numerator of each of which is 1, equals the sum of their denominators divided by the product of their denominators.

XVIII. PROPOSITION.—Prove that the number of decimal places in the dividend less the number in the divisor equals the number in the quotient.

XXIX. PROPOSITION.—Prove that annexing ciphers to an integer and moving the decimal point to the right in a decimal, are, in effect, like processes in multiplication.

XXX. PROPOSITION.—Prove that multiplying a fraction by its denominator cancels the denominator.

What uses are made

XXXI. Of each of the six cases of reductions of fractions ?

XXXII. Of each of the two cases of converse reductions of decimals and fractions ?

How is the value of an improper fraction affected

XXXIII. By adding the same number to both terms ?

XXXIV. By subtracting the same number from both terms ?

XXXV. PROPOSITION.—Prove that division, in each of the four general classes of numbers, may be performed by subtraction.

XXXVI. PROPOSITION.—Prove that in dividing one of two dissimilar fractions by the other, the two fractions are actually reduced to similar fractions.

XXXVII. PROPOSITION.—Prove that the division in Proposition 26 consists in dividing the numerator of the equivalent dividend by the numerator of the equivalent divisor; and

XXXVIII. PROPOSITION.—Prove that the steps in the two preceding Propositions (26, 27) are accomplished by multiplying the dividend by the reciprocal of the divisor.

XXXIX. Show that the sum of the sum and the difference of two numbers is twice the greater number; and

XXX. That the difference between the sum and the difference of two numbers is twice the less number.

XXXI. What is the difference between any of the fundamental processes in decimal numbers and the corresponding processes in compound numbers ?

XXXII. In dividing by units of any order greater than ones, what is done with the figures cut off from the right of the dividend in the written work ?

XXXIII. Show that, in practice, either factor may be used as a multiplicand; but that when one factor is concrete, that factor is the *logical* multiplicand.

How is the value of a proper fraction affected

XXXIV. By adding the same number to both terms ?

XXXV. By subtracting the same number from both terms ?

XXXVI. Show that a fraction expresses either unexecuted division ; or, the quotient of an executed division.

XXXVII. PROPOSITION.—Prove that the difference between two fractions—the numerator of each of which is 1,—equals the difference between their denominators divided by the product of their denominators.

XXXVIII. PROPOSITION.—Prove that the value of a complex fraction equals the value of a simple fraction whose numerator equals the product of the extreme terms of the complex fraction (upper numerators and lower denominators), and whose denominator equals the product of the middle terms.

XXXIX. PROPOSITION.—Prove, without using fractions, that the number of decimal places in a product equals the number in the factors.

XL. PROPOSITION.—Prove that the product of each of two or more fractions multiplied by a multiple of their denominators, is an integer.

XLI. PROPOSITION.—Prove that writing the first figure of each partial product directly under the figure of the multiplier used, is writing units of like order in the same column.

Using the same multiplier in each instance,—

XLII. Show that the sum of the products of two or more numbers by any multiplier equals the product of their sum by the same multiplier ; and

XLIII. That the difference of the products of two numbers by any multiplier equals the product of their difference by the same multiplier.

XLIV. PROPOSITION.—Prove that if the dividend is a fraction and the division is exact, the divisor must also be a fraction.

XLV. PROPOSITION.—Prove that the product of two proper fractions is less than the quotient of either fraction divided by the other.

XLVI. PROPOSITION.—Prove that the sum of the partial dividends and the final remainder equals the given dividend.

XLVII. PROPOSITION.—Prove that one half the sum of two numbers plus one half their difference equals the greater number ; and

XLVIII. PROPOSITION.—Prove that one half the sum of two numbers minus one half their difference equals the less number.

What uses are made

XLIX. Of each of the six cases in the reductions of compound numbers ?

L. Of each of the reductions of decimals, and of the reductions of decimal currency ?

285. BLACKBOARD OUTLINES.

GENERAL TERMS. { 1. _____
2. _____
3. _____
4. _____

A. ADDITION. { **I. TERMS.** — 1. _____ 2. _____ 3. _____
II. BASIS. { 1. *Axiom.* 2. *Principle.*
3. *Important Facts.* { a. _____
b. _____
c. _____
III. RULES. { 1. _____
2. _____
IV. SHORT METHODS. { 1. _____
2. _____
3. _____

C. MULTIPLICATION. { **I. TERMS.** { 1. _____ 4. _____
2. _____ 5. _____
3. _____ 6. _____
7. _____
8. _____
II. BASIS. { 1. *Axiom.* { 1st. _____
2d. _____
3d. _____
4th. _____
2. *Principles.* { a. *General.* { 1st. _____
2d. _____
3d. _____
4th. _____
b. *Integers and Decimals.* { 1st. _____
2d. _____
3d. _____
4th. _____
5th. _____
c. *Fractions.* { 1st. _____
2d. _____
3. *Important Facts.* { 1. _____
2. _____
III. RULES. { 1. _____
2. _____
3. _____
IV. SHORT METHODS. { 1. _____
2. _____
3. _____
4. _____
5. _____
6. _____
7. _____

B. SUBTRACTION.

I. TERMS. { 1. _____ 2. _____ 3. _____
4. _____ 5. _____

II. BASIS. { 1. *Axiom.* 2. *Principle.*
3. *Important Facts.* { a. _____
b. _____
c. _____
d. _____

III. RULES. { 1. _____
2. _____

IV. SHORT METHODS. { 1. _____
2. _____
3. _____

D. DIVISION.

I. TERMS. { 1. _____ 5. _____ 9. _____
2. _____ 6. _____ 10. _____
3. _____ 7. _____ 11. _____
4. _____ 8. _____ 12. _____

II. BASIS. { 1. *Axiom.* { 1st. _____
2d. _____
3d. _____
4th. _____
5th. _____
6th. _____
7th. _____
2. *Principles.* { a. *General.* { 1st. _____
2d. _____
3d. _____
4th. _____
5th. _____
6th. _____
7th. _____
b. *Integers and Decimals.* { 1st. _____
2d. _____
3d. _____
c. *Fractions.* { 1st. _____
2d. _____
3. *Important Fact.* _____

III. RULES. { 1. _____
2. _____
3. _____

IV. SHORT METHODS. { 1. _____
2. _____
3. _____
4. _____
5. _____
6. _____

CHAPTER V.
COMMERCIAL CALCULATIONS,—
NOT INVOLVING MEASUREMENTS OR PERCENTAGE.

SECTION I.
ARITHMETICAL EQUATIONS.

The use of arithmetical equations greatly facilitates computations in business affairs. Hence, pupils should master their transformations and reductions.

286. Define an arithmetical equation (39).

An arithmetical equation consists of two members; each member consists of one or more terms; and each term may consist of factors, of equal parts, or of both factors and equal parts.

287. Any computation indicated by symbols of operation may be considered as one member of an equation, and the result of the computations as the other member.

a. The *members* of an equation are separated by the sign = ; and
The *terms* of the members are separated by the signs + or -.

b. The *factors* of a *term* are connected by the sign × ; and
The *equal parts* of a *term* are connected by the sign ÷.

i. e. $\left\{ \begin{array}{l} = \text{ shows divisions of equations into members.} \\ + \text{ and } - \text{ show divisions of members into terms.} \\ \times \text{ and } \div \text{ show divisions of terms into factors and equal parts.} \end{array} \right.$

The equation, $8 \div 4 + 6 \times 7 \div 3 - 28 \div 7 \times 3 + 6 = 10$
is read "8 ÷ 4, plus 6 × 7 ÷ 3, minus 28 ÷ 7 × 3, plus 6 = 10."

The terms of this equation and their values are as follows:—

Terms: $8 \div 4 + 6 \times 7 \div 3 - 28 \div 7 \times 3 + 6 = 10$

Values: $\underbrace{2} + \underbrace{14} - \underbrace{12} + 6 = 10$

288. To find the value of a numerical expression consisting of two or more terms,—

RULE.—I. Compute the value of each term separately.

II. Unite the values of the terms as indicated by the signs that precede them.

PROBLEMS.

Separate into terms and prove equality of members in

1. $7 \times 5 \times 8 - 64 \div 16 \times 40 - 12 + 17 \times 3 = 159$
2. $20 + 18 \times 3 \div 6 \times 11 - 150 \div 2 + 11 = 4 \times 3 \times 5 - 15 \div 3$
3. $17 \times 2 + 8 \div 4 \times 6 \div 8 \times 20 = 2 \times 2 \times 2 \times 2 \times 2 + 2 \times 2 \times 2 \times 2 \times 2$
4. $117 - 84 \div 7 - 36 \times 4 \div 9 - (6 \times 3 + 2) \times 4 = 9$
5. $3 \times 7 \times 12 \div 4 \times 13 \times 5 \div 9 = 225 \times 4 \div 45 \times 52 \times 21 \div 48$

Find the second member in

6. $(72 - 46) + (24 - 8) \times 6 \times (14 + 6) \div 10 - 45 \times 4 + (12 \times 3 - 8) \times 2 = (?)$
7. $8 + 6 \times 5 - 12 \div 3 + 8 \times 14 + 16 - 14 \div 7 \times 6 + 8 = (?)$
8. $(8 + 6) \times 5 - 12 \div (3 + 8) \times (14 + 16) - 14 \div 7 \times (6 + 8) = (?)$
9. $8 + (6 \times 5 - 12) \div (3 + 8 \times 14) + (16 - 14) \div (7 \times 6 + 8) = (?)$
10. $(8 + 6) \times (5 - 12 \div 3) + 8 \times 14 + (16 - 14) \div (7 \times 6) + 8 = (?)$
11. $\overline{8 + 6 \times 5 - 12 \div 3 + 8 \times 14 + 16} - \overline{14 \div 7 \times 6 + 8} = (?)$
12. $(8 + 6) \times 5 + [14 \div (7 \times 6 + 8) - 12 \div (3 + 8 \times 14 + 16)] = (?)$

Supply the missing number in

13. $46 \div 2 - 18 \times 4 \div 6 + 72 \times 5 \div 8 + (?) = 80$
14. $(36 - 27) \times (18 \div 3 + 4) + (?) + 21 \div 7 \times (6 \times 5 - 9) = 200$
15. $27 + (9 \times 4 + 7) - (52 - 41) \times (16 + 8) \div (?) = 20$
16. $37 \times 2 \times (?) - (84 + 37 \times 3) = 16 \times (3 \times 5 - 2) + 100$

Show that

17. $146 - 12 - 15 - 84 = 146 - (12 + 15 + 84)$
18. $324 - 67 \times 2 + 84 \div 7 - 8 \times 6 \times 2 = 324 - (67 \times 2 - 84 \div 7 + 8 \times 6 \times 2)$
19. $845 - (32 \times 5 + 84 \times 7 - 38 \times 5 \div 2 + 16 \times 7) =$
 $845 - 32 \times 5 - 84 \times 7 + 38 \times 5 \div 2 - 16 \times 7$

Show by examples	{	1. That the signs of factors are not affected by introducing or rejecting signs of aggregation.
		2. That the signs of terms are not affected by introducing or rejecting signs of aggregation preceded by +.
		3. That the signs of terms are reversed by introducing or rejecting signs of aggregation preceded by -.

SECTION II.

CURRENCY, and ACCOUNTS AND BILLS.

I. CURRENCY.

289. Money is the universal equivalent of commercial values.

290. Currency is money or its equivalent, commonly used in buying and selling.

291. Coin is metal stamped with certain characters, by authority of government, to give it a fixed legal value.

292. Bullion is gold or silver in the bar or lump.

a. The **circulating medium** of a country is the money of that country.

b. **Paper money** is a substitute for coin. It has no intrinsic value.

The paper money of the United States consists of—

1. **Treasury Notes** issued by the government, and known as *greenbacks*;

2. **Certificates of Deposit** issued by the government, and known as *silver certificates* and *gold certificates*; and

3. **Bank-notes** issued by national banks.

c. **Fine gold** or **fine silver** is pure metal. Fine gold and silver being soft, rapidly diminish by abrasion. They are consequently compounded with a baser metal, or *alloy*, that hardens the compound and renders it more suitable for coin.

293. The coins of the United States are of gold, silver, nickel, and bronze.

1. The **gold coins** are the twenty-dollar, ten-dollar, five-dollar, and two-and-one-half-dollar pieces.

2. The **silver coins** are the dollar, half-dollar, quarter-dollar, and dime.

3. The **nickel coin** is the five-cent piece.

4. The **bronze coin** is the cent.

a. Coins in circulation, but not now coined :—

Gold.—Three-dollar, dollar.

Silver.—Twenty-cent, half-dime, three-cent.

Nickel.—Three-cent, one-cent.

Bronze.—Two-cent.

b. The gold and silver coins are nine tenths fine and one tenth alloy; and the nickel coins are one fourth nickel and three fourths copper.

- c. The alloy of gold coin is composed of equal parts of silver and copper. The alloy of silver coin is pure copper.
- d. The *intrinsic value* of a coin is the value of the finer metal. The alloy in coin is not considered of any value.
- e. The weight of United States gold coins is 25.8 grains to the dollar; of the silver dollar, 412½ grains; of subsidiary silver coin (fractions of a dollar) 835.8 grains to the dollar; and of the nickel five-cent piece, 77.16 grains.
- f. The standard fineness of French coins is the same as of the coins of the United States.

294. Legal tender is money which, if offered in payment, is a legal satisfaction of a debt.

PROBLEMS.

What is the weight of fine silver

- 1. In 36 half-dollars? | 2. In 18 silver dollars?
- 3. What is the weight of \$100 in gold coin and 75 silver dollars?
- 4. What is the weight of the fine silver in 100 dollars of gold coin and 75 silver dollars?
- 5. Which of the United States gold coins contains 116.1 gr. of fine gold?

II. ACCOUNTS AND BILLS.

295. Merchandise is personal property of any kind, bought or sold in the ordinary course of trade.

296. An account current is a formal written statement of debts and credits between two parties.

297. An item is a single debt or credit of an account. When it relates to merchandise, it includes quantity, price, and cost.

298. A statement is a transcript of that part of an account from, and including the balance of, the last preceding statement.

299. An abstract of account is a summary or copy, that gives total amounts of separate transactions without the items.

300. A bill of goods is a written statement given by the seller to the buyer. It should contain the name of the place and the date of the purchase, the names of buyer and seller, a list of items with their prices, and the total cost.

301. An *invoice* is a bill of merchandise sold at wholesale or consigned to an agent.

a. A *freight bill*, *shipper's bill*, or *bill of lading*, is an itemized list of freight delivered to a common carrier for transportation.

(See Supplement, page 349.)

b. A *duplicate bill* or *invoice* is a copy of an original bill; it should always be marked *duplicate* on its face.

c. To receipt a bill the seller or creditor writes at the bottom of the bill,—*Received payment*, and signs his name.

302. SIGNS AND ABBREVIATIONS

Commonly used in accounts, bills, and business correspondence.

<i>Ac</i> or <i>acct.</i> , account.	<i>Cr.</i> , credit or creditor.	<i>Mds.</i> , merchandise.
<i>amt.</i> , amount.	<i>Dr.</i> , debit or debtor.	<i>pc.</i> , piece or pieces.
<i>bal.</i> , balance.	<i>F. O. B.</i> , free on board.	<i>pay't</i> , payment.
<i>Bo't</i> , bought.	<i>int.</i> , interest.	<i>Pd.</i> , paid.
<i>Co.</i> , company.	<i>inst.</i> , this month.	<i>recd.</i> or <i>rec'd</i> , received.
<i>Contra</i> , opposite.	<i>prox.</i> , the coming month.	
<i>C. O. D.</i> , collect on delivery.	<i>ult.</i> , the past month.	

*, number, when placed before a number; pounds, when placed after a number.
e. g. * 19 is No. 19; and 19 * is 19 lb.

The abbreviations of denominations of compound numbers.

PROBLEMS.

BILLS OF PURCHASE.

Extend all items, and make the footings in these bills.

6. Bill, charged in account.

Mr. GEO. SHIPPEN

PHILADA., PA., Mar. 9, 1885.

Bought of JOHN WANAMAKER.

Terms,—On *Ac* at 20 da.

Cases	pc.								
* 39	5	American prints, 31 ¹ , 33 ² , 35 ³ , 30 ² , 40 ² , @ 6¢,							
* 28	2	Cassimere, 38 ² , 47 ³ ,	@	\$1 ¹⁰ ,					
* 44	3	N. Y. Mills muslin, 30 ¹ , 43 ² , 48 ² ,	@	11¢,					
	4	Alpaca, 35 ² , 35 ³ , 43 ³ , 38,	@	70¢,					
	3	Cotton flannel, 45 ² , 50, 54 ¹ ,	@	20¢,					
	2	American black silk, 47 ³ , 54 ² ,	@	\$1 ²⁵ ,					

The small figures at the right indicate fourths. e. g. 81¹ is 81 $\frac{1}{4}$ yda.

303. The only parties who may legally receipt a bill are the party who sells the goods or some person duly authorized to receipt for him.

For forms of receipting bills, see bills numbered 7, 8, and 9.

7. Bill, receipted.

NEW YORK, Mar. 9, 1885.

Messrs. JONES & THOMPSON, Scranton, Pa.,

Bought of N. Y. HARDWARE Co.

Terms,—Cash.

10	doz. adz-eye nail hammers,	1, \$11.50, 2, \$8.75, 3, \$7.25,				
6	" trimmers'	" 1, \$8, 2, \$7, 3, \$6,				
9	" bricklayers'	" 1, \$12, 2, \$11, 3, \$10,				
10	" coopers'	" 3, \$11.50, 4, \$10.50, 5, \$9.50,				429.75

Recd. payment,

N. Y. HARDWARE Co.

NOTE.—1, \$11.50, 2, \$8.75, 3, \$7.25 means 3 doz. No. 1 @ \$11.50 per doz., 4 doz. No. 2 @ \$8.75, etc.

8. Bill, receipted by cashier.

CHICAGO, ILL., May 11, 1887.

A. C. PETERSON & Co.

Bo't of RICHMOND, BRUCE & Co.

Case	Pieces		No. yd.	Price	Amount
* 86	10.	Lowell sheetings—			
		34 ^s 33 29 ^s 30 ⁱ 35 ^s			
		28 ⁱ 36 ^s 32 ⁱ 29 ^s 31 ^s		6½¢	
* 24	13	Fancy shirtings—			
		27 ^s 29 ⁱ 28 ⁱ 32 ^s 28 ^s			
		31 ⁱ 28 ^s 30 ^s 31 31 ^s			
		29 ^s 32 ⁱ 28		8¼¢	

Recd payment by note at 90 days.

RICHMOND, BRUCE & Co.,

per A. L., Cashier.

ACCOUNTS AND BILLS.

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9. Bill for Services, receipted by clerk.

Mr. JOHN R. CORNELL DENVER, COL., May 21, 1887.
To WM. H. JENKINS, Dr.

	To 5 days' labor hauling stone and sand, @ \$2,	10	—	
	" 3½ " use of team in hauling, @ \$3,	10	50	20 50
Recd. payment, May 28, 1887.				

WM. H. JENKINS,
per S. POOLE.

10. Statement of Account.

Mr. HENRY PHILLIPS ST. LOUIS, MO., Feb. 21, 1887.
In a/c with JOHNSON & HARDING, Dr.

1887						
Jan.	20	To mdse., as per bill rendered,	1,975	—		
Feb.	13	" " " " " "	1,550	—		
"	19	" " " " " "	380	—	3,905	—
— Cr. —						
1887						
Jan.	20	By note @ 30 da.,	1,000	—		
"	22	" cash,	800	—		
Feb.	19	" "	300	—		
"	20	" note @ 60 da.,	1,000	—	3,100	—
Balance due,					805	—

11. Account Current.

J. B. PETERSON NEW ORLEANS, Dec. 1, 1887.
In acc't with C. S. HOLTON & Co.

1887			Dr.	Cr.
Nov. 1	To bal. due on acc't rendered Oct. 31, /87,		317	
" 5	" 10% sugar @ 7½¢; ½% tea @ 80¢; 2% coffee @ 30¢,		175	
" 15	By cash,			4
" 18	To ½ bar. flour, \$2.15; 2% rice @ 9¢; 1 pk. potatoes, 25¢,		318	
" 25	" 5% butter @ 26¢; 12½% ham @ 16¢,		326	
" 30	By cash, \$3.20; order on F. C. Case to balance, \$3.20,			736

Put the following narratives into the form of bills:

12. BALTIMORE, MD., Aug. 29, 1886. Mrs. E. C. Harris bought of R. C. Snow & Co.,—3 cans tomatoes @ 16¢; $\frac{1}{2}$ bu. of potatoes @ 80¢; 3 doz. eggs @ 20¢; 1 doz. lemons 45¢; 2 lb. crackers @ 10¢; 1 lb. coffee 36¢; $1\frac{1}{4}$ lb. dried beef @ 25¢; $3\frac{1}{2}$ lb. butter @ 26¢; 1 can cloves 15¢. Paid in full to George Ellis, Snow & Co.'s delivery clerk.

13. Oct. 11, 1887. Mr. Elliot Shepard of Kansas City, Mo., bo't of Minor and Hamilton of New York City,—8 pr. ladies' French kid shoes @ \$28¹⁵ per doz.; $1\frac{3}{4}$ doz. gent's Oxford ties @ \$17⁵⁰; $2\frac{3}{4}$ doz. children's pebble-goat spring-heel Cong. shoes @ \$12; 7 pr. ladies' opera slippers @ \$24 per doz. Paid by draft at 10 da.

14. ALBANY, N. Y., June 8, 1886. Henry Jones bought of Whitney & Co.,—10 doz. pr. cotton hose @ \$1⁵⁰; 5 doz. lace neckties @ \$5; 4 doz. lace pouches @ \$3; 2 doz. leather belts @ \$3⁵⁰; 1 pc. Irish linen, 40 yd., @ 50¢; 2 pc. bleached linen damask, 81 yd., @ 45¢; 1 pc. jaconet, 39 yd., @ 25¢; 3 pc. cottonade, 93 yd., @ 30¢; and 2 pc. brown duck, 65 yd., @ 23¢.

15. OMAHA, NEB., Jan. 17, 1888. William Smith & Co. bought of Henry Bates & Bro.,—4 bar. Minn. flour @ \$7; 10 bar. spring-wheat flour @ \$6⁵⁰; 20 bar. superior extra flour @ \$7⁵⁰; 24 bar. superfine flour @ \$5²⁵; 12 pkg., 1,325*, @ 4 $\frac{1}{2}$ ¢; and 7.38 tons of ground feed @ \$12⁰⁰.

16. BOSTON, MASS., April 11, 1887. H. L. Gleason bought of Jordan, Marsh & Co.,—16 pc. fancy prints 34², 29¹, 42, 38², 35², 30², 40¹, 37², 33¹, 39, 31², 41¹, 32², 36², 35¹, 34, @ 4 $\frac{1}{2}$ ¢; 12 pc. seersucker 46², 44², 45, 41¹, 38², 48¹, 40², 47², 39², 43¹, 44², 42¹, @ 11 $\frac{1}{2}$ ¢; and 8 pc. sheeting 26², 28¹, 33², 31¹, 30, 27², 33¹, 29², @ 14¢. H. L. Gleason gave his note @ 60 da. in payment.

17. SAN FRANCISCO, CAL., June 10, 1887. Harvey L. King bought of Henry L. Dodge & Co.,—3 bar. of sugar 283—17=266, 254—16 $\frac{1}{2}$ =237 $\frac{1}{2}$, 271—18=253, @ 7 $\frac{1}{2}$ ¢; 4 sacks of coffee, net 84, 80¹, 82², 87, @ 13 $\frac{1}{2}$ ¢; and 3 chests Y. H. tea, net 38, 40², 35², @ 39 $\frac{1}{2}$ ¢.

18. Make a duplicate of bill,—problem 13.

SECTION III.

AVERAGE.

304. A *mean* or *medial number* is a number that expresses the value of an equal part of the sum of two or more unequal numbers.

305. *Average* is the process of finding a mean or medial number.

a. Average is used to find the mean value, quantity, or quality of a unit of a mixture or compound composed of ingredients or elements of different unit values, quantities, or qualities.

b. Average is also called *alligation medial* or *medial alligation*.

306. The process of average is based upon this

AXIOM.—*A whole, formed by uniting unequal parts, may be divided into equal parts.*

Ex. 1. What is the average weight of 5 bales of cotton, weighing, respectively, 378 lb., 456 lb., 512 lb., 482 lb., and 390 lb.?

Process.

378 lb.
456
512
482
390

5) 2,218 lb.

Average weight, 443.6 lb.

Ex. 2. Find the average length of the pieces, and the average price per yard, of 3 pc. of Eng. broadcloth, $43\frac{1}{2}$ yd. each, @ \$2.50 $\text{\textyen}$ yd.; 2 pc. of Fr. broadcloth, 44.5 yd. and 39.25 yd., @ \$3.16 $\text{\textyen}$ yd.; and 1 pc. of Amer. broadcloth, 35 yd., @ \$2.13 $\frac{1}{2}$ $\text{\textyen}$ yd.

FULL SOLUTION.

3 pc., $43\frac{1}{2}$ yd. ea.	131 $\frac{1}{2}$ yd. @ \$2.50	\$328.12 $\frac{1}{2}$
2 " { 44.5 } yd.	83 $\frac{1}{2}$ " " 3.16	264.65
1 " { 39.25 } yd.	35 " " 2.13 $\frac{1}{2}$	74.72 $\frac{1}{2}$
<u>6 pc.</u>	<u>250 yd.</u>	<u>\$667.50</u>

$250 \text{ yd.} \div 6 = 41\frac{2}{3} \text{ yd.}$, average length;

$\$667.50 \div 250 = \$2.67$, average price per yd.

State the process for finding $\left\{ \begin{array}{l} 1. \text{ The average of two like numbers.} \\ 2. \text{ The average of three like numbers.} \\ 3. \text{ The average of any number of like numbers.} \end{array} \right.$

Ex. 3. If 336 lb. of Maracaibo coffee @ $10\frac{1}{4}\phi$, 288 lb. of Rio @ $11\frac{1}{8}\phi$, 306 lb. of Java @ $18\frac{1}{4}\phi$, and 7 bu. of Canada peas @ $\$1.35$ ∇ bu. of 60 lb. are ground and mixed, at what price per lb. must the "pure ground coffee" be sold to gain $\frac{1}{4}$ as much as the ingredients cost?

FULL SOLUTION.

336 lb. @ $10\frac{1}{4}\phi$,	$\$34.02$
288 $11\frac{1}{8}\phi$,	32.22
306 $18\frac{1}{4}\phi$,	56.61
420 $2\frac{1}{2}\phi$,	9.45
<u>1,350 lb.</u>	<u>$\\$132.30$</u>
Add profits (= $\frac{1}{4}$ cost),	$33.07\frac{1}{2}$
1,350 lb. sell for	$\$165.37\frac{1}{2}$
$\$165.37\frac{1}{2} \div 1,350 = 12\frac{1}{4}\phi \nabla$ lb.	

4. Point out the steps found in the solution of Ex. 2 not found in the solution of Ex. 1.

5. State the method for finding the average or mean value, quantity, or quality of two or more ingredients, when the value, quantity, or quality of each is given.

Find the average of

PROBLEMS.

1	2	3	7	8
985	$1\frac{1}{2}$ and $\frac{2}{3}$	$113\frac{7}{8}$	472.45 gal.	13 mi. 62 rd. 8 ft. 4 in.
467		$68\frac{4}{5}$	87.075	72 115 15
4	5	6	309.5	319 10 6
46.328	326 da.	2,976 lb.	8.231	7 208 11
354.9	219	8,054	4,000	168 13 8
8.76	93	818	632.35	319 16 3

Find the average price per unit

9. Of oats @ 35 ϕ , 42 ϕ , 28 ϕ , 36 ϕ , 54 ϕ , and $37\frac{1}{4}\phi \nabla$ bu.
10. Of apples @ $\$2.10$, $\$1.75$, $\$2.25$, $\$1.62\frac{1}{2}$, and $\$3.15 \nabla$ bar.
11. Of sugars @ $7\frac{1}{2}\phi$, $6\frac{1}{4}\phi$, 5 ϕ , $8\frac{1}{8}\phi$, $5\frac{1}{2}\phi$, $6\frac{3}{4}\phi$, $9\frac{1}{16}\phi$, and $7\frac{1}{2}\phi \nabla$ lb.
12. Of teas @ $\$1.95$, 72 ϕ , 80 ϕ , 65 ϕ , 50 ϕ , 42 ϕ , and $87\frac{1}{4}\phi \nabla$ lb.
13. Of flour @ $\$4.65$, $\$7.12\frac{1}{2}$, $\$9.15$, $\$6.50$, $\$5.75$, and $\$8.25 \nabla$ bar.
14. Of horses @ $\$231.25$, $\$162.50$, $\$475$, and $\$337.50$ per head.
15. Of land @ $\$62\frac{1}{2}$, $\$77\frac{1}{2}$, $\$100$, $\$115\frac{1}{4}$, $\$180$, and $\$15$ per acre.

16. The register of a certain graded school shows the attendance for one week to be as follows: Monday 685, Tuesday 710, Wednesday 756, Thursday 772, Friday 730. What is the average daily attendance?

17. The contents and valuation of the several parcels of land in a certain farm are here given. What is the average value per acre of the whole farm?

35	A. meadow	@ \$	72
62	" pasture	"	48
70½	" grain land	"	106½
45	" swamp	"	33½
52½	" woodland	"	120

Abstract of my bank account for one week.

<i>Days</i>	<i>Deposited</i>	<i>Drew out</i>	<i>Balances</i>
Mon.	\$729.50	\$415.25	
Tues.	486.25	234.81	
Wednes.	632.70	300.—	
Thurs.	756.—		
Fri.	567.19	362.50	
Sat.	932.36	511.72	

What is the daily average

18. Of deposits? | **19.** Of sums drawn out? | **20.** Of balances?

Memorandum of seven lots of butter bought by Jones & Co., Oct. 28, 1887.

21. What was the average weight of the tubs?	12 tubs	56 lb. ea.	@ 31½¢
	7	51	28
22. What was the average price per tub?	15	53½	30
	8	48½	33½
23. What was the average price per pound?	10	54½	29½
	21	50	27½
	11	52½	26½

Find the averages of rain-fall in the time named,

<i>Months</i>	<i>Rainy Days</i>	<i>Depth of Rain-fall</i>
Mar.	13	8.03 in.
Apr.	22	11.4
May	19	9.65

24. Daily, for the days in which rain fell.

25. Daily, for all the days in the three months. | **26.** Monthly.

27. A grocer mixed 71 gal. of N. O. molasses @ 65¢, 96 gal. of Porto Rico molasses @ 81½¢, and a barrel of water; he sold the mixture at ½ more per gal. than it cost him. At what price per gal. did he sell it?

28. A wholesale dealer in clothing filled an order for 36 light-weight overcoats @ \$7.25, 30 coarse overcoats @ \$4.90, 52 fine overcoats @ \$13.50, and 39 heavy ulsters @ \$9.37½. What was the average price of the coats?

Memoranda of temperature for the cold week in January, 18—.

Days	A. M.			P. M.		
	2:00	6:00	10:00	2:00	6:00	10:00
Sun.	+ 8°	+12.5°	+16°	+22.25°	+10°	0
Mon.	-10.5+	-16	-12.25	- 4	+ 2	- 5
Tues.	-14	-22.25	-18	- 7.5	-16	-24
Wed.	-27.5	-34	-26.5	-20	-12	-10
Thurs.	-10	- 8.5	- 4	0	+ 4	+ 4
Fri.	+ 6	+10	+15	+18	+16	+14.5
Sat.	+16	+22	+30	+36	+32.5	+40

* +, above zero; —, below zero.

29-35. What was the average for each day of the week?

36-41. What was the average for each hour of observation?

42. What was the average for the week?

The workmen in an edge-tool factory are 3 forgers @ \$3 $\frac{1}{2}$ day, 5 helpers @ \$1.25, 3 welders @ \$2.25, 2 formers @ \$2.50, 4 grinders @ \$2.75, 2 polishers @ \$2, and 2 packers and shippers @ \$1.50.

43. How many workmen are employed?

44. What are their average daily wages?

The meats used in six months in a Saratoga hotel were as follows:

	Beef	Pork	Veal	Mutton	Lamb	Fish	Poultry
May	1,050 lb.	675 lb.	512 $\frac{1}{2}$ lb.	465 lb.	381 $\frac{1}{2}$ lb.	300 lb.	293 $\frac{1}{2}$ lb.
June	1,105	612 $\frac{1}{2}$	632	510	442 $\frac{1}{2}$	378	350
July	967 $\frac{1}{2}$	328	597	483 $\frac{1}{2}$	456	400	437 $\frac{1}{2}$
Aug.	813	271	517 $\frac{1}{2}$	526 $\frac{1}{2}$	482	407	500
Sept.	702 $\frac{1}{2}$	310	416 $\frac{1}{2}$	501	448 $\frac{1}{2}$	368	488 $\frac{1}{2}$
Oct.	889	419	332 $\frac{1}{2}$	359	391	297 $\frac{1}{2}$	341

What was the average number of pounds of meat used

45-50. Of all the kinds for each month?

51-57. Of each kind per month?

58. Of all the kinds per month for the six months?

59. A board is 2 ft. 8 in. wide at one end and 1 ft. 1 in. at the other. What is its average width?

60. The heights of a railroad embankment at seven equi-distant points are 9 ft. 4 in., 21 ft. 10 in., 38 ft. 6 in., 54 ft. 1 $\frac{1}{2}$ in., 47 ft., 22 ft. 4 in., and 12 ft. 3 in. What is its average height?

The pay-roll of a railroad contractor for one week showed 5 hands employed $9\frac{1}{2}$ h. per day for 5 days @ \$1.15 p day, 7 hands 11 h. per day for 4 days @ \$1.25, 12 hands $8\frac{1}{2}$ h. per day for 6 days @ \$1.40, and 32 hands $10\frac{1}{2}$ h. per day for 7 days @ \$.93 $\frac{1}{2}$.

Allowing 10 hours for a day's work,

Find the average $\left\{ \begin{array}{l} 61-65. \text{ Of days worked per man for the week.} \\ 66. \text{ Of time actually worked per day per man.} \\ 67. \text{ Of daily wages.} \end{array} \right.$

68. What is the cost per bushel of a quantity of ground feed the ingredients of which are 25 bu. of oats @ 35¢, 15 bu. of barley @ 62 $\frac{1}{2}$ ¢, and 31 bu. of corn @ 75¢?

A miller sold 5 lots of flour as follows:—436 bar. @ \$5.25, 650 bar. @ \$4.87 $\frac{1}{2}$, 175 bar. @ \$5.50, 390 bar. @ \$6.37 $\frac{1}{2}$, and 280 bar. @ \$5.75.

Average $\left\{ \begin{array}{l} 69. \text{ The number of barrels per lot.} \\ 70. \text{ The price per lot.} \\ 71. \text{ The price per barrel.} \end{array} \right.$

72. A surveyor finds that at one station or corner of a survey the needle is influenced by local attraction, and different trials of the bearing of the same course give readings as follows:—S. 34° 56' E., S. 35° 1' E., S. 34° 58' E., and S. 35° E. What is the average reading of the compass?

73. An exploring party took observations to determine their latitude, with the following results:—North,—47° 22' 17", 47° 22' 20", 47° 22' 22", 47° 22' 19", and 47° 22' 24". The conditions of the third observation were twice as favorable as the others. What should be recorded as the true latitude?

74. A sea-captain has three chronometers, each set at Greenwich time when he left port. After being some weeks at sea he finds that they vary slightly. How will he determine the time?

75. Write a problem to illustrate problem 74.

SECTION IV.

PRICE, QUANTITY, AND COST.

307. In business transactions involving the money value of property or labor, three elements,—*price, quantity, and cost*,—are considered.

When two of these elements are known, the unknown element may be found by computation.

308. A *commercial unit* is either a fixed number, or a definite quantity considered in connection with money value.

309. *Price* is the value of a commercial unit.

310. *Quantity* is the number of units or parts of a commercial unit of a commodity.

311. *Cost* is the value of a quantity.

a. 1 horse, 1 dozen, 1 hundred, 1 pound, 1 ton, 1 yard, are commercial units.

b. Commercial units are also called *units of price*.

c. When quantity is expressed in other than commercial units, it must be reduced to commercial units before making the computation.

312. Demonstrate the truth of the following

$$\text{FORMULÆ. } \left\{ \begin{array}{l} \text{I. } \textit{Price} \times \textit{quantity} = \textit{cost}. \\ \text{II. } \textit{Cost} \div \textit{quantity} = \textit{price}. \\ \text{III. } \textit{Cost} \div \textit{price} = \textit{quantity}. \end{array} \right.$$

What process is employed in computing

1. Cost? | 2. Price? | 3. Quantity?

4. From 244, II, 1, and 267, II, 2, show that, in computing price or cost, quantity must always be considered an abstract number.

Show that quantity must be expressed in commercial units

5. In Formula I, before multiplying;

6. In Formula II, before dividing;

7. In Formula III, after dividing.

Name five commodities the commercial unit of each of which is

8. A hundred. | 9. A thousand. | 10. A ton. | 11. A dozen. | 12. A gross.

PROBLEMS.

State (1) what elements are given, (2) what element is required, and (3) how it is found, with reasons for the process, in each of the 3 problems following:—

1. Find the cost of 215.31 tons of steel at \$162.50 per ton.
2. Find the price of silk hats that cost \$51.75 per case of 12 hats.
3. Find the number of lap-robcs that can be bought for \$1,032 at \$21.50 each.

Reduce the following quantities to units of price:—

4. 8,275 feet of lumber, sold by the C.
5. 21,462 square feet of roofing, put on by the square.
6. 6 barrels of eggs, containing 1,050 eggs each, bought by the dozen.
7. 3,854,125 bricks, made by the M.
8. 16,385 pounds of hay, bought by the ton.
9. 31,728 lead-pencils, sold by the gross.
10. 100,000 clothes-pins, sold by the great gross.

313. PROPOSITION.—*Prove, using illustrative examples, that price, quantity, and cost, correspond respectively to multiplicand, multiplier, and product.*

314. Deduce brief rules or formulæ for finding cost, price, and quantity

- I. *When the unit of price is 100.*
- II. *When the unit of price is 1,000.*
- III. *When the unit of price is the net ton.*

PROBLEMS.

Find the cost of

- | | |
|---------------------------------|-------------------------------------|
| 11. 38½ yd. @ \$2.15. | 17. 13,964 hop poles @ \$9½ ¢ M. |
| 12. ¼ lb. @ \$.81½. | 18. 78,215 bricks @ \$7.50 ¢ M. |
| 13. 3 qt. 1 pt. @ \$2½ ¢ gal. | 19. 954 lb. of coal @ \$5.25 ¢ T. |
| 14. 5 bu. 3 pk. @ \$5.56½ ¢ bu. | 20. 16,720 lb. of iron @ \$67½ ¢ T. |
| 15. 4,968 lb. @ \$8.50 ¢ cwt. | 21. 3,060 pens @ 62½ ¢ gross. |
| 16. 678 pickets @ \$1.75 ¢ C. | 22. 825 pails @ \$1.80 ¢ doz. |

The cost of	{	23. 95 bar. of sugar is \$1,095.35 ;	{	what is the price	per bar. ?
		24. 743 young cattle is \$13,931 $\frac{1}{4}$;			per head ?
		25. 2,146 drain tiles is \$64.38 ;			per C ?
		26. 1,675 lb. of ground feed is \$14.07 ;			per C ?
		27. 12,382 roofing slates is \$278.59 $\frac{1}{4}$;			per M ?
		28. 3,640 grape roots is \$67.34 ;			per M ?
		29. 13,460 lb. of phosphate is \$222.09 ;			per T. ?
		30. 5,278 lb. of hay is \$55.42 ;			per T. ?

Find the quantity, in units of price

31. Of wheat	{	that can be bought for	\$6,108 at 93 $\frac{1}{4}$ ¢ per bu.
32. Of molasses			\$144 at 64 $\frac{1}{16}$ ¢ per gal.
33. Of fruit trees			\$125 $\frac{1}{2}$ at \$16 $\frac{1}{2}$ per C.
34. Of dressed beef			\$607.25 at \$8.75 per C.
35. Of lumber			\$214.90 at \$25 per M.
36. Of shingles			\$117.90 at \$3.60 per M.
37. Of ice			\$258.75 at \$3.75 per T.
38. Of land plaster			\$72 $\frac{3}{8}$ at \$4 $\frac{1}{2}$ per T.

39. When the price of wheat in Chicago is \$1.12 $\frac{1}{2}$ per bu., what is the cost in Chicago of 23 car-loads of wheat of 650 bushels each ?

40. What is the cost of the same wheat delivered in New York, freight 22 $\frac{1}{2}$ ¢ per cental ?

41. Last year a merchant's gas bill was \$171.36, at \$2.25 per 1,000 feet. For how much gas did he pay ?

42. A Troy shirt manufacturer received \$400 for linen shirts at \$16 $\frac{1}{2}$ per dozen. How many dozen shirts did he sell ?

43. A hosiery manufacturer paid \$430.25 for Saxony yarn at 82¢ per pound. How many pounds did he buy ?

44. A merchant paid \$708.75 for 135 dozen pairs of ladies' gloves. What was the average price per pair ?

45. The bricks for my house cost me \$288.75 @ \$8.33 $\frac{1}{3}$ per M. How many bricks did I buy ?

46. A cooper paid \$3,226.52 for 117,328 barrel staves. What was the price per M?

47. Find the cost of 375 feet of oak plank at \$15.75 per M.

48. Find the cost of $24\frac{3}{4}$ gross of lead-pencils at $31\frac{1}{4}$ ¢ per doz.

49. A wood-turner receives $18\frac{3}{4}$ ¢ per hundred for turning chair legs, and he turns 120 per hour. How much can he earn in a week, if he works 10 hours per day?

50. A hotel-keeper's meat bill for a month was \$88.55 for 770 lb. of meat. What was the average price per cwt.?

51. At 85¢ per ton, what is the freight on 4,680 lb. of household goods?

52. A printer received \$4.42 $\frac{1}{2}$ for a lot of quarter-sheet hand-bills, at \$1.50 for the first hundred, and 45¢ per hundred for the remainder of the lot. How many hand-bills did he print?

53. The mailing clerk in the office of a newspaper receives \$10.95 for mailing 32,850 papers weekly. How much is that per hundred papers?

54. \$34,320 for 132,000 railroad ties is how much per C?

55. If .45 of a ton of hay cost \$9.90, what is the price per ton?

56. A laborer received \$17.91 for splitting rails at 60¢ per C. How many rails did he split?

57. What is the value of 46,950 cu. ft. of ship timber at \$19.50 per M?

58. An iron founder paid \$99.45 for 7 T. 15 cwt. 1 qr. 4 lb. of pig-iron. What was the price per ton?

59. At \$3.25 per C, how many broom handles can be bought for \$79.56?

60. How much is the freight on 102,400 lb. of merchandise from Boston to New Orleans, at \$5.25 per ton?

61. A show-case maker paid \$13.64 for $243\frac{3}{4}$ ft. of black walnut lumber. What was the price per M?

62. A paper maker paid \$160.38 for rags @ $\frac{1}{3}$ ¢ per lb. How many tons of rags did he buy?

63. An upholsterer paid \$36.75 for Spanish moss at \$25 per ton. How many pounds did he buy?

SECTION V.

RATIO OR COMPARISON.

315. *Ratio* is the relative value (determined by division or subtraction) that exists between any two unequal like numbers.

The relative value is

- a. *Comparative Ratio*,—i. e., the quotient of the former number divided by the latter ;
- b. *Arithmetical Ratio*,—i. e., the difference between the numbers ;
- c. *Geometrical Ratio*,—i. e., the quotient of the latter number divided by the former.

316. In expressing a comparative ratio the terms used are

1. *Couplet*,—i. e., the two numbers connected by the sign of ratio (see page 13) ;

2. *Antecedent*,—i. e., the first term of a couplet ;

3. *Consequent*,—i. e., the second term of a couplet.

The couplet 8 : 14 is read " 8 is to 14," or " the ratio of 8 to 14."

NOTE.—For discussion and applications of arithmetical and geometrical ratios, see pages 234, 238.

317. *Comparison* is the method of finding a comparative ratio.

318. PRINCIPLE.—*Only numbers of the same unit value can be compared.*

319. PROPOSITIONS I–VI.—*State in three propositions, and demonstrate each, the Propositions (page 146), so modified as to apply to comparative ratio.*

ILLUSTRATIVE EXAMPLES.

(See Solutions, next page.)

- | | |
|--|----------------------------------|
| 1. Compare 12 with 60. | 3. .6 is how many times .04 ? |
| 2. Compare $\frac{3}{4}$ with $\frac{2}{3}$. | 4. 15 h. is what part of 2 da. ? |
| 5. What is the relative value of \$875 to \$52 $\frac{1}{2}$? | |
| 6. What is the relative value of \$52 $\frac{1}{2}$ to \$875 ? | |

ANALYTICAL SOLUTIONS.

- (1) $12 \div 60 = \frac{1}{5} = \frac{1}{5}$; hence, 12 is $\frac{1}{5}$ of 60.
 (2) $\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1} = 2$; hence, $\frac{1}{2}$ is 2 times $\frac{1}{4}$.
 (3) $.6 \div .04 = \frac{.6}{.04} = 15$; hence, .6 is 15 times .04.
 (4) 2 da. = 48 h., and $15 \text{ h.} \div 48 \text{ h.} = \frac{15}{48} = \frac{5}{16}$; hence, 15 h. are $\frac{5}{16}$ of 2 da.
 (5) $\$875 \div \$52\frac{1}{2} = 16\frac{2}{3}$; hence, \$875 are $16\frac{2}{3}$ times \$52 $\frac{1}{2}$.
 (6) $\$52\frac{1}{2} \div \$875 = \frac{1}{16\frac{2}{3}}$; hence, \$52 $\frac{1}{2}$ are $\frac{1}{16\frac{2}{3}}$ of \$875.

PROBLEMS.

Compare the cost

1. Of 3 pairs of shoes with 4 pairs of shoes.
2. Of 4 pairs of shoes with 3 pairs of shoes.
3. Of 10 bushels of apples with 25 bushels of apples.
4. Of 25 bushels of apples with 10 bushels of apples.
5. Of 20 sheep with 45 sheep. | 6. Of 45 sheep with 20 sheep.

Compare	What part	Express the comparative ratio
7. 6 with 36.	16. Of $\frac{2}{3}$ is $\frac{1}{3}$?	25. Of 36 bu. and 15 bu.
8. 36 with 6.	17. Of $13\frac{1}{2}$ is $\frac{1}{3}$?	26. Of 112 lb. and 196 lb.
9. 72 with 1,728.	18. Of $\frac{3}{4}$ is $5\frac{1}{4}$?	27. Of $31\frac{1}{2}$ gal. and 42 gal.
10. 144 with 21.	19. Of $321\frac{1}{2}$ is 56?	28. Of 14 wk. 2 da. and 6 wk. 3 da.
11. .375 with .8125.	20. Of $41\frac{3}{4}$ is $104\frac{1}{4}$?	29. Of $\frac{2}{3}$ mi. and $\frac{2}{3}$ rd.
12. .0729 with 9.9.	21. Of $\frac{2}{3}$ is .21?	30. Of 3 A. and 9.6 sq. rd.
13. .9 with .215.	22. Of .852 is $2\frac{2}{3}$?	31. Of $8\frac{1}{3}$ gro. and $15\frac{1}{2}$ doz.
14. 36.15 with .0735.	23. Of $\$9.31\frac{1}{2}$ is \$.68 $\frac{2}{3}$?	32. Of 5 yr. 10 mo. and 2 yr. 3 mo.
15. 54.2 with 3.07.	24. Of $\$93.12\frac{1}{2}$ is \$385.03 $\frac{1}{4}$?	33. Of $964\frac{2}{3}$ bu. and 516 bu. 6 qt.

34. 1 is what part of $3\frac{1}{3}$? | 36. What part of 7.5 is .0075?
 35. 15 is what part of $\frac{1}{3}$? | 37. What part of $\frac{1}{3}$ of $3\frac{1}{2}$ is $\frac{1}{3}$ of 36?

38, 39. A man's salary is \$1,500 a year, and he spends \$800. What part of his salary does he spend? What part of it does he save?

40, 41. I bought a horse for \$175, and sold him for \$115. For what part of the cost did I sell him? What part of the cost did I lose?

42. If a piece of work can be done in 9 days, what part of the work can be done in 5 days?

43, 44. I bought 20 yards of silk and used $17\frac{1}{2}$ yards. What part of the silk did I use? What part of it had I left?

45. $68\frac{1}{2}$ yards are what part of $113\frac{3}{8}$ yards?

46. What is the comparative ratio of the couplet $86:34$?

47. What is the comparative ratio of 2 ft. 4 in. : 9 ft. 6 in.?

48. 64 bu. 3 pk. 5 qt. are what part of 973 bu. 2 pk. 3 qt.?

49. $973 \text{ bu. } 2 \text{ pk. } 3 \text{ qt.} : 64 \text{ bu. } 3 \text{ pk. } 5 \text{ qt.} = \text{what ratio?}$

50. A and B own a herd of 2,028 cattle, and A owns 780 of them. What part of the herd does each own?

51. In problem 50 what is the comparative ratio of A's number of cattle to B's?

52-54. Compare the respective quantities, prices, and values of $156\frac{3}{8}$ bushels of wheat @ $\$1.06\frac{1}{2}$, and $587.93\frac{3}{4}$ bushels of corn @ $\$5$.

55. A farm is divided among three heirs, the oldest receiving $256\frac{1}{4}$ A.; the second 189.61 A.; and the youngest 193 A. $72\frac{3}{4}$ sq. rd. What part of the farm does each receive?

56. In problem 55 what is the comparative ratio of the share of the youngest to the share of the oldest?

57. I owe $\$3,500$ on a house that cost me $\$7,500$. What part of the cost have I paid?

58. Two men can do $\frac{1}{2}$ of a piece of work in 1 day, and one of them can do twice as much as the other. What part of the whole work can each do in 1 day?

59. Four men hire a pasture for the season. The first man pastures 8 horses $5\frac{1}{2}$ months; the second pastures 7 horses 6 months; the third, 12 horses $4\frac{1}{2}$ months; and the fourth, 16 horses 3.9 months. What part of the rent should each pay?

SECTION VI.

ANALYSIS.

Define *analysis* (Art. 34).

320. The *solution of problems by analysis* is a general method, including both the reasoning and the process; it is independent of special rules.

Give the reasons and the process for finding any number, when there is given

- | | | | |
|---|--|---------------------------|--|
| 1. $\frac{1}{2}$ of it. | 5. $\frac{1}{18}$ of it. | 9. $\frac{5}{18}$ of it. | 13. 5 times $\frac{2}{3}$ of it. |
| 2. $\frac{2}{3}$ of it. | 6. $\frac{2}{3}$ of it. | 10. $\frac{3}{10}$ of it. | 14. 9 times $\frac{4}{11}$ of it. |
| 3. $\frac{1}{4}$ of it. | 7. $\frac{5}{8}$ of it. | 11. .07 of it. | 15. $3\frac{1}{2}$ times $\frac{7}{18}$ of it. |
| 4. $\frac{1}{4}$ of it. | 8. $\frac{4}{5}$ of it. | 12. .812 of it. | 16. $\frac{1}{3}$ of $\frac{4}{5}$ of it. |
| 17. $\frac{1}{11}$ of .16 of the number. | 20. $\frac{1}{18}$ of 12 times the number. | | |
| 18. .6 of .045 of the number. | 21. $\frac{2}{108}$ of $6\frac{1}{2}$ times the number. | | |
| 19. $\frac{1}{3}$ of 5 times the number. | 22. $\frac{2}{3}$ of $3\frac{1}{2}$ times $\frac{2}{3}$ of the number. | | |
| 23. 4 times $\frac{4}{5}$ of $6\frac{7}{11}$ times $\frac{2}{11}$ of 33 times the number. | | | |

321. In solving problems by analysis, the reasoning and computation are commonly

- 1st. *From* the given number to a unit of the same; then
- 2d. *From* this unit to the required number.

Ex. 1. \$1,836.25 is $\frac{5}{8}$ of the yearly rent of an iron foundry. For how much per year does the foundry rent?

EXPLANATION.—1st. *Reasons for process.*— $\frac{1}{8}$ of the rent is $\frac{1}{5}$ as much as $\frac{5}{8}$ of it; and the whole rent is 8 times as much as $\frac{1}{8}$ of it; i. e., the whole rent is 8 times $\frac{1}{5}$ of $\frac{5}{8}$ of it.

2d. *Application of reasons to process.*— $\frac{1}{8}$ of the rent is $\frac{1}{5}$ of $\frac{5}{8}$ of it, or $\frac{1}{8}$ of \$1,836.25, which is \$367.25; and the whole rent is 8 times $\frac{1}{8}$ of it, or 8 times \$367.25, which is \$2,938.

ANALYTICAL PROCESS.

$$\begin{aligned} \$1,836.25 \div 5 &= \$367.25 \\ 8 \times \$367.25 &= \$2,938 \end{aligned}$$

COMMON PROCESS.

$$\frac{8}{5} \times \frac{5}{8} \text{ of } \$1,836.25 = \$2,938$$

Ex. 2. The distance between New York and Chicago increased by $\frac{1}{17}$ of itself is 1,292 miles. What is the distance?

EXPLANATION.—1st. *Reasons for process.*—Any number increased by $\frac{1}{17}$ of itself is $\frac{18}{17}$ of the number. $\frac{1}{17}$ of any number is $\frac{1}{17}$ of $\frac{18}{17}$ of that number; and the number is 18 times $\frac{1}{17}$ of itself; i. e., the number is 18 times $\frac{1}{17}$ of $\frac{18}{17}$ of the number.

2d. *Application of reasons to process.*— $\frac{1}{17}$ of the distance is $\frac{1}{17}$ of $\frac{18}{17}$ of it, or $\frac{1}{17}$ of 1,292 miles, which is 76 miles; and the whole distance is 18 times $\frac{1}{17}$ of it, or 18 times 76 miles, which is 988 miles.

ANALYTICAL PROCESS.

$$1 = \frac{18}{17}; \quad \frac{18}{17} + \frac{1}{17} = \frac{19}{17} \\ 1,292 \text{ mi.} \div \frac{19}{17} = 76 \text{ mi.} \\ 18 \times 76 \text{ mi.} = 988 \text{ mi.}$$

COMMON PROCESS.

$$\frac{18}{17} \times \frac{1}{17} \text{ of } 1,292 \text{ mi.} = 988 \text{ mi.}$$

Ex. 3. If 32 sheep cost \$114, how much will 168 sheep cost?

EXPLANATION.—1st. *Reasons for process.*—1 sheep costs $\frac{1}{32}$ as much as 32 sheep, and 168 sheep cost 168 times as much as 1 sheep; i. e., 168 sheep cost 168 times $\frac{1}{32}$ of the cost of 32 sheep.

2d. *Application of reasons to process.*— $\frac{1}{32}$ of \$114, the cost of 32 sheep, is \$3.56 $\frac{1}{4}$, the price of 1 sheep; and 168 times \$3.56 $\frac{1}{4}$, the price of 1 sheep, is \$598.50, the cost of 168 sheep.

ANALYTICAL PROCESS.

$$\begin{aligned} \$114 \div 32 &= \$3.56\frac{1}{4} \\ 168 \times \$3.56\frac{1}{4} &= \$598.50 \end{aligned}$$

COMMON PROCESS.

$$\frac{168}{32} \times \frac{1}{32} \text{ of } \$114 = \$598.50$$

Ex. 4. If 9 coopers make 27 barrels in 2 hours, how many barrels will 30 coopers make in 15 hours?

EXPLANATION.—1st. *Reasons for process.*—(From the given number to a unit of the same). In any given time 1 cooper will make $\frac{1}{9}$ as many barrels as 9 coopers will make; and in 1 hour 1 cooper will make $\frac{1}{2}$ as many barrels as he will make in 2 hours. (From a unit of the given number to the number of units in the required number). In 1 hour 30 coopers will make 30 times as many barrels as 1 cooper will make; and in 15 hours any number of coopers will make 15 times as many barrels as they will make in 1 hour; i. e., in 15 hours, 30 coopers will make 15 times 30 times $\frac{1}{2}$ of $\frac{1}{9}$ of as many barrels as 9 coopers will make in 2 hours.

2d. *Application of reasons to process.*—In 2 hours 1 cooper will make $\frac{1}{9}$ as many barrels as 9 coopers will make in the same time, or $\frac{1}{9}$ of 27 barrels, which is 3 barrels; and in 1 hour he will make $\frac{1}{2}$ as many barrels as in 2 hours, or $\frac{1}{2}$ of 3 barrels, which is 1 $\frac{1}{2}$ barrels. In 1 hour 30 coopers will make 30 times as many barrels as 1 cooper will make in the same time, or 30 times 1 $\frac{1}{2}$ barrels, which are 45 barrels; and in 15 hours they will make 15 times as many barrels as in 1 hour, or 15 times 45 barrels, which are 675 barrels.

ANALYTICAL PROCESS.

$$\begin{aligned} \frac{1}{9} \text{ of } 27 \text{ bar.} &= 3 \text{ bar.} \\ \frac{1}{2} \text{ of } 3 \text{ bar.} &= 1\frac{1}{2} \text{ bar.} \\ 30 \times 1\frac{1}{2} \text{ bar.} &= 45 \text{ bar.} \\ 15 \times 45 \text{ bar.} &= 675 \text{ bar.} \end{aligned}$$

$$\text{COMMON PROCESS.}—\frac{15}{2} \times \frac{30}{9} \times \frac{1}{9} \text{ of } 27 \text{ bar.} = 675 \text{ bar.}$$

322. By using the comparative ratios of like given numbers in making computations, the analytic process may often be shortened.

Compare the following solutions of Ex. 4 with the preceding solution of the same example :

EXPLANATION.—1st. *Reasons for process.*—The required number of barrels depends upon (1) the relative number of coopers, and (2) the relative number of hours.

Since 30 coopers are $\frac{3}{2}$ or $\frac{3}{2}$ times 9 coopers, in any given time they will make $\frac{3}{2}$ as many barrels as 9 coopers will make in the same time.

Since 15 hours are $\frac{1}{2}$ times 2 hours, any given number of coopers will make $\frac{1}{2}$ times as many barrels in 15 hours as they will make in 2 hours.

2d. *Application of reasons to process.*— $\frac{3}{2}$ times 27 barrels (that 9 coopers will make in 2 hours) are 90 barrels that 30 coopers will make in 2 hours; and $\frac{1}{2}$ times 90 barrels (that 30 coopers will make in 2 hours) are 675 barrels that 30 coopers will make in 15 hours.

ANALYTICAL PROCESS.

$$(3^{\text{rd}}) = \frac{3}{2} \times 27 \text{ bar.} = 90 \text{ bar.}$$

$$\frac{1}{2} \times 90 \text{ bar.} = 675 \text{ bar.}$$

SHORTER PROCESS.

$$\frac{1}{2} \times \frac{3}{2} \times 27 \text{ bar.} = 675 \text{ bar.}$$

PROBLEMS.

Find the number which

1. Minus $\frac{1}{4}$ of itself is 684.
2. Plus $\frac{3}{4}$ of itself is 13,472.
3. Plus $\frac{1}{11}$ of itself is 2,256.

Find the number

4. $\frac{2}{3}$ of which is 627 $\frac{4}{11}$ more than $\frac{1}{3}$ of it.
5. $\frac{3}{4}$ of which is 1,054.41.
6. .32 of which is 90 less than .4375 of it.

7. Which is 7 times $\frac{1}{10}$ of 15 yd. 2 ft. 9 in.

8. Which is 195 more than $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$ of itself.

9. $\frac{1}{11}$ of which is \$1,323.05 more than $\frac{1}{11}$ of it.

10. $\frac{1}{12}$ of which is $\frac{2}{3}$ of $3\frac{1}{2}$ times 185 gal. 2 qt.

11. $31\frac{2}{3}$ times $\frac{3}{4}$ of which is 15.25 more than $24\frac{2}{3}$ times $\frac{1}{11}$ of it.

12. At 55¢ for two quarts of maple sirup, how much will $2\frac{1}{2}$ gallons cost?

13. Two men paid \$7.50 for the use of a horse. One of them drove him 75 miles, and the other 90 miles. How much should each man pay?

14. If an 8-lb. sack of N. Y. dairy salt costs 5¢, what is the price per bushel at the same rate?

15. Two men buy a barrel of beef for \$11 $\frac{1}{2}$, and one of them pays \$5. How many pounds of the beef should each man have?

16. If 53 men can do a piece of work in 60 days, 42 men can do the same work in how many days?

17. If the shadow of a post 6 feet high is 4 ft. 6 in. long, what is the height of a tree whose shadow is 125 ft. long at the same time?

18. Three farmers send their butter to market together, E sending 475 lb., F 325 lb., and G 250 lb. The butter was sold for \$380.93 $\frac{1}{2}$, and the expenses were \$40.68 $\frac{1}{2}$. How much were the net receipts of each?

19. If 15 barrels of flour will last 54 persons 3 months, how many barrels will last 21 persons 6 months?

20. A owns 12 Sec., B 9 Sec., and C the remainder of a Township of Western land. They sell the land at \$1.25 per acre above cost. How much is each one's share of the profits?

21. In a safe are a certain number of half-eagles, six times as many dollars, and twice as many dimes as dollars. The whole number of coins is 570. How much money is in the safe?

22. A fruit-dealer bought oranges at the rate of 15¢ per doz., and sold them at the rate of 3 for a dime. His profits were \$72. How many boxes of 24 doz. oranges each did he buy?

23. A calf eats 3 times as much hay as a sheep, a horse 2 times as much as a calf, and a cow 1 $\frac{1}{2}$ times as much as a horse. If a horse eats 24 pounds of hay per day, worth \$12.50 per ton, how much will it cost for the hay for 50 sheep, 20 cows, 10 calves, and 3 pairs of horses, from Nov. 1st to April 1st?

24. Two men build a store, one of them working 8 $\frac{1}{2}$ h. per day for 30 da., and the other 9 h. 45 min. per day for 22 da. What part of the work does each man do?

25. M's age equals $\frac{1}{3}$ of $\frac{1}{4}$ of N's age, and the sum of their ages is 78 years. What is the age of each?

26. The sum of $\frac{1}{3}$ and $\frac{1}{4}$ of a certain number is 2 more than $\frac{1}{2}$ of the number. What is the number?

27. Divide 123 into two parts, one of which diminished by 5 shall equal the other diminished by $\frac{1}{3}$ of itself.

28. Divide the number 246 into two parts, one of which increased by 12 shall equal $\frac{1}{2}$ of the other.

29. If 6 men or 9 boys can do a certain piece of work in 5 days, in how many days can 3 men and 2 boys do the work?

30. A manufacturer pays boys 50¢ per day, women \$1 per day, and men \$1.50 per day, and his pay-roll is \$198 per week. He employs twice as many women as men, and twice as many boys as women. How many persons does he employ?

31. The total length of 3 pieces of carpeting is 185 yards. The second piece is $\frac{2}{3}$ as long as the first, and $\frac{1}{2}$ as long as the third. What is the length of each piece?

32. At 6 o'clock A.M. the Eastern Local Mail train leaves St. Louis, and runs at the rate of 24 miles per hour for 3 h., when the rate of speed is increased 1 mile per hour. At 8 o'clock A.M. the Fast Mail follows at the rate of $31\frac{1}{2}$ miles per hour till 12 o'clock noon, when the rate of speed is increased to $33\frac{1}{2}$ miles per hour. Find the time at which the Fast Mail will overtake the Local Mail.

33. From the conditions of the preceding problem, find the distance from St. Louis at which the Fast Mail passes the Local Mail.

34. Divide 270 lb. of maple sugar among three persons, giving to the second $12\frac{1}{2}$ lb. less than to the first, and $21\frac{1}{2}$ lb. more than to the third.

35. M can walk a mile in 15 minutes, and N can walk the same distance in 4 min. 30 sec. less time. How many yards the start must N give M, that the two may finish the mile at the same instant?

36. Three men can paint a church in 2 weeks. If after they have worked 7 days one of the men quits, and the other two finish the painting in 9 days more, how much of the work does the man do who works only 7 days?

37. Divide 4,560 into three such parts that the third part shall be two times as great as the second, and the second shall be 40 more than two times the first.

38. Three bricklayers received \$862.50 for building the walls of a school-house. R and S did $\frac{1}{2}$ of the work, S and T $\frac{1}{4}$ of it, and R and T $\frac{1}{4}$ of it. What part of the work did each man do?

39. In problem 38, to how much of the money was each man entitled?

The value of a hat factory equals the value of the manufactured stock and the cash on hand. The value of the manufactured stock equals the cash on hand (\$3,625) and $\frac{1}{4}$ of the value of the factory. What is the value

40. Of the manufactured stock? | **41.** Of the factory?

42. A boy in filling a barrel with water, spilled $\frac{3}{4}$ of all that he attempted to put into the barrel. How many gallons of water did he spill?

43. A contractor sold 340 cd. of green wood and 195 cd. of dry wood for \$1,100 $\frac{1}{2}$, receiving 50¢ more per cd. for the dry wood than for the green. What was the price per cord of each kind of wood?

44. In two orchards are 490 apple-trees, and in one of the orchards are 35 trees less than $\frac{1}{4}$ as many as there are in the other. How many trees are there in the larger orchard?

45, 46. D and E build a stone wall; for 9 days work by D and $11\frac{1}{2}$ days work by E they receive \$39.90; and for 8.2 days work by D and 5.6 days work by E they receive \$21.72. How much does each man earn? What are the daily wages of each?

47. $\frac{3}{4}$ of $\frac{1}{4}$ of W's age equals $\frac{1}{4}$ of Y's age, and the sum of their ages is 99 years. What is the age of each?

48. I bought a house and lot, and, after making repairs at an expense of \$620, I sold the property for one fourth more than I paid for it, thereby losing \$82.50. What did I pay for the property?

49. L can do a certain piece of work in 7 days, and M and N can do the same work in 5 days. In how many days can the three working together do it?

50. A real-estate agent has 5 farms for sale. $\frac{1}{4}$ of all the land is in the first farm, $\frac{1}{4}$ in the second, $\frac{1}{4}$ in the third, $\frac{1}{4}$ in the fourth, and 160 acres in the fifth. How many acres of land has he to sell?

51. Four men are employed to dig a drain. The first, second, and third can dig it in 9 days; the first, second, and fourth in $9\frac{1}{4}$ days; the first, third, and fourth in $8\frac{1}{4}$ days; and the second, third, and fourth in 8 days. Which one of the men can do the work in the least number of days?

SECTION VII.

REVIEW.

323. TEST QUESTIONS.

- I.** State five business transactions to which average is applied.
 - II.** Show that a commercial unit is also a unit of price.
 - III.** Distinguish between an item and an invoice.
 - IV.** Which metal is the alloy in nickel coins?
 - V.** What is the order of computation in the solution of problems by analysis?
 - VI.** What are the essentials of a bill of purchase?
 - VII.** Why must quantity always be considered an abstract number in problems involving quantity, price, and cost?
 - VIII.** Show that a comparative ratio is the reciprocal of an inverse ratio.
 - IX.** From the definition of a bill of purchase, write a definition of a bill of sale.
 - X.** How can the general principles of division be made applicable to ratio?
-
- XI.** Wherein do a statement and an abstract of account differ?
 - XII.** When is the cost less than the price? When greater?
 - XIII.** Distinguish between an account current and an abstract of account.
 - XIV.** What is the customary commercial unit in this State for transactions in lumber? In stone? In salt?
 - XV.** Rule form for bill or invoice to show case or package, number of pieces, number of yards in each piece, price, and total yards and cost.
 - XVI.** Which coin is finer, English or United States?
 - XVII.** How may a bill be receipted without the seller or creditor writing his name after the words acknowledging payment?
 - XVIII.** Distinguish between a comparative ratio and a geometrical ratio.
 - XIX.** What is the intrinsic value of a 20-dollar gold piece, when an ounce of fine gold is worth \$20.62½?
 - XX.** When may the footing of an account and the balance of an account be the same?

324. BLACKBOARD OUTLINES.

A. ARITHMETICAL EQUATIONS.

I. TERMS.	{	1. _____	3. _____	5. _____		II. RULE.
		2. _____	4. _____	_____		

B. CURRENCY, AND ACCOUNTS AND BILLS.

TERMS.	{	I. <i>Currency.</i>	{	1. _____	5. _____	7. _____		
				2. _____	6. _____	8. _____		
				3. _____	a. _____	9. _____		
				4. _____	b. _____	10. _____		
				_____	c. _____	_____		
				II. <i>Accounts and Bills.</i>	{	11. _____	14. _____	17. _____
						12. _____	15. _____	a. _____
						13. _____	16. _____	b. _____

C. AVERAGE.

I. TERMS.	{	1. _____		II. AXIOM.
		2. _____		
		a. _____		

D. PRICE, QUANTITY, AND COST.

I. TERMS.	{	1. _____	3. _____		II. FORMULÆ.	{	1. _____
		2. _____	_____				2. _____
		_____	_____				3. _____

E. RATIO OR COMPARISON.

I. TERMS.	{	1. _____	2. _____		II. PRINCIPLE.
		a. _____	3. _____		
		b. _____	4. _____		
		c. _____	5. _____		
		_____	_____		

F. ANALYSIS.

I. TERMS.	{	1. _____		II. PROCESS.
		2. _____		

CHAPTER VI. POWERS AND ROOTS.

SECTION I. INVOLUTION.

I. DEFINITIONS AND PRINCIPLES.

325. Define a power (141). Define a root (142).

326. *Involution* is the process of finding powers from roots.

327. An *exponent* is a number placed at the right and above a root, to indicate the required power of the root.

a. The fifth power of $7 = 7^5 = 7 \times 7 \times 7 \times 7 \times 7 = 16,807$;
the third power, or cube, of $28 = 28^3 = 28 \times 28 \times 28 = 12,167$.

Conversely,—7 is the fifth root of 16,807; 28 is the cube root of 12,167.

b. The exponent indicates the number of times the root is to be used as a factor.

c. Powers are *indicated* by exponents; indicated powers are *expanded* by multiplication.

d. The root itself is commonly called the *first power*; its exponent (¹) is seldom written.

328. Ex. 1. What is PROCESS.
the 7th power of 8? $8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 = 2,097,152$

The written expression in the foregoing process is an arithmetical equation (39); it may be varied by introducing signs of aggregation (38, IV) as follows:—

$$\begin{aligned}
 (8 \times 8) \times (8 \times 8) \times (8 \times 8) \times 8 &= 8^2 \times 8^2 \times 8^2 \times 8^1 = 8^7 = 2,097,152; \text{ or} \\
 (8 \times 8 \times 8) \times (8 \times 8 \times 8) \times 8 &= 8^3 \times 8^3 \times 8^1 = 8^7 = 2,097,152; \text{ or} \\
 (8 \times 8 \times 8) \times (8 \times 8 \times 8 \times 8) &= 8^3 \times 8^4 = 8^7 = 2,097,152; \text{ or} \\
 (8 \times 8 \times 8) \times (8 \times 8) \times (8 \times 8) &= 8^3 \times 8^2 \times 8^2 = 8^7 = 2,097,152; \text{ or} \\
 (8 \times 8) \times (8 \times 8 \times 8 \times 8 \times 8) &= 8^2 \times 8^5 = 8^7 = 2,097,152.
 \end{aligned}$$

In each of these five equations the sum of the exponents of the aggregations is 7, which equals the exponent of the required power. Again,—

$$8^2 \times 8^4 = 8^{2+4} = 8^6; \quad 5^3 \times 5^2 \times 5^1 = 5^6; \quad 19^3 \times 19^2 \times 19^1 = 19^6. \quad \text{Hence,}$$

PRINCIPLE I. *Any power of a root equals the product of any lower powers of the same root, the sum of whose exponents equals the exponent of that power.*

Ex. 2. Expand 15^4 ,—i. e.,
find the 4th power of 15.

PROCESS.

$$15 \times 15 \times 15 \times 15 = 50,625$$

Since $15 = 3 \times 5$, the process may be written

$$\overbrace{3 \times 5 \times 3 \times 5 \times 3 \times 5 \times 3 \times 5} = 50,625; \text{ or, grouping like factors,}$$

$$3 \times 3 \times 3 \times 3 \times 5 \times 5 \times 5 \times 5 = 3^4 \times 5^4 = 50,625. \text{ Again,—}$$

$$3^4 \times 5^4 \times 7^4 = 105^4; 2^4 \times 3^4 \times 7^4 = 42^4; 4^4 \times 9^4 = 36^4. \text{ Hence,}$$

PRINCIPLE II. *Any power of a root equals the product of the like powers of any set of factors that produce the root.*

PROBLEMS.

In solving these problems, use short methods of multiplication and combinations of powers or factors, when they shorten the work of computation.

1. What is the 5th power of 7? | 2. Expand 18^3 .
 3. Prove by computation that the product of the 3d and 4th powers of 5 is the 7th power of 5.
 4. Find the 32d power of 2, using only five multiplications.
 5. Find the cube of 35 from the cubes of its factors.
 6. Express all the combinations for the 5th power of 7. (See Ex. 1.)
- Expand $\left\{ \begin{array}{l} 7. 13^6 \\ 8. 7^4 \end{array} \right| \begin{array}{l} 9. 31^5 \\ 10. 135^3 \end{array} \left| \begin{array}{l} 11. 24^8 \\ 12. 2.7^5 \end{array} \right| \begin{array}{l} 13. .5^{10} \\ 14. 11^8 \end{array} \left| \begin{array}{l} 15. 2.56^4 \\ 16. (\frac{2}{3})^7 \end{array} \right| \begin{array}{l} 17. (3\frac{1}{2})^5 \\ 18. (\frac{1}{4})^8 \end{array} \right.$

Complete and learn the following table :

Roots	1	2	3	4	5	6	7	8	9
Squares	—	—	—	—	—	—	—	—	—
Cubes	—	—	—	—	—	—	—	—	—
Roots	.1	.2	.3	.4	.5	.6	.7	.8	.9
Squares	—	—	—	—	—	—	—	—	—
Cubes	—	—	—	—	—	—	—	—	—

329. The least and the greatest integer that can be expressed by one figure are respectively 1 and 9; and the least and the greatest decimal that can be expressed by one figure are respectively .1 and .9; hence, if

1 and 9, and .1 and .9 be raised to the same power, the results will show the least and the greatest limit to the number of places produced by raising to that power a number expressed by any one figure.

$1^2 = 1$	$9^2 = 81$	$.1^2 = .01$	$.9^2 = .81$
$1^3 = 1$	$9^3 = 729$	$.1^3 = .001$	$.9^3 = .729$
$1^4 = 1$	$9^4 = 6,561$	$.1^4 = .0001$	$.9^4 = .6561$
$1^5 = 1$	$9^5 = 59,049$	$.1^5 = .00001$	$.9^5 = .59049$

Comparing roots with their powers, we see that

(1) The second power of an integer expressed by one figure contains one or two places; the third power contains one to three places; the fourth power, one to four places; the fifth power, one to five places; and so on.

(2) The second power of a decimal expressed by one figure contains two places; the third power three places; the fourth power, four places; the fifth power, five places; and so on. Hence,

PRINCIPLE III. *The limits of the number of places in any power of any integer expressed by one figure, are one and the number expressed by the exponent of the power. And*

PRINCIPLE IV. *The limit of the number of places in any power of any decimal expressed by one figure, is the number expressed by the exponent of the power.*

330. The least and the greatest integer that can be expressed by two figures are respectively 10 and 99; and the least and the greatest decimal that can be expressed by two figures are respectively .01 and .99.

$10^2 = 100$	$99^2 = 9,801$	$.01^2 = .0001$	$.99^2 = .9801$
$10^3 = 1,000$	$99^3 = 970,299$	$.01^3 = .000001$	$.99^3 = .970299$
$10^4 = 10,000$	$99^4 = 96,059,601$	$.01^4 = .00000001$	$.99^4 = .96059601$
$10^5 = 100,000$	$99^5 = 9,509,900,499$	$.01^5 = .0000000001$	$.99^5 = .9509900499$

Comparing these roots and powers with the like powers of roots expressed by one figure (329), we see that each figure after the left-hand figure in the root produces as many additional places in the power as equal the ones in the exponent. Hence,

PRINCIPLE V. *The limits of the number of places in any required power of any root are the limits of the same power of a root expressed by one figure, plus as many places as are indicated by the product of the exponent by the number of places less one in the root.*

EXERCISES.

How many figures will be required to write

The square		The cube		The fourth power	
1. Of 28!	3. Of 152!	5. Of 17!	7. Of 200!	9. Of 9!	11. Of 61!
2. Of 63!	4. Of 807!	6. Of 46!	8. Of 530!	10. Of 20!	12. Of 824!

To express the expanded equivalents for the following indicated powers:

13. 37^2 !	15. 12^6 !	17. 56^5 !	19. $.7^2$!	21. 1.1^4 !	23. 4.1^3 !
14. 15^4 !	16. 22^3 !	18. 8^{10} !	20. $.9^3$!	22. $.57^2$!	24. 6.59^2 !

II. SQUARES AND CUBES OF DECIMAL NUMBERS.

331. To determine the parts that make up the square of a root expressed by two figures, we will square 34 by different processes.

$34 = 30 + 4$, or 3 tens and 4 ones; $34^2 = \overline{30 + 4}^2 = (30 + 4) \times (30 + 4)$.

FIRST PROCESS.	SECOND PROCESS.	THIRD PROCESS.	GENERAL PROCESS.
$34 = 30 + 4$	$= 30 + 4$	$= (tens + ones)$	
$34 = 30 + 4$	$= 30 + 4$	$= (tens + ones)$	
$102 = 900 + 120$	$= 30^2 + (30 \times 4)$	$= tens^2 + (tens \times ones)$	
$136 = 120 + 16$	$= (30 \times 4) + 4^2$	$= (tens \times ones) + ones^2$	
$1,156 = 900 + 240 + 16$	$= 30^2 + 2 \times (30 \times 4) + 4^2$	$= tens^2 + 2 \times (tens \times ones) + ones^2$	

In the first process 34 is squared by the ordinary method of multiplication, with this exception,—the digits of the multiplier are taken in order from left to right.

In the second process the products of the units of the different orders are written separately, and the additions are indicated.

In the third process the parts of each of the factors that form the different parts of the power are kept separate, and both the multiplications and additions are indicated.

The General Process is the third process expressed in words.

The final result obtained by each of the first three processes is formulated in the General Process, and is expressed in

PRINCIPLE VI. *The square of any number regarded as tens and ones is composed of the square of the tens, plus two times the product of the tens and the ones, plus the square of the ones.*

Applying Principle VI in squaring the number 34, the work may be written in either of the forms following:

FIRST PROCESS.

$$\begin{aligned} 30^2 &= 900 \\ 2 \times (30 \times 4) &= 240 \\ 4^2 &= 16 \\ \hline 34^2 &= 1,156 \end{aligned}$$

SECOND PROCESS.

$$\begin{aligned} 3 \text{ tens}^2 &= 9 \times \times \\ 2 \times (3 \text{ tens} \times 4 \text{ ones}) &= 24 \times \\ 4 \text{ ones}^2 &= 16 \\ \hline 34^2 &= 1,156 \end{aligned}$$

332. To determine the parts that make up the cube of a root expressed by two figures, cube 34 by a continuation of the third process and of the general process (**331**).

In continuing these processes, always combine the multiplier with its like factor (if any) in each term of the multiplicand, and indicate the multiplication, by a change in the exponent of that factor.

THIRD PROCESS.

$$\begin{aligned} \text{Root or 1st power,} \quad 30+4 &= \\ \quad 30+4 &= \\ \quad 30^2+(30 \times 4) &= \\ \quad (30 \times 4)+4^2 &= \\ \text{Square,} \quad 30^2+2 \times (30 \times 4)+4^2 &= \\ \quad 30+4 &= \\ \quad 30^3+2 \times (30^2 \times 4)+(30 \times 4^2) &= \\ \quad (30^2 \times 4)+2 \times (30 \times 4^2)+4^3 &= \\ \text{Cube,} \quad 30^3+3 \times (30^2 \times 4)+3 \times (30 \times 4^2)+4^3 &= \end{aligned}$$

GENERAL PROCESS.

$$\begin{aligned} & \text{tens+ones} \\ & \text{tens+ones} \\ & \text{tens}^2 + (\text{tens} \times \text{ones}) \\ & \quad (\text{tens} \times \text{ones}) + \text{ones}^2 \\ & \text{tens}^3 + 2 \times (\text{tens}^2 \times \text{ones}) + \text{ones}^2 \\ & \quad \text{tens+ones} \\ & \text{tens}^3 + 2 \times (\text{tens}^2 \times \text{ones}) + (\text{tens} \times \text{ones}^2) \\ & \quad (\text{tens}^2 \times \text{ones}) + 2 \times (\text{tens} \times \text{ones}^2) + \text{ones}^3 \\ & \text{tens}^3 + 3 \times (\text{tens}^2 \times \text{ones}) + 3 \times (\text{tens} \times \text{ones}^2) + \text{ones}^3 \end{aligned}$$

The final result, as shown in the General Process, is expressed in

PRINCIPLE VII. *The cube of any number regarded as tens and ones is composed of the cube of the tens, plus three times the product of the square of the tens and the ones, plus three times the product of the tens and the square of the ones, plus the cube of the ones.*

Applying Principle VII in cubing the number 34, the work may be written in either of the forms following:

FIRST PROCESS.

$$\begin{aligned} 30^3 &= 27,000 \\ 3 \times (30^2 \times 4) &= 10,800 \\ 3 \times (30 \times 4^2) &= 1,440 \\ 4^3 &= 64 \\ \hline 34^3 &= 39,304 \end{aligned}$$

SECOND PROCESS.

$$\begin{aligned} 3 \text{ tens}^3 &= 27 \times \times \times \\ 3 \times (3 \text{ tens}^2 \times 4 \text{ ones}) &= 108 \times \times \\ 3 \times (3 \text{ tens} \times 4 \text{ ones}^2) &= 144 \times \\ 4 \text{ ones}^3 &= 64 \\ \hline 34^3 &= 39,304 \end{aligned}$$

III. GENERAL FORMULÆ FOR SQUARES AND CUBES.

333. The formulæ for the squares and cubes of decimal numbers regarded as tens and ones are,—

1. For squares :— $tens^2 + 2 \times tens \times ones + ones^2$ (331, Prin. VI).
2. For cubes :— $tens^3 + 3 \times tens^2 \times ones + 3 \times tens \times ones^2 + ones^3$ (332, Prin. VII).

Substituting t for tens and o for ones in the formulæ given above, we have, for the expansion of indicated squares and cubes of decimal numbers regarded as tens and ones, the following

$$\begin{array}{l} \text{GENERAL} \left\{ \begin{array}{l} \text{I. For squares :—} t^2 + 2 \times t \times o + o^2 \\ \text{FORMULÆ.} \left\{ \begin{array}{l} \text{II. For cubes :—} t^3 + 3 \times t^2 \times o + 3 \times t \times o^2 + o^3 \end{array} \right. \end{array} \right.$$

These formulæ will be used in the theory of square and cube roots.

Ex. 1. Expand 34^2 .

PROCESSES.

$$34 = 30 + 4 = 3 \text{ tens} + 4 \text{ ones}$$

$$\text{Formula I. } 34^2 = t^2 + 2 \times t \times o + o^2$$

1. FULL PROCESS.

$$\begin{array}{r} t^2 = 30^2 = 900 \\ 2 \times t \times o = 2 \times 30 \times 4 = 240 \\ o^2 = 4^2 = 16 \\ \hline \text{Hence, } 34^2 = 1,156 \end{array}$$

2. SHORT PROCESS.

$$\begin{array}{r} t^2 = 3^2 = 9 \times \times \text{ hund.} \\ 2 \times t \times o = 2 \times 3 \times 4 = 24 \times \text{ tens} \\ o^2 = 4^2 = 16 \text{ ones} \\ \hline \text{Hence, } 34^2 = 1,156 \end{array}$$

3. PROCESS BY EQUATIONS.

$$\begin{array}{l} 34^2 = t^2 + 2 \times t \times o + o^2 \\ = 30^2 + 2 \times 30 \times 4 + 4^2 \\ = 900 + 240 + 16 \\ = 1,156 \end{array}$$

Ex. 2. Expand 34^3 .

PROCESSES.

$$34 = 30 + 4 = 3 \text{ tens} + 4 \text{ ones}$$

$$\text{Formula II. } 34^3 = t^3 + 3 \times t^2 \times o + 3 \times t \times o^2 + o^3$$

1. FULL PROCESS.

$$\begin{array}{r} t^3 = 30^3 = 27,000 \\ 3 \times t^2 \times o = 3 \times 30^2 \times 4 = 10,800 \\ 3 \times t \times o^2 = 3 \times 30 \times 4^2 = 1,440 \\ o^3 = 4^3 = 64 \\ \hline \text{Hence, } 34^3 = 39,304 \end{array}$$

2. SHORT PROCESS.

$$\begin{array}{r} t^3 = 3^3 = 27 \times \times \times \text{ thous.} \\ 3 \times t^2 \times o = 3 \times 3^2 \times 4 = 108 \times \times \text{ hund.} \\ 3 \times t \times o^2 = 3 \times 3 \times 4^2 = 144 \times \text{ tens} \\ o^3 = 4^3 = 64 \text{ ones} \\ \hline \text{Hence, } 34^3 = 39,304 \end{array}$$

3. PROCESS BY EQUATIONS.

$$\begin{array}{l} 34^3 = t^3 + 3 \times t^2 \times o + 3 \times t \times o^2 + o^3 \\ = 30^3 + 3 \times 30^2 \times 4 + 3 \times 30 \times 4^2 + 4^3 \\ = 27,000 + 10,800 + 1,440 + 64 \\ = 39,304 \end{array}$$

Apply proper formulæ and expand the indicated powers in the following

PROBLEMS.

19. 13^3	21. 57^3	23. 17^3	25. 31^3	27. 6.5^3	29. $.86^3$
20. 64^3	22. 92^3	24. 46^3	26. 75^3	28. 2.3^3	30. $.48^3$
31. 114^3	$114 = 11 \text{ tens} + 4 \text{ ones.}$			33. 21.3^3	35. 58.6^3
32. 3.27^3	$3.27 = \left\{ \begin{array}{l} 32 \text{ tens of hundredths} + 7 \text{ ones of} \\ \text{hundredths.} \end{array} \right.$			34. $.142^3$	36. 409^3

Prove the following

PROPOSITIONS.

- I. *All powers of 1 are 1.*
- II. *All powers of roots greater than 1 are greater than their roots.*
- III. *All powers of roots less than 1 are less than their roots.*
- IV. *There must be as many ciphers in the final period of a perfect power ending with ciphers, as there are ones in the exponent.*

SECTION II.

EVOLUTION.

I. DEFINITIONS, AND EVOLUTION BY FACTORING.

334. *Evolution* is the process of finding roots from powers.

Evolution is commonly called *extraction of roots*.

335. The *radical sign* ($\sqrt{}$) is the sign placed before a number, to denote that a root is to be extracted.

a. An *index* is a number placed in the opening of the radical sign, to indicate the root to be extracted. Thus,—

$\sqrt[3]{125}$, $\sqrt[5]{843}$, are read, “the cube root of 125,” and “the fifth root of 843.”

b. The index of the square root is commonly omitted. Thus,—

$\sqrt{49}$ is read “the square root of 49.”

c. A *fractional exponent* is also called an *index*. Thus, $82^{\frac{1}{5}}$ is read “the square of the fifth root of 82,” or “the fifth root of the square of 82.” In this example the fraction $\frac{1}{5}$ is a fractional exponent or index.

336. A *perfect power* is a number that has an exact root.

337. An *imperfect power* is a number that has not an exact root.

a. A number may be a perfect power of a root of one degree and an imperfect power of a root of some other degree. Thus, 64 is a perfect power of a root of either the second, third, or sixth degree, but an imperfect power of a root of the fourth or the fifth degree.

b. Perfect powers are also called *rational numbers*, and imperfect powers are called *radicals* or *surds*.

338. The *square root* of a number is one of two equal factors that produce the number.

339. The *cube root* of a number is one of three equal factors that produce the number.

340. Since $5 \times 5 = 25$, 5 is the square root of 25; and since $54 \times 54 = 2,916$, 54 is the square root of 2,916. Also, since $4 \times 4 \times 4 = 64$, 4 is the cube root of 64; and since $54 \times 54 \times 54 = 157,464$, 54 is the cube root of 157,464.

$$\text{Ex. 1. } 5 \times 5 = 25; 7 \times 7 = 49; 25 \times 49 = 1,225 = 5^2 \times 7^2 = 35^2$$

$$\text{Ex. 2. } 2 \times 2 \times 2 = 8; 3 \times 3 \times 3 = 27; 8 \times 27 = 216 = 2^3 \times 3^3 = 6^3$$

Conversely,—

$$\text{Ex. 3. } \sqrt{1,225} = \sqrt{25 \times 49} = \sqrt{(5 \times 5) \times (7 \times 7)} = 5 \times 7 = 35$$

$$\text{Ex. 4. } \sqrt[3]{216} = \sqrt[3]{8 \times 27} = \sqrt[3]{(2 \times 2 \times 2) \times (3 \times 3 \times 3)} = 2 \times 3 = 6$$

From the foregoing examples is deduced

PRINCIPLE I. *Each prime factor of the root of a perfect power is a factor in the power as many times as there are ones in the index of the root.*

Ex. 5. What is the cube root of 1,728?

PROCESS.

$$1,728 = (2 \times 2 \times 2) \times (2 \times 2 \times 2) \times (3 \times 3 \times 3); 2 \times 2 \times 3 = 12$$

STEPS. 1st,—Resolve 1,728 into its prime factors;

2d,—Separate the factors into groups, each consisting of as many like factors as are indicated by the index of the root,—viz., 3.

3d,—For the required root, find the continued product of one factor from each group.

PROBLEMS.

1. $\sqrt{4,900}$	4. $\sqrt[3]{592,704}$	7. $\sqrt{207.36}$	10. $\sqrt[3]{16,581.375}$
2. $\sqrt{5,184}$	5. $\sqrt[4]{5308416}$	8. $\sqrt[3]{21.952}$	11. $\sqrt[3]{\frac{3276}{24878}}$
3. $\sqrt[3]{46,656}$	6. $\sqrt[5]{14,348,907}$	9. $\sqrt{.6084}$	12. $\sqrt{44\frac{3140}{17889}}$

Ex. 6. $\sqrt{64} = \sqrt{4 \times 16} = \sqrt{4} \times \sqrt{16} = 2 \times 4 = 8$

Ex. 7. $\sqrt[3]{216} = \sqrt[3]{8 \times 27} = \sqrt[3]{8} \times \sqrt[3]{27} = 2 \times 3 = 6$

From the foregoing illustrative examples is deduced

PRINCIPLE II. *Any root of a perfect power equals the product of like roots of the rational factors that produce the power.*

Ex. 8. Apply Principle II in finding the square root of 1,296.

ANALYTICAL PROCESS.

$$\sqrt{1,296} = \sqrt{4 \times 4 \times 81} = \sqrt{4} \times \sqrt{4} \times \sqrt{81} = 2 \times 2 \times 9 = 36$$

EXPLANATION.—I divide 1,296 by the perfect square, or rational factor, 4; the result, 324, by the rational factor 4; and obtain the rational factor 81. I then write the roots, 2, 2, and 9, of these rational factors in a line, and find their continued product, 36, which is the required root.

PROCESS.

$$\begin{array}{r|rr} 1,296 & 324 & 81 \\ \hline 4 & 4 & \\ 2 & \times & 2 \times 9 = 36 \end{array}$$

Ex. 9. Find the cube root of 74,088.

STEPS. 1st.—Divide by rational factor 8;

2d.—Divide result by rational factor 27;

3d.—Extract cube roots of rational factors, 8, 27, 843;

4th.—Find product of the roots, 2, 3, 7. Result, 42.

PROCESS.

$$\begin{array}{r|rr} 74,088 & 9,261 & 843 \\ \hline 8 & 27 & \\ 2 & \times & 3 \times 7 = 42 \end{array}$$

Apply Principle II in extracting the indicated roots in the following

PROBLEMS.

13. $\sqrt{3,969}$	16. $\sqrt{12.25}$	19. $\sqrt[3]{.250047}$	22. $\sqrt{12\frac{1}{4}}$
14. $\sqrt{11,025}$	17. $\sqrt[3]{42,875}$	20. $\sqrt[3]{9.261}$	23. $\sqrt[3]{\frac{3}{1111}}$
15. $\sqrt{.1764}$	18. $\sqrt[3]{216,000}$	21. $\sqrt{\frac{49}{144}}$	24. $\sqrt[3]{2,460\frac{1}{8}}$

II. NUMBERS OF PLACES IN ROOTS OF DECIMAL NUMBERS.

341. It is evident that $587 = 58 \text{ tens} + 7 \text{ ones}$, $109 = 10 \text{ tens and } 9 \text{ ones}$, $3,245 = 324 \text{ tens and } 5 \text{ ones}$, and so on.

It is also evident that $.25 = 2 \text{ tens and } 5 \text{ ones of hundredths}$, $.587 = 58 \text{ tens and } 7 \text{ ones of thousandths}$, $.3245 = 324 \text{ tens and } 5 \text{ ones of ten-thousandths}$, and so on. Also, that $2.5 = 2 \text{ tens and } 5 \text{ ones of tenths}$, $58.7 = 58 \text{ tens and } 7 \text{ ones of tenths}$, $5.87 = 58 \text{ tens and } 7 \text{ ones of hundredths}$, and so on. Hence,

Any decimal number expressed by more than one figure may be resolved into two parts, viz.: *ones*, expressed by the right-hand figure; and *tens*, expressed by all the figures at the left of ones.

342. From 329, Prin. III, IV, and 330, Prin. V, are deduced the following

IMPORTANT FACTS.

I. *The number of places in any required root of an integer can be determined, by pointing off the integer into periods of as many places each as are indicated by the index, counting from the ones toward the left.*

The left-hand period may be either *full* or *partial*; i. e., it may or may not contain as many places as are indicated by the index.

II. *The number of places in any required root of any decimal can be determined, by pointing off the decimal into periods of as many places each as are indicated by the index, counting from tenths toward the right.*

All decimal periods must be *full*; i. e., each must contain as many places as are indicated by the index.

III. *The number and kinds of places, integral or decimal, in any required root of any mixed decimal number can be determined, by pointing off the number into periods, as stated in Facts I and II.*

IV. *The required power of the left-hand figure of the root is found wholly in the first or left-hand period of the number; of the first two figures of the root, in the first two periods of the number; of the first three figures, in the first three periods; and so on.*

V. *Any power of the first figure of a root is the greatest like power found in the first period of a power of the same degree.*

Ex. 1. How many integral places are there in the cube root of 75,350,946,733,343 ?

PROCESS.

EXPLANATION.—Beginning at ones and pointing the number off into periods of three figures each (the number indicated by the index), I have five periods,—four full and one partial. Hence, there are five places in the root.

Ex. 2. What is the first figure of the required root in example 1 ?

Having determined the figures that compose the first period of the power, we see by inspection that the greatest integral cube number in the left-hand period (75) is 64; and the cube root of 64 is 4. Hence, 4 is the first figure of the required root.

PROCESS.

Ex. 3. How many places are there in $\sqrt{8,654,378.934625}$, and what is the first figure of the required root?

$8'65'43'78'.93'46'25'$
Hence, 7 places,—4 integral and 3 decimal.
 $2^2 = 4$, and $3^2 = 9$; hence,
2 is the first figure of the required root.

PROBLEMS.

Find the number of integral places, and the first figure of the required root,

In the square root { 25. Of 784 | 27. Of 148,225 | 29. Of 7,215,019
26. Of 6,849 | 28. Of 99,325 | 30. Of 40,300,805

In the cube root { 31. Of 1,331 | 33. Of 1,860,867
32. Of 91,125 | 34. Of 1,331,000,000

35. In the 4th root of 148,225. | 36. In the 6th root of 99,325.

37. In the 5th root of 187,642,875,945,815,438,125.

Find the number of places of each kind—integral and decimal—and the first figure of the required root, in

38. The square root of 7,835,468.9344. | 40. $\sqrt[4]{18,675.93462376}$.
39. The cube root of 18,643,257.94856. | 41. $\sqrt[5]{345.861432789463172}$.

Demonstrate the following

PROPOSITIONS.

- I. All roots of 1 are 1.
- II. All roots of powers greater than 1 are less than their powers.
- III. All roots of powers less than 1 are greater than their powers.

SECTION III.

EXTRACTION OF SQUARE ROOT.

The practical applications of square and cube roots are much more numerous than those of roots of higher degrees. Also, a knowledge of the principles upon which the extraction of square and cube roots is based is necessary to a thorough understanding of the principles and laws that govern roots of higher degrees. Hence, these two subjects require special consideration.

All roots expressed by one figure, and their squares and cubes, will be found in the completed table (328). When expressed by two or more figures, the methods of determining the number of figures, and the first figure, have already been shown (330).

343. $34^2 = (30+4)^2 = 30^2 + 2 \times (30 \times 4) + 4^2 = 900 + 240 + 16 = 1,156$. Pointing off this power into two-figure periods, 11'56, and comparing the parts with the power, it will be seen

(1) That the second power (9) of the tens (3) of the root is wholly in the first period of the power (11).

(2) That the product of 2 times the tens (3) and the ones (4), *i. e.*, $(2 \times 3 \times 4 =)$ 24 is wholly in that part of the power expressed by the first period of the power minus the square of the tens of the root $(11 - 9 =)$ 2 with the tens (5) of the second period of the power annexed, 25; and $25 - (2 \times 3 \times 4 = 24 =)$ 1.

(3) That the second power (16) of the ones (4) is wholly in that part of the power expressed by the 25 minus 2 times the product of the tens (3) and the ones (4) of the root, *i. e.*, $(2 \times 3 \times 4 = 24)$, and $25 - 24 =$ 1, with the ones (6) of the second period annexed, 16; and $16 - 4^2 = 0$. Hence,

344. IMPORTANT FACTS.

I. The product of two times the tens and the ones of the square root of a decimal number is wholly in the number formed by subtracting the square of the tens from the first period of the power, and to the remainder annexing the tens of the second period of the power.

II. *The square of the ones is wholly in the number formed by subtracting the product of two times the tens and the ones from the preceding minuend, and to the remainder annexing the ones of the second period of the power.*

345. Ex. 1. Extract the square root of 4,489.

ANALYSIS.—Separating the number into periods of two figures each, I find that the root will be expressed by two figures (342, I).

Since the square of the tens of the root is wholly in the first period of the given number or power, and is the greatest square in that period (342), I write 36 (the greatest square in 44) under 44; and 6 (the square root of 36) for the tens of the root.

Subtracting 36 from 44, and to the remainder annexing the tens of the second period, I have the minuend 88.

This number wholly contains the second part of the power, or the part that is composed of the two factors ($2 \times$ the tens and the ones) (344, I). Two times the tens, or 12, being one factor of 88, the other factor is found by dividing 88 by ($2 \times$ the tens =) 12, which gives 7 and a remainder. Hence, I write 7 for the ones of the root.

($3 \times$ tens $\times$ ones =) $2 \times 6 \times 7 = 84$. Subtracting 84 from 88 and to the remainder annexing the ones of the second period of the power, I have the minuend 49 (344, II).

(Ones² =) $7^2 = 49$, which I subtract from 49, and all of the given number has been used.

Hence, 67 is the required root.

Ex. 2. Extract the square root of 610.09.

OUTLINE OF ANALYSIS.—Three periods,—two integral and one decimal; hence, three places in the root,—two integral and one decimal.

Greatest square in first period, 4; first figure of root, ($\sqrt{4} =$) 2.

First figure of root regarded as tens, 2; first divisor, ($2 \times 2 =$) 4; second figure of root, 4; first remainder with tens of second period of power annexed, 21; $2 \times$ tens $\times$ ones, 16; second remainder with ones of second period of power annexed, 50.

PROCESS.

$$\begin{array}{r}
 44'89 \overline{)67} \\
 t^2 = 6^2 = 36 \\
 2 \times t = 2 \times 6 = 12 \overline{)88} \\
 2 \times t \times o = 2 \times 6 \times 7 = 84 \\
 o^2 = 7^2 = 49
 \end{array}$$

PROCESS.

$$\begin{array}{r}
 6'10.09 \overline{)24.7} \\
 t^2 = 2^2 = 4 \\
 2 \times t = 2 \times 2 = 4 \overline{)21} \\
 2 \times t \times o = 2 \times 2 \times 4 = 16 \\
 o^2 = 4^2 = 50 \\
 2 \times t = 2 \times 24 = 48 \overline{)340} \\
 2 \times t \times o = 2 \times 24 \times 7 = 336 \\
 o^2 = 7^2 = 49
 \end{array}$$

Ones³, 16; third remainder with tens of third period of power annexed, 340,—which becomes a dividend in finding the units of the next order (or the third figure) of the root.

First two figures of root regarded as tens, 24; second divisor ($3 \times 24 =$), 48; third figure of root, 7; $2 \times$ tens (24) $\times$ ones (7), 336; first remainder with ones of third period of power annexed, 49; square of ones, 49; no remainder.

Hence, required root, 24.7.

PROCESS.

Ex. 3. Extract the square root of $4\frac{1}{4}$.

$$\sqrt{4\frac{1}{4}} = \frac{\sqrt{529}}{\sqrt{164}} = \frac{23}{12}$$

346. STEPS IN EXTRACTING SQUARE ROOT.

- I. *Separate the number into periods of two figures each.*
 - II. *Find tens of root; subtract square of tens from first period; and form dividend.*
 - III. *Divide by two times tens of root, for ones of root; subtract two times tens times ones of root; form new subtrahend; and subtract square of ones of root.*
- If there is a remainder, regarding that part of the root now found as tens of the required root,
- IV. *Form new dividend and new divisor, and proceed as with first dividend and first divisor for units of next and each succeeding place in the root.*

a. If any dividend is less than the corresponding divisor:—

Write a cipher in the root, and for a new dividend annex to the dividend the ones figure of the period used.

b. If there is a remainder after all the periods have been used:—

Annex periods of decimal ciphers, extend the work to any required degree of exactness, and write + after the root obtained, to indicate that it is not exact.

c. If the right-hand decimal period contains but one figure:—

Make it a full period by annexing a decimal cipher.

d. To extract the square root of a fraction:—

Extract the square root of each term separately.

e. To extract the square root of a mixed fractional number:—

First reduce it to a mixed decimal number or to an improper fraction.

In extraction of roots, the decimal point is disregarded after the number is separated into periods.

PROBLEMS.

1. Supply the omitted parts, and write outlines of the completed process of extracting the square root of 10,485.76.

PROCESS.

$$\begin{array}{r}
 1'0485.76 \quad | \quad 102.4 \\
 = 2 \sqrt{0} \\
 = \quad 0 \\
 \quad \quad 4 \\
 = \quad 0 \\
 = 20 \sqrt{48} \\
 = \quad 40 \\
 \quad \quad 85 \\
 = \quad 4 \\
 = 204 \sqrt{817} \\
 = \quad 816 \\
 \quad \quad 16 \\
 = \quad 16
 \end{array}$$

Extract the square root of

2. 5,329	7. .46537
3. 205,209	8. 177.1561
4. 6,270,016	9. 9.9435½
5. 7,824,192	10. 43½
6. .00394384	11. ½

Find the value of

12. $\sqrt{2}$	17. $\sqrt{3.164164}$
13. $\sqrt{512}$	18. $\sqrt{111111}$
14. $\sqrt{330,625}$	19. $\sqrt{151111}$
15. $\sqrt{.910116}$	20. $\sqrt{86.111}$
16. $\sqrt{4,515.84}$	21. $\sqrt{32711}$

SHORT METHOD.

347. In the following method, three steps of the general method are omitted without increasing the difficulties of the process.

Ex. 1. Extract the square root of 4,489.

The process by the general method is fully explained in 345.

In the process by the general method it will be seen that the factors (3 and 6) of the divisor (12) and the first two factors (3 and 6) of the subtrahend (84) are the same, and that the ones (7) of the root is the third factor of 84. The process may therefore be shortened by omitting the equation $2 \times t \times o = 2 \times 6 \times 7$, and multiplying the divisor 12 ($= 2 \times 6$) by the ones (7) of the root and writing the product (84) directly under the 88. (See next page.)

PROCESSES.

1. BY THE GENERAL METHOD.

$$\begin{array}{r}
 t^2 = 6^2 = 36 \quad | \quad 67 \\
 2 \times t = 2 \times 6 = 12 \quad | \quad 88 \\
 2 \times t \times o = 2 \times 6 \times 7 = 84 \\
 o^2 = 7^2 = 49
 \end{array}$$

2. FIRST CONTRACTION.

$$\begin{array}{r}
 t^2 = 6^2 = \begin{array}{r} 44'89 \overline{)67} \\ 36 \\ \hline 2 \times t = 2 \times 6 = 12 \overline{)88} \\ 24 \\ \hline 49 \\ 49 \\ \hline \end{array} \\
 o^2 = 7^2 = \begin{array}{r} 49 \\ 49 \\ \hline \end{array}
 \end{array}$$

SHORT PROCESS.

$$\begin{array}{r}
 t^2 = 6^2 = \begin{array}{r} 44'89 \overline{)67} \\ 36 \\ \hline 2 \times t = 2 \times 6 = 12 \overline{)88:9} \\ 24 \times \\ \hline 49 \\ 49 \\ \hline \end{array} \\
 o^2 = 7^2 = \begin{array}{r} 84 \times \\ 49 \\ \hline \end{array}
 \end{array}$$

It will also be seen, on examination, that all of the power remaining after the square of the tens (36) has been subtracted, is 889; that 84 tens (= 840) of this remainder is the second part subtracted (i. e., $2 \times t \times o$), and that 49 (i. e., o^2) is the third part subtracted. The sum of 840 and 49 is 889. The process may therefore be still further shortened, by adding the two subtrahends (840 and 49) and subtracting their sum from the remainder obtained by subtracting the square of the tens of the root from the power. See Short Process, and the process illustrated on page 199.

Ex. 2. Extract the square root of 610.09.

PROCESS.

SUGGESTION.—Carefully compare the corresponding steps in the process on page 199 and the process here given, and determine the reasons

(1) For omitting from the full process the equation $2 \times t \times o$.

(2) For the use of the $\times$ in the partial remainders 16 and 336.

(3) For not writing the sums of $16 \times$ and 16, and $336 \times$ and 49.

$$\begin{array}{r}
 6'10.09 \overline{)24.7} \\
 4 \\
 \hline
 t^2 = 2^2 = 4 \overline{)24:0} \\
 16 \times \\
 \hline
 o^2 = 4^2 = 16 \\
 2 \times t = 2 \times 24 = 48 \overline{)340:9} \\
 336 \times \\
 \hline
 o^2 = 7^2 = 49
 \end{array}$$

348. Write out the steps in the short method of extracting square root, on the plan given in **346**.

PROBLEMS.

Extract the square root

22. Of 5,476.

23. Of 45.0241.

24. Of .555025.

25. Of $\frac{111111}{111111}$.

26. Of 954.

Find results, correct to three decimal places, of

27. $\sqrt{859}$.

28. $\sqrt{7,744}$.

29. $\sqrt{327.61}$.

30. $\sqrt{.6}$.

31. $\sqrt{1,000}$.

32. $\sqrt{\frac{111}{111}}$.

33. $\sqrt{5\frac{1}{2}}$.

34. $\sqrt{3}$.

SECTION IV.

EXTRACTION OF CUBE ROOT.

$$\begin{aligned}
 349. \quad 34^3 &= (30 + 4)^3 \\
 &= 30^3 + 3 \times (30^2 \times 4) + 3 \times (30 \times 4^2) + 4^3 \\
 &= 27,000 + 10,800 + 1,440 + 64 = 39,304.
 \end{aligned}$$

Pointing off this power (39,304) into three-figure periods, 39'304, and comparing the parts with the power, it will be seen

(1) That the cube (27) of the tens (3) of the root is wholly in the first period of the power (39).

(2) That the product of 3 times the square of the tens (3) and the ones (4) is wholly in that part of the power expressed by the first period of the power minus the cube of the tens of the root ($39 - 27 =$) 12 with the hundreds (3) of the second period of the power annexed (123); and $123 - (3 \times 3^2 \times 4 =) 108 = 15$.

(3) That the product of 3 times the tens (3) and the square of the ones (4) is wholly in that part of the power expressed by ($123 - 108 =$) 15 with the tens (0) of the second period of the power annexed (150); and $150 - (3 \times 3 \times 4^2 =) 144 = 6$.

(4) That the cube of the ones (4) is wholly in that part of the power expressed by ($150 - 144 =$) 6 with the ones (4) of the second period of the power annexed, 64; and $64 - 4^3 = 0$. Hence,

IMPORTANT FACTS.

I. *The product of three times the square of the tens and the ones of the cube root of a decimal number is wholly in the number formed by subtracting the cube of the tens of the root from the first period of the power, and to the remainder annexing the hundreds of the second period of the power.*

II. *The product of three times the tens and the square of the ones is wholly in the number formed by subtracting the product of three times the*

square of the tens and the ones from the preceding minuend, and to the remainder annexing the tens of the second period of the power.

III. *The cube of the ones is wholly in the number formed by subtracting the product of three times the tens and the square of the ones from the last preceding minuend, and to the remainder annexing the ones of the second period of the power.*

350. Ex. 1. Extract the cube root of 103,823.

ANALYSIS.—Separating the number into periods of three figures each, I find that the root will be expressed by two figures.

Since the cube of the tens of the root is wholly in the first period of the power and is the greatest cube in that period (342, V), I write 64, the greatest cube in 103, under 103; and 4, the cube root of 64, for the tens of the root.

Subtracting 64 from 103, and to the remainder annexing the hundreds of the second period of the power, I have 398.

This number (398) wholly contains the second part of the power—or the part that is composed of the two factors $3 \times \text{tens}^2$ and ones (349, I).—3 times the square of the tens, or 48, being one factor of 398, the other factor is found by dividing 398 by 48, which—making allowance for the remaining parts of the power—gives 7 and a remainder. Hence, I write 7 for the ones of the root.

($3 \times \text{tens}^2 \times \text{ones} =$) $3 \times 4^2 \times 7 = 336$. Subtracting 336 from the first minuend 398, and to the remainder annexing the tens of the second period of the power, I have 622, the second minuend (349, II).

($3 \times \text{tens} \times \text{ones}^2 =$) $3 \times 4 \times 7^2 = 588$. Subtracting 588 from the second minuend 622, and to the remainder annexing the tens of the second period of the power, I have 343, the third minuend (349, III).

($\text{Ones}^3 =$) $7^3 = 343$. Subtracting 343 from the third minuend, there is no remainder, and all of the given power has been used.

Hence, 47 is the required root.

Ex. 2. Extract the cube root of 51,686.703125; write analysis of the process through the first two places of the root, and outlines of analysis through the remaining two places of the root.

PROCESS.

$$\begin{array}{r}
 103'823 \quad | \quad 47 \\
 \underline{64} \\
 398 \\
 \underline{336} \\
 622 \\
 \underline{588} \\
 343 \\
 \underline{343} \\
 0
 \end{array}$$

$t^3 = 4^3 = 64$
 $3 \times t^2 = 3 \times 4^2 = 48$
 $3 \times t^2 \times o = 3 \times 4^2 \times 7 = 336$
 $3 \times t \times o^2 = 3 \times 4 \times 7^2 = 588$
 $o^3 = 7^3 = 343$

PROCESS.

$$\begin{array}{r}
 51686.703125 \quad | \quad 37.25 \\
 \underline{27} \\
 3 \times t^2 = 3 \times 3^2 = 27 \quad | \quad 246 \\
 3 \times t^2 \times o = 3 \times 3^2 \times 7 = 189 \\
 \underline{578} \\
 3 \times t \times o^2 = 3 \times 3 \times 7^2 = 441 \\
 \underline{1376} \\
 o^3 = 7^3 = 343 \\
 3 \times t^2 = 3 \times 37^2 = 4107 \quad | \quad 10337 \\
 3 \times t^2 \times o = 3 \times 37^2 \times 2 = 8214 \\
 \underline{21230} \\
 3 \times t \times o^2 = 3 \times 37 \times 2^2 = 444 \\
 \underline{207863} \\
 o^3 = 2^3 = 8 \\
 3 \times t^2 = 3 \times 372^2 = 415152 \quad | \quad 2078551 \\
 3 \times t^2 \times o = 3 \times 372^2 \times 5 = 2075760 \\
 \underline{27912} \\
 3 \times t \times o^2 = 3 \times 372 \times 5^2 = 27900 \\
 \underline{125} \\
 o^3 = 5^3 = 125
 \end{array}$$

Ex. 3. Extract the cube root of $\frac{11576}{205375}$.

PROCESS.

The cube root of a fraction equals the cube root of its numerator divided by the cube root of its denominator.

$$\sqrt[3]{\frac{11576}{205375}} = \frac{\sqrt[3]{11576}}{\sqrt[3]{205375}} = \frac{22}{55}$$

351. STEPS IN EXTRACTING CUBE ROOT.

- I. *Separate the number into periods of three figures each.*
- II. *Find tens of root; subtract cube of tens from first period; and form dividend.*
- III. *Divide by three times square of tens of root, for ones of root; subtract product of three times square of tens times ones; form second subtrahend; subtract product of three times tens times square of ones; form third subtrahend; and subtract cube of ones of root.*

If there is a remainder, regarding that part of the root now found as tens of the required root,

IV. *Form new dividend and new divisor, and proceed as with first dividend and first divisor, for the units of the next place, and of each succeeding place in the root.*

a. In extracting the cube root of a number, if any subtrahend is greater than the corresponding minuend:—

Diminish the ones of the root, until a complete subtrahend is obtained less than the corresponding minuend.

b. Adapt remarks a, b, c, d, e of square root (346) to cube root.

Extract the cube root

PROBLEMS.

1. Of 2,744	6. Of 938,313,739	11. Of $\frac{1}{1111111111}$
2. Of 658,503	7. Of 350.402625	12. Of .000000125
3. Of .941192	8. Of 1,003,003,001	13. Of 2,430,968,440.536
4. Of 5,929.741	9. Of 22,069,810,125	14. Of 58,414,878.582336
5. Of 406,869,021	10. Of $\frac{1}{1000000000}$	15. Of 219.256227

SHORT METHOD.

352. Ex. 1. Extract the cube root of 103,823, writing the full process as shown in 350.

Inspecting this process, we perceive

1. That in obtaining the first root figure one subtraction is required, and in obtaining a subsequent root figure three subtractions are required.

2. That each of the three minuends consists of the remainder obtained by the last preceding subtraction, with the next figure of the power annexed.

SUGGESTION. — Carefully compare the corresponding steps in the two processes, and determine the reasons

(1) For omitting the changing of the equation in the full process,

$3 \times t^2 \times o = 3 \times 4^2 \times 7 = 336$,
to the equation in the short process,

$o \times div. = 7 \times 48 = 336$.

SHORT PROCESS.

$$\begin{array}{r}
 103'823 \quad | \quad 47 \\
 t^3 = 4^3 = \quad 64 \\
 3 \times t^2 = 3 \times 4^2 = 48 \quad | \quad 398:23 \\
 o \times div. = 7 \times 48 = \quad 336 \times \\
 3 \times t \times o^2 = 3 \times 4 \times 7^2 = \quad 588 \times \\
 o^3 = 7^3 = \quad 343
 \end{array}$$

(2) For writing the three subtrahends, adding them, and subtracting their sum from the dividend as in the short process, instead of performing the three subtractions as in the common process.

(3) For omitting the sum of the three subtrahends (89,828), see 350.

Ex. 2. Extract the cube root of 10,174, correct to 2 decimal places.

ANALYTICAL PROCESS.

	10'174'000'000	21.66 +		
$t^3 = 2^3 =$	8			
$3 \times t^2 = 3 \times 2^2 =$	12	21:74	{ 2, 1st remainder. 21, 1st dividend. 2174, 1st minuend.	
$o \times \text{divisor} = 1 \times 12 =$		12 × ×		
$3 \times t \times o^2 = 3 \times 2 \times 1^2 =$		6 ×		
$o^3 = 1^3 =$		1	{ Parts of 1st subtrahend.	
$3 \times t^2 = 3 \times 21^2 =$	1323	9130:00		{ 913, 2d remainder. 9130, 2d dividend. 913000, 2d minuend.
$o \times \text{divisor} = 6 \times 1323 =$		7938 × ×		
$3 \times t \times o^2 = 3 \times 21 \times 6^2 =$		2268 ×		
$o^3 = 6^3 =$		216	{ Parts of 2d subtrahend.	
$3 \times t^2 = 3 \times 216^2 =$	139968	963040:00		{ 96304, 3d remainder. 963040, 3d dividend. 96304000, 3d minuend.
$o \times \text{divisor} = 6 \times 139968 =$		839808 × ×		
$3 \times t \times o^2 = 3 \times 216 \times 6^2 =$		23328 ×		
$o^3 = 6^3 =$		216	{ Parts of 3d subtrahend.	
		12089704		
			Final remainder.	

OUTLINES OF ANALYSIS.—Two integral periods; hence, two integral places in root. Annex two decimal periods of ciphers.

Greatest cube in first period, 8; first figure of root, ($\sqrt[3]{8} =$) 2; first remainder, 2; first dividend, 21.

First figure of root regarded as tens, 2; first trial divisor, ($3 \times 2^2 =$) 12; second figure of root, 1; first minuend, 2,174; (8 times tens² times ones =) 1 time divisor = 12 (hundreds), first part of first subtrahend; 3 times tens times square of ones = 6 (tens), second part of first subtrahend; cube of ones = 1, third part of first subtrahend; subtracting the sum of three parts (1,261) from the minuend = second remainder, 913.

Regarding that part of the root now found as 21 tens, the pupil should be required to continue the outlines through two more periods according to the outlines here given for two periods, closing with this statement: "Two decimal periods having been used, the work has been extended to the required degree of accuracy.

"Hence, 21.66 + is the required root, and 12.089704 is the final remainder."

353. Write out the steps in the short method of extracting cube root, on the plan given in **351**.

PROBLEMS.

16. Supply the omitted parts in the incomplete short process here given of extracting the cube root of 51,686,703,125, and write outlines of analysis of the process.

PROCESS.	
51'686'703'125	3,725
27	
27 $\overline{) 246:86}$	
189 × ×	
441 ×	
343	
4107 $\overline{) 10337:03}$	
8214 × ×	
444 ×	
8	
415152 $\overline{) 2078551:25}$	
2075760 × ×	
27900 ×	
125	

Extract the cube root

17. Of 89,184
18. Of 970,299
19. Of 10,941,048
20. Of .000019983
21. Of 256,895.581293

Find results, correct to 3 decimal places, of

- | | |
|----------------------------------|--------------------------------------|
| 22. $\sqrt[3]{2}$ | 27. $\sqrt[3]{871,164}$ |
| 23. $\sqrt[3]{3}$ | 28. $\sqrt[3]{93,437.245}$ |
| 24. $\sqrt[3]{5}$ | 29. $\sqrt[3]{461,002.0056}$ |
| 25. $\sqrt[3]{7}$ | 30. $\sqrt[3]{\frac{44127}{155316}}$ |
| 26. $\sqrt[3]{\frac{112}{1008}}$ | 31. $\sqrt[3]{8,663\frac{1}{8}}$ |

For roots of higher degrees,
see Supplement, pages 413-415.

SECTION V.

REVIEW.

354. PROBLEMS.

Find the fourth power $\left\{ \begin{array}{l|l|l} 1. \text{ Of } 3^3 & 3. \text{ Of } 2^5 & 5. \text{ Of } (\frac{3}{4})^3 \\ 2. \text{ Of } 5^4 & 4. \text{ Of } .03^3 & 6. \text{ Of } \frac{21}{4}^3 \end{array} \right.$

Expand 7. $.4^3$. | 8. $.9^4$. | 9. 12^3 . | 10. 1.2^3 . | 11. $.12^3$. | 12. $.012^3$.

13. Find the difference between 4^3 and 3^4 .

14. $7^3 \times 7^5 \times 7^2 \times 7^4$ is equivalent to what power of 7?

15. $\frac{15^3 \times 15^3}{15^5}$ = what power of 15? | 17. What is the sixth power of $4\frac{1}{16}$?
 16. $2.3^7 \div 2.3^4$ = what power of 2.3? | 18. $9^3|^5$ = what power of 9?
 19. $\frac{33^3 \times 33^5 \times 33^8}{33^4 \times 33^6}$ = what power of 33?

20. The area of a public park that is square is 5 acres. What is the length of each side?

21. What are the dimensions of a square floor whose area is 540 square feet?

22. The capacity of a cubical water tank at a RR. station is 18 cubic inches less than 9,078 gallons. Find the inside dimensions of the tank.

23. To pave a square court with Belgian pavement, 59,876,471 paving stones—each an 8-inch cube—are required. What is the size of the court?

24. If the paving stones (problem 23) were piled in a cubical pile, what would be its size?

25. The base of the Great Pyramid of Cheops is a square which covers 13 acres. What is the length of a side of the base?

What is the length of one side of a square piece of land that contains

26. 1 acre? | 27. $17\frac{1}{2}$ acres? | 28. 2,304 square rods?

29. A merchant sold dress goods for \$37.82 $\frac{1}{2}$. The number of yards sold equalled the price per yard. What was the price?

30. Find the interior edge of each of two cubical boxes, one of which holds a gallon of water, and the other a bushel of wheat.

31. Find one dimension of a cube whose volume is 27 times that of a cube 2 feet 9 inches on each edge.

32. The entire surface of the 6 equal square sides of a box is 373,248 square inches. What are the dimensions of the box?

33. If the box (problem 32) is made of 1-inch lumber, what is its capacity, in bushels?

34. Last year the aggregate attendance at a certain school was 38,025 days, and the average daily attendance equalled the number of days in the school year. What was the average daily attendance?

35. A cubical bin in a grain elevator holds 28,400 bushels. What are its dimensions, inside measurement?

36. A grocer shipped 126,736 eggs in barrels, and the number of eggs in a barrel was equal to the number of barrels. How many barrels of eggs did he ship?

37. The pedestal of a monument is a cube of granite whose volume is 10 cu. yd. 4 cu. ft. 1,080 cu. in. What is the area of one of its faces?

38. A general placed the soldiers of his division in a square. Being reinforced by 8 times the first number of men, he placed the whole number in a square; and then being reinforced by 3 times his second number, he placed the total number in a square that had 300 men on a side. How many men had he at first?

39. 435,600 cabbages are growing in a field, and the number of rows equals the number of cabbages in a row. How many cabbages are in each row?

40. What is the length of one edge of a cubical block that contains 7 cu. ft. 71 cu. in.?

41. In a cornfield are 55,696 hills, and the number of hills in a row equals the number of rows. How many rows are there in the field?

42. A square roof is covered with 24,649 shingles, laid 4 inches to the weather. What is the size of the roof?

43. The annual expenses of a club are \$7,569, and each member's share is as many dollars as there are members. What is the number of members?

44. A lumberman paid \$56,406.25 for a valuable tract of timber land, the price per acre being as many dollars as there were acres in the tract. What was the price per acre?

45. 76 A. 18 sq. rd. 18.75 sq. yd. of land is divided into three equal squares. How many chains and links does a side of each measure?

46-48. A merchant paid \$911.25 for a case of dress goods. The numbers expressing the cost per yard, the number of yards in a piece, and the number of pieces, were equal. How many pieces were there? How many yards were there? What was the price per yard?

49. Some of the military townships in the State of New York, granted by the General Government to soldiers of the Revolution, contain 64,000

acres each. What is the length, in rods, of a side of one of these townships that is square?

50. What is one dimension of a cubical pile of common bricks sufficient for the walls of a house 22 feet front, 36 feet deep, and 24 feet high above the water-tables,—the walls to be $1\frac{1}{2}$ bricks thick for the first 12 feet in height, and 1 brick thick the remainder of the height, allowing $\frac{1}{4}$ of the solid contents of the walls to be of mortar, and making no allowance for openings (windows and doors)?

355. TEST QUESTIONS.

Pupils should now be required to carefully prepare twenty questions, in sets of ten each, for the review of Powers and Roots.

356. BLACKBOARD OUTLINES.

In Section I of this chapter are five terms, seven principles, and two general formulæ; in Section II are ten terms, two principles, and three important facts; in Section III are two important facts, and the steps in the general method—and also in the short method—of extracting square root; and in Section IV are three important facts, and the steps in the general method—and also in the short method—of extracting cube root.

The pupil should now be required to collect these terms, principles, formulæ, facts, and steps, and arrange them properly as blackboard outlines of Powers and Roots.

CHAPTER VII.

RATIO, PROPORTIONS, AND PROGRESSIONS.

It is unfortunate that in the science of Arithmetic some terms are used in entirely different senses. This is especially true of the terms *ratio* and *proportion*, each being used in both an abstract sense and a concrete or applied sense. The pupil must be prepared to distinguish readily in which use these terms are used.

SECTION I.

RATIO.

357. Define comparative ratio (315).

358. The *sign* of ratio is the colon (:) placed between two numbers.

Thus, 8:4 expresses the ratio of 8 to 4, and is read,—“the ratio of 8 to 4;” “8 is to 4;” or “8 divided by 4.”

a. A *couplet* is two numbers connected by the colon to express ratio.

b. In its abstract sense, *ratio* means the measure of relation, and is sometimes called *direct ratio*. In this sense ratio is always simple.

Thus, the ratio of 8 to 4 is 2; of \$8 to \$4 is 2; of 8 ft. to 4 ft. is 2.

c. In its concrete sense, *ratio* means a couplet.

359. The *terms* of a ratio, or couplet, are the numbers compared.

The *antecedent* is the first term; the *consequent* is the second term.

a. In expressing the ratio of 12 to 4,—12:4 is a couplet; 12 and 4 are the terms; 12 is the antecedent; and 4 is the consequent.

b. The *antecedent* is the number to be measured or compared; and

The *consequent* is the unit or standard with which the antecedent is measured or compared.

c. *Ratio is found by dividing the antecedent by the consequent.**

d. Ratio may be expressed in the form of a fraction by making the antecedent the numerator and the consequent the denominator.

Some arithmeticians teach that ratio is found by dividing the consequent by the antecedent, and erroneously name the process “*The French Method*.” A majority of the French mathematicians use “*The English Method*,” which is the method used in this work.

EXERCISES.

Write—first as a couplet and then as a fraction—the ratio of

- | | | | | |
|------------|------------------------------------|---------------------------------------|-------------------|-----------------------|
| 1. 10 to 5 | 5. 7 to 5 | 9. $\frac{2}{3}$ to $\frac{1}{2}$ | 13. .3 to .16 | 17. 3.105 to 2.1 |
| 2. 17 to 9 | 6. 8 to 12 | 10. $4\frac{2}{3}$ to $18\frac{1}{2}$ | 14. .97 to 345 | 18. 84.125 to 4.875 |
| 3. 8 to 26 | 7. $\frac{3}{4}$ to $\frac{1}{2}$ | 11. $25\frac{1}{2}$ to $\frac{2}{3}$ | 15. .001 to .0005 | 19. 18 bu. to 45 bu. |
| 4. 5 to 7 | 8. $\frac{2}{3}$ to $1\frac{1}{2}$ | 12. $\frac{2}{3}$ to $5\frac{1}{2}$ | 16. 18.2 to .07 | 20. .45 ft. to 31 ft. |
21. $7\frac{1}{2}$ lb. to 40 lb. | 22. 9 da. to 13 h. 42 min.

The use of the term *ratio* in the concrete sense has led to the expressions *simple ratio* and *compound ratio*. Using the term in this sense,

360. A *simple ratio* is a single couplet;—as 9:5; and

361. A *compound ratio* is two or more couplets the products of whose corresponding terms form a single couplet;—as

$$(4:3) \times (6:5) = (4 \times 6):(3 \times 5) = 24:15; \text{ or}$$

$$\left. \begin{array}{l} 4:3 \\ 6:5 \end{array} \right\} = 24:15; \text{ or } \left. \begin{array}{l} 4:3 \\ 6:5 \end{array} \right\} = \frac{4}{6} \times \frac{3}{5} = \frac{2}{5} = 24:15.$$

362. An *inverse ratio* is the reciprocal of a true or direct ratio. Thus, the direct ratio of 15 to 3 is 5; the inverse ratio of 15 to 3 is $\frac{1}{5}$,—the reciprocal of 5.

a. Inverse ratio is sometimes called *reciprocal ratio*.

b. An inverse or reciprocal ratio can be found, by dividing the consequent of a direct ratio by the antecedent.

c. The ratio of similar fractions is the ratio of their numerators.

PROBLEMS.

Find the ratio expressed by each of the following couplets:—

- | | | | | |
|-----------|-----------|-----------|-------------|----------------------------------|
| 1. 24:8 | 4. 810:90 | 7. 5:65 | 10. 125:500 | 13. $16\frac{2}{3}:100$ |
| 2. 27:3 | 5. 256:8 | 8. 18:72 | 11. 91:367 | 14. 90: $7\frac{1}{2}$ |
| 3. 144:24 | 6. 324:18 | 9. 34:170 | 12. 31:121 | 15. $66\frac{2}{3}:8\frac{1}{2}$ |
16. .6:.25 | 17. 8.6:172 | 18. 19.25:4 $\frac{1}{2}$

Reduce each of the following couplets to its lowest terms:—

$$\text{Thus, } 16:44 = 16 \div 4:44 \div 4 = 4:11.$$

19. 32:96	21. $18\frac{1}{2}:150$	23. 62.5:1.25	25. $94\frac{2}{3}:107\frac{1}{3}$	27. 8.26: $7\frac{1}{3}$
20. 810:270	22. $37\frac{1}{2}:6\frac{1}{2}$	24. .875:6.25	26. 26.5: $43\frac{1}{2}$	28. 3,456:13,184

Find the ratio expressed in each of the following couplets (by making the fractions in a couplet similar), and prove the result:—

Thus: { Finding the ratio, $-\frac{1}{2}:\frac{1}{4} = \frac{1\frac{1}{2}}{1\frac{1}{2}}:\frac{1\frac{1}{2}}{1\frac{1}{2}} = 35:24 = 1\frac{1}{4}$.
 { Proving the process, $-\frac{1}{2}:\frac{1}{4} = \frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1} = \frac{4}{2} = 1\frac{1}{4}$.

29. $\frac{2}{3}:\frac{1}{4}$	31. $\frac{1}{2}:\frac{1}{15}$	33. $3\frac{1}{2}:10$	35. $4\frac{1}{2}:8\frac{1}{2}$	37. $8\frac{1}{2}:\frac{1}{15}$	39. $\frac{1}{4}:7\frac{1}{15}$
30. $\frac{2}{3}:\frac{1}{11}$	32. $\frac{1}{2}:\frac{1}{5}$	34. $6\frac{1}{2}:25$	36. $7\frac{1}{2}:11\frac{1}{2}$	38. $9\frac{1}{10}:5\frac{1}{2}$	40. $25\frac{1}{2}:\frac{1}{2}$

363. Ratio may exist between numbers that can be reduced to the same unit value; but they must be reduced to like numbers before the ratio can be determined.

Thus,
 3 yd.: 9 in. = 108 in.: 9 in. = 12;—or
 3 yd.: 9 in. = 3 yd.: $\frac{1}{4}$ yd. = 12.

Find the ratio of

41. 4 yd. to 3 ft.	45. 4 wk. to 20 min.
42. 2 in. to 5 yd.	46. 7 gal. to 5 pt.
43. 6 cwt. to 12 lb.	47. 4 ch. to 36 ft.
44. 8 h. to 3 da.	48. 3 bu. to 3 pk.
49. 18 lb. 6 oz. to 1 T. 670 lb.	
50. 3 mi. 157 rd. 2 yd. to 65 rd. 1 yd. 2 ft.	
51. 3 yr. 5 mo. 20 da. to 7 yr. 1 mo. 16 da.	
52. 3 bar. 16 gal. 1 qt. to 9 gal. 3 qt. 1 pt.	

Combine into a compound ratio, and reduce to a simple ratio in its lowest terms,—

53. 3:5, 4:6, 8:9, 5:12.
54. 24:25, 17:9, 5:34, 18:8.
55. 7:21, 15:63, 42:35, 18:28, 9:1.
56. 5:17, 13:42, 34:65, 3:11, 88:21.

Reduce to a single couplet in its lowest terms,—

57. $(18:7) \times (81:36) \times (24:27)$	59. $\overline{84:35} \times \overline{70:21} \times \overline{5:2} \times \overline{30:14}$		
58. $\overline{3:126} \times \overline{93:31} \times \overline{15:80}$	60. $(\frac{1}{2}:\frac{1}{3}) \times (\frac{1}{4}:\frac{1}{5}) \times (\frac{1}{6}:\frac{1}{7}) \times (\frac{1}{8}:\frac{1}{9})$		
61. $\left\{ \begin{array}{l} 4:13 \\ 9:5 \\ 39:64 \end{array} \right.$	62. $\left\{ \begin{array}{l} 22:81 \\ 9:2 \\ 27:33 \end{array} \right.$	63. $\left\{ \begin{array}{l} 45 \text{ to } 86 \\ 12 \text{ to } 35 \\ 129 \text{ to } 18 \end{array} \right.$	64. $\left\{ \begin{array}{l} 14 \text{ ft. } 9 \text{ in.} \\ 16 \text{ yd. } 3 \text{ mi.} \\ 1 \text{ mi. } 88 \text{ rd. } 5\frac{1}{2} \text{ ft.} \end{array} \right.$

Find the inverse ratio of	65. 3:7	68. 8:19	71. 60: $7\frac{1}{2}$	74. .65:39.5
	66. 8:15	69. $\frac{1}{2}:\frac{1}{15}$	72. $66\frac{2}{3}:13\frac{1}{3}$	75. 5 bu.:7 qt.
	67. 17:4	70. $13\frac{1}{2}:40$	73. 9.1:13	76. 45 min.:3 da.

364. Since the ratio is the quotient of the antecedent divided by the consequent,

The antecedent equals the product of the consequent multiplied by the ratio. And.

The consequent equals the quotient of the antecedent divided by the ratio.

Find the antecedent, when

- 77.** The consequent is 5 and the ratio is 13. | **79.** Consequent 26, and ratio 7.
78. The consequent is 17 and the ratio is 11. | **80.** Consequent $37\frac{1}{2}$, and ratio .25.

Find the consequent, when

- 81.** The antecedent is 72 and the ratio is 9. | **83.** Antecedent 1,296, and ratio 72.
82. The antecedent is 414 and the ratio is 23. | **84.** Antecedent 3,184, and ratio .76.

365. APPLICATIONS OF THE GENERAL PRINCIPLES OF DIVISION.

i. *Multiplying*

multiplies

- (1) *The dividend* }
 (2) *The numerator* } *by any number* { *the quotient*
 (3) *The antecedent* } { *the fraction* } *by that number.*
 { *the ratio*

ii. *Dividing*

divides

- (1) *The dividend* }
 (2) *The numerator* } *by any number* { *the quotient*
 (3) *The antecedent* } { *the fraction* } *by that number.*
 { *the ratio*

iii. *Multiplying*

divides

- (1) *The divisor* }
 (2) *The denominator* } *by any number* { *the quotient*
 (3) *The consequent* } { *the fraction* } *by that number.*
 { *the ratio*

iv. *Dividing*

multiplies

- (1) *The divisor* }
 (2) *The denominator* } *by any number* { *the quotient*
 (3) *The consequent* } { *the fraction* } *by that number.*
 { *the ratio*

v. *Multiplying or dividing*

does not change

- (1) *Both dividend and divisor* }
 (2) *Both numerator and denominator* } *by the same number* { *the quotient.*
 (3) *Both antecedent and consequent* } { *the fraction.*
 { *the ratio.*

SECTION II.

PROPORTION.

366. Proportion, in its abstract sense, is an equality of ratios.

Comparing *numbers* determines *ratio*, or *relation*;

Comparing *ratios* determines *proportion*, or *equality*.

367. The *sign of proportion* is the double colon ($::$) written between two ratios. It is read "as" or "equals."

a. The ratios of $9:3$ and $12:4$ are equal. This equality may be expressed by the sign $=$ or by the double colon.

Thus, $9:3 = 12:4$; or $9:3::12:4$.

b. The expression $9:3::12:4$ is read in full,—“the ratio of 9 to 3 equals the ratio of 12 to 4;” but ordinarily it is read “9 is to 3 as 12 is to 4.”

I. SIMPLE PROPORTION.

368. A simple proportion, in its concrete sense, is an expression of the equality of two simple ratios.

Four numbers (*i.e.*, two couplets) are required, to express a simple proportion. These numbers are classified,—

1. *As antecedents and consequents*,—when considered parts of the couplets that form the proportion;—

2. *As extremes and means*,—when considered parts of the proportion without regard to the couplets.

- | | | | | |
|----------------------------------|---|-------------------------|---|--------------------------|
| <i>a.</i> The <i>antecedents</i> | } | in a proportion are the | { | first and third terms. |
| <i>b.</i> The <i>consequents</i> | } | | { | second and fourth terms. |
| <i>c.</i> The <i>extremes</i> | } | of a proportion are the | { | first and fourth terms. |
| <i>d.</i> The <i>means</i> | } | | { | second and third terms. |

In the proportion $6:9::10:15$, which terms

- | | |
|--------------------------------|-----------------------|
| 1. Are antecedents of ratios ? | 3. Are the extremes ? |
| 2. Are consequents of ratios ? | 4. Are the means ? |

The fourth term of a simple proportion is also called the *fourth proportional*.

369. Three terms of a simple proportion being known, the fourth or unknown term can be found by computation.

Writing the proportion $18 : 6 :: 21 : 7$ in the fractional form and making the fractions similar—keeping the factors separate—we have

$$\frac{18}{6} :: \frac{21}{7} = \frac{18 \times 7}{6 \times 7} :: \frac{21 \times 6}{7 \times 6}.$$

The factors 18 and 7 of the numerator of the first similar fraction are the extremes of the given proportion, the factors 21 and 6 of the numerator of the second similar fraction are the means, and the products of the two sets of factors are equal. Hence, in any proportion,

PRINCIPLE I.—*The product of the extremes equals the product of the means.*

The unknown or missing term of a proportion is commonly found by one of the three methods following.

First Method.

370. Applying Prin. I, **369**, to the proportion $18 : 6 :: 21 : 7$; $18 \times 7 = 6 \times 21$,—we obtain $18 = \frac{6 \times 21}{7}$; and $7 = \frac{6 \times 21}{18}$. Also $6 = \frac{18 \times 7}{21}$; and $21 = \frac{18 \times 7}{6}$. Hence,

1. To find either extreme :— $\left\{ \begin{array}{l} \text{Divide the product of the means by the} \\ \text{given extreme.} \end{array} \right.$
2. To find either mean :— $\left\{ \begin{array}{l} \text{Divide the product of the extremes by the} \\ \text{given mean.} \end{array} \right.$

Second Method.

371. By the definition of a proportion, it is evident that

The ratio of the first couplet equals the ratio of the second couplet.

In the incomplete proportion $3 : 9 :: \text{—} : 36$, the third term of the proportion,—i. e., the first term of the second couplet—is missing; but in the first couplet 3 is $\frac{1}{3}$ of 9, therefore the missing term is $\frac{1}{3}$ of 36. Hence,

To find the missing term :—*Find the ratio of the given couplet; if the other given term is an antecedent, divide it by this ratio; if a consequent, multiply it by this ratio.*

Third Method.

372. It is evident upon inspection that, in any proportion

The ratio of the antecedents is equal to the ratio of the consequents.

Thus, in the proportion $5 : 20 :: 35 : 140$, the ratio of the antecedents ($5 : 35$) equals the ratio of the consequents ($20 : 140$). Hence,

To find the missing term:—*Find a number that has the same ratio to its corresponding term (antecedent or consequent) as the ratio between the other two terms taken in the same order.*

a. In general applications of the foregoing methods the numbers are considered as abstract.

b. The Second Method (371) most nearly conforms to the principles of ratio.

373. Ex. Find the missing term of the proportion $3 : 15 :: 4 : -$. PROCESSES.

EXPLANATIONS.—(1) The product of the means, 15 and 4, divided by the given extreme, 3, equals 20, the missing term (370). Or,

(2) The ratio of the complete couplet, $3 : 15$, is $\frac{1}{5}$; 4, the antecedent of the incomplete couplet, divided by $\frac{1}{5}$ equals 20, the missing term (371). Or,

(3) The antecedents (3 and 4) of the proportion are given, and their ratio is $\frac{3}{4}$; 15, the first consequent, is $\frac{3}{4}$ of 20, the second consequent and missing term (372).

$$\begin{array}{r} 1 \\ \frac{15 \times 4}{3} = 20 \\ 2 \\ \frac{3 : 15 = \frac{1}{5}}{4 \div \frac{1}{5} = 20} \\ 3 \\ \frac{3 : 4 = \frac{3}{4}}{15 \text{ is } \frac{3}{4} \text{ of } 20} \end{array}$$

PROBLEMS.

By each of the processes above shown, find the missing term in

- | | | |
|-------------------------|--|---|
| 1. $5 : 25 :: 9 : -$ | 4. $- : 468 :: 24 : 288$ | 7. $5 \text{ bu.} : - \text{ bu.} :: \$8 : \$56$ |
| 2. $7 : 63 :: - : 27$ | 5. $12 : - :: 18 : \frac{1}{3}$ | 8. $3 \text{ lb. } 5 \text{ oz.} : 36 \text{ lb. } 7 \text{ oz.} :: - : 55$ |
| 3. $51 : - :: 87 : 435$ | 6. $\frac{1}{4} : \frac{1}{3} :: .6 : -$ | 9. $85.5 : 4\frac{1}{2} :: 18 \text{ gal.} : - \text{ gal.}$ |

Using the shortest method for each, find the missing term in

- | | | |
|---|---|--|
| 10. $- : 37 :: 441 : 63$ | 12. $36 : - :: 108 : 12$ | 14. $5 : \frac{1}{5} :: - : \frac{1}{3}$ |
| 11. $- : .49 :: 36 : 28$ | 13. $42 \text{ lb.} : - \text{ lb.} :: 3\frac{1}{2} : 1\frac{1}{2}$ | 15. $7 : 41 :: - : 82$ |
| 16. $.45 : \frac{1}{10} :: 17 \text{ bu.} : - \text{ bu.}$ | 19. $5 \text{ da.} : 25 \text{ da.} :: 71 \text{ oz.} : - \text{ oz.}$ | |
| 17. $842 : 9,521 :: 421 : -$ | 20. $12.5 : .625 :: - \text{ gr.} : 100 \text{ gr.}$ | |
| 18. $4,563 : 171 :: - : 982$ | 21. $12 \text{ bu. } 3 \text{ pk.} : 3 \text{ qt.} :: \$17 : - \text{ ¢}$ | |
| 22. $1 \text{ yr. } 7 \text{ mo. } 15 \text{ da.} : - \text{ yr. } - \text{ mo. } - \text{ da.} :: \$11.70 : \$34.10$ | | |
| 23. $5 \text{ yr. } 4 \text{ mo. } 17 \text{ da.} : 2 \text{ yr. } 11 \text{ mo. } 2 \text{ da.} :: \$184.80 : \$-$ | | |

374. From problems 1 to 23, this section, may be deduced

PRINCIPLE II.—*The third term of a proportion is greater or less than the fourth, according as the first is greater or less than the second.*

375. A *mean proportional* to two numbers is a number to which the first bears the same ratio as the mean proportional bears to the second.

Thus, in $5:10::10:20$, 10 is a mean proportional to 5 and 20. This proportion is also written $5:10:20$; in this form it is known as a *continued proportion*. The last term of such a proportion is called the *third proportional*.

376. In the proportion (375) $5:10::10:20$, by Principle I (369) $10 \times 10 = 5 \times 20$; or $10^2 = 5 \times 20$. From $10^2 = 5 \times 20$ we obtain $10 = \sqrt{5 \times 20}$. Hence,

A mean proportional to two given numbers is the square root of the product of the numbers.

In making the computations, consider the numbers as abstract.

Ex. 1. Find the mean proportional to 3 and 48.

EXPLANATION.—Since a mean proportional to two numbers is the square root of their product, I find the product of 3 and 48, which is 144, and then extract the square root of 144. The result is 12, the mean proportional required.

PROCESS.

$$\begin{aligned} 3 \times 48 &= 144 \\ \sqrt{144} &= 12 \end{aligned}$$

The proportion may be expressed as a continued proportion, $3:12:48$; or as a simple proportion, $3:12::12:48$.

Ex. 2. Find the third proportional of $8:24:—$.

PROCESSES.

EXPLANATIONS.—(1) Since the square of the second or mean proportional is equal to the product of the extremes (i. e., the product of the first and third proportionals),—I square the second proportional, 24, and obtain 576; I then divide 576 by the first proportional, 8, and obtain 72, the third proportional.

$$\begin{aligned} 1 \\ 24^2 &= 576 \\ 576 \div 8 &= 72 \end{aligned}$$

(2) Since the ratio of the first proportional to the second is as the second to the third (375), I find the ratio of 8 to 24, which is $\frac{1}{3}$; then considering 24 as the antecedent of a second couplet of which the ratio is $\frac{1}{3}$, I find the consequent to be 72, the third proportional required.

$$\begin{aligned} 2 \\ 8:24 &= \frac{1}{3} \\ 24 \div \frac{1}{3} &= 72 \end{aligned}$$

PROBLEMS.

Find the mean proportional to

24. 4 and 9 26. 3 and 12 28. 1 and $\frac{1}{2}$ 30. .15 and 7.35 32. $\frac{2}{3}$ and $1\frac{1}{3}$
25. 9 and 16 27. 4 and 16 29. $2\frac{2}{3}$ and $1\frac{1}{3}$ 31. 1.905 and 8 33. 91 and 17

Find the third proportional in

34. $8:16:—$ 36. $19:133:—$
35. $23:115:—$ 37. $11:88:—$

Find the second proportional in

38. $7:—:175$ 40. $100\frac{1}{4}:—:4\frac{1}{2}$
39. $3:—:1,323$ 41. $8\frac{1}{2}:—:19\frac{1}{2}$

42. Find the length of a line that is a mean proportional to two lines that are respectively 10 rods and 160 rods long.

43. Find the side of a square that is a mean proportional to two other squares whose sides are respectively 11 feet and 275 feet.

377. Use abstract numbers and demonstrate the truth of each of the following

PROPOSITIONS.*

I. *Multiplying both terms of either couplet of a proportion by the same number does not destroy the proportion (i.e., does not destroy the equality of the ratios).*

II. *Dividing both terms of either couplet of a proportion by the same number does not destroy the proportion.*

III. *Multiplying the antecedents in a proportion by the same number does not destroy the proportion.*

IV. *Dividing the antecedents in a proportion by the same number does not destroy the proportion.*

V. *Multiplying the consequents in a proportion by the same number does not destroy the proportion.*

VI. *Dividing the consequents in a proportion by the same number does not destroy the proportion.*

VII. *Multiplying every term of a proportion by the same number does not destroy the proportion.*

VIII. *Dividing every term of a proportion by the same number does not destroy the proportion.*

IX. *Inverting the terms of a proportion (i.e., taking consequents for antecedents and antecedents for consequents) does not destroy the proportion.*

X. *Alternating the terms of a proportion (i.e., taking the antecedents for one couplet and the consequents for the other) does not destroy the proportion.*

XI. *Raising every term of a proportion to the same power does not destroy the proportion.*

XII. *Extracting the same root of every term of a proportion does not destroy the proportion.*

* These sixteen propositions are sometimes called "*The Doctrines of Proportion.*"

XIII. *If the product of two factors equals the product of two other factors, either set of factors may be made the means of a proportion and the other set the extremes.*

XIV. *Any two couplets that are each in proportion to a third couplet are in proportion to each other.*

XV. *If the four terms of a proportion are multiplied by the corresponding terms of another proportion the products form a proportion.*

XVI. *If the four terms of a proportion are divided by the corresponding terms of another proportion the quotients form a proportion.*

378. In any problem in simple proportion, three terms are given and a fourth term is required.

a. Two of the three given terms must be like numbers; and the ratio of these two terms equals the ratio of the other known term and the required term.

b. Making a statement in proportion consists in arranging the three given terms so that two of them form one couplet; the remaining term and the required term, another couplet; and the two couplets, a proportion.

Ex. 1. When 5 tons of hay cost \$20, how much will 8 tons cost?

ANALYSIS.—The required term in the proportion is the cost of 8 tons of hay; and since the cost of 5 tons is expressed in dollars, the required term will also be dollars; hence, I write \$20, the cost of 5 tons of hay, for the third term of the proportion, or for the first term of the second couplet.

Since the cost of 8 tons is greater than the cost of 5 tons, the fourth proportional will be greater than the third term,—hence, I write 5 tons and 8 tons for the first and second terms respectively of the first couplet. I now have the incomplete proportion, 5 T.: 8 T.: \$20: \$—, in which are given two means and one extreme, to find the other extreme or fourth proportional.

Solving the proportion, I have \$32, the cost of 8 tons of hay.

PROCESS.

$$\begin{aligned} 5 \text{ T.} : 8 \text{ T.} :: \$20 : \$— \\ 5 : 20 = \frac{1}{4} \\ 8 \text{ is } \frac{1}{4} \text{ of } 32 \\ \text{Hence, } \$32 \end{aligned}$$

Ex. 2. In how many hours will a railroad train run the same distance, at the rate of 30 miles an hour, that another train will run in 15 hours at the rate of 20 miles an hour?

PROCESS.

$$\begin{aligned} 30 \text{ mi.} : 20 \text{ mi.} :: 15 \text{ h.} : — \text{ h.} \\ 3 \text{ mi.} : 2 \text{ mi.} :: 15 \text{ h.} : — \text{ h.} \\ 3 \text{ is } \frac{1}{5} \text{ of } 15; \text{ and} \\ 2 \text{ is } \frac{1}{5} \text{ of } 5 \times 2 = 10 \\ 10 \text{ h., the required time.} \end{aligned}$$

ANALYSIS.—Since the required result is time, I write 15 hours, the given time, for the third term of a proportion.

Since less time is required to travel the same distance at 30 miles an hour than at 20 miles an hour, the required term or fourth proportional will be less than the third

term;—hence, I write 30 miles and 20 miles for the first and second terms respectively of the first couplet.

Solving the proportion (applying 377, Prop. I, and 372) I have 10 hours, the required time.

379. RULE FOR MAKING STATEMENTS IN SIMPLE PROPORTION.

I. For the third term:—*Write the number that is of the same kind as the required term.*

II. For the first couplet:—*Write the other two numbers,—the greater for the second term when the required term is to be greater than the third term; and the less for the second term when the required term is to be less than the third term.*

a. Before solving the proportion, the terms of the first couplet must be of the same unit value.

b. Apply propositions (377) to simplify proportions and shorten processes.

PROBLEMS.

44. If a horse trots 3 miles in 10 minutes, how far can he trot in 27 minutes, at the same rate?

45. If the rent of a farm of 320 acres is \$1,225, what should be the rent of a farm of 400 acres, of the same quality?

46. If 160 acres of land produce 3,000 bushels of wheat, 216 acres of land of the same quality should produce how many bushels?

47. If I pay a mason \$52.50 for $17\frac{1}{2}$ days work, how much should I pay him for $29\frac{1}{2}$ days work?

48. How many days can I employ the mason (Problem 47) for \$60?

49. If a staff 5 feet long casts a shadow $7\frac{1}{2}$ feet in length, what is the length of a pole that casts a shadow $79\frac{1}{2}$ feet long at the same time?

50. $\frac{4}{5}$ of my farm is worth \$1,008; what is the value of $\frac{4}{5}$ of it?

51. At 85¢ for 25 lb., what is the cost of a barrel of flour?

52. If your pulse beats 9 times in $7\frac{1}{2}$ seconds, how many times does it beat in a minute?

53. If 90 men consume 15 barrels of flour in 2 weeks, how many barrels will the same men consume in 52 weeks, at the same rate?

54. If 16° on the equator equal $1,106\frac{1}{4}$ miles, what is the equatorial circumference of the earth?

55. Sound travels 4,484 feet in 4 seconds. In how many seconds does it travel 36,993 feet?

56. If light moves 601,000 miles in $3\frac{1}{2}$ seconds, in how many seconds will it pass from the sun to the earth—a distance of 93,000,000 miles?

57. C's property is valued at \$7,500 and he pays \$63.75 taxes. What is the valuation of D's property if his taxes, at the same rate, are \$382.50?

58. If \$400 gain \$42 in a given time, what sum will gain \$132.60 in the same time, at the same rate?

59. If a certain quantity of food lasts 60 men 8 days, how many days should the same quantity last 80 men?

60. If 21 men can build a barn in $13\frac{1}{2}$ days, in how many days can 7 men build it?

61. A man borrows \$3,500 for 4 mo. 15 da. For how much time should he lend \$2,800 to balance the favor?

62. A certain sum of money will buy 36 tons of coal @ \$3.25; how many tons will the same sum buy @ \$4.50?

63. If $1\frac{1}{2}$ lb. of tea are worth \$1.12, how much are $7\frac{1}{2}$ lb. worth?

64. If $\frac{3}{8}$ yd. of cloth cost 81¢, how much will 80 yd. cost?

65. If I walk $8\frac{1}{2}$ mi. in $2\frac{1}{2}$ h., how many mi. can I walk in $8\frac{1}{2}$ h.?

66. If 18d. is worth 37¢, what is the value in U. S. money of £5 sterling?

67. On a relief map, a country 240 miles long is represented as 4 feet long. A mountain 15,480 feet high should appear how high on the same map?

II. COMPOUND PROPORTION.

380. A *compound proportion*, in its concrete sense, is an expression of equality between two ratios one or both of which are compound.

$\begin{matrix} 5 : 6 \\ 3 : 4 \end{matrix} \} :: 10 : 16$ and $\begin{matrix} 4 : 9 \\ 12 : 8 \end{matrix} \} :: \begin{matrix} 5 : 7 \\ 28 : 30 \end{matrix}$ are compound proportions.

381. A compound ratio is reduced to a simple ratio by multiplying together all the antecedents for a new antecedent and all the consequents for a new consequent (361). Hence, Principle I (369) applies to compound proportion.

382. Since the product of all the terms in the extremes equals the product of all the terms in the means,

PRINCIPLE II. *Any term in an extreme equals the quotient of the product of all the terms in the means divided by the product of all the other terms in the extremes.* And

PRINCIPLE III. *Any term in a mean equals the quotient of the product of all the terms in the extremes divided by the product of all the other terms in the means.*

383. From Principles II, III, it is evident that all the terms but one in a compound proportion being known, that term can be found by computation.

Ex. Find the missing term in $\left\{ \begin{array}{l} 4 : 8 \\ 6 : 9 \end{array} \right\} :: \left\{ \begin{array}{l} - : 3 \\ 7 : 35 \end{array} \right\}$ the compound proportion,

EXPLANATION.—I write all the terms of the extremes (4, 6, 8, 35) as factors of a dividend, and all the given terms (8, 9, 7) of the means as factors of a divisor.

Cancelling, I obtain 5, the missing term.

PROCESSES.

(1)

$$\begin{array}{c|c|c} 2 & 8 & 4 \\ 8 & 9 & 6 : 8 \\ 7 & & 3 \\ \hline & 8 & 5 \end{array}$$

(2)

$$\frac{4 \times 6 \times 8 \times 35}{8 \times 9 \times 7} = 5$$

PROBLEMS.

Find the missing term in each of the following compound proportions:

- | | | |
|--|--|---|
| 68. $\left\{ \begin{array}{l} 7 : 35 \\ 9 : 27 \end{array} \right\} :: 12 : -$ | 72. $\left\{ \begin{array}{l} 20 : 3 \\ 18 : 5 \end{array} \right\} :: \left\{ \begin{array}{l} 8 : 5 \\ - : 96 \end{array} \right\}$ | 76. $\left\{ \begin{array}{l} 2 : 2\frac{1}{2} \\ \frac{4}{5} : 1\frac{1}{10} \end{array} \right\} :: 17 : -$ |
| 69. $\left\{ \begin{array}{l} 81 : 9 \\ 50 : 25 \end{array} \right\} :: - : 72$ | 73. $\left\{ \begin{array}{l} 11 : 36 \\ 4 : - \end{array} \right\} :: \left\{ \begin{array}{l} 55 : 12 \\ 17 : 85 \end{array} \right\}$ | 77. $\left\{ \begin{array}{l} 5 : 19 \\ 8 : 24 \\ - : 35 \end{array} \right\} :: \left\{ \begin{array}{l} \frac{3}{4} : \frac{1}{11} \\ 1\frac{2}{11} : 1\frac{1}{4} \\ 5\frac{1}{2} : 17 \end{array} \right\}$ |
| 70. $\left\{ \begin{array}{l} 3 : 4 \\ 15 : 7 \\ 8 : 9 \end{array} \right\} :: - : 14$ | 74. $\left\{ \begin{array}{l} 20 : 18 \\ - : 24 \\ 48 : 40 \end{array} \right\} :: \left\{ \begin{array}{l} 52 : 15 \\ 60 : 13 \end{array} \right\}$ | 78. $\left\{ \begin{array}{l} 14 : 12 \\ .9 : 35 \end{array} \right\} :: \left\{ \begin{array}{l} 4\frac{1}{2} : 2\frac{1}{3} \\ - : \frac{8}{13} \end{array} \right\}$ |
| 71. $30 : 6 :: \left\{ \begin{array}{l} 15 : - \\ 12 : 5 \end{array} \right\}$ | 75. $29 : - :: \left\{ \begin{array}{l} 48 : 14 \\ 21 : 96 \end{array} \right\}$ | 79. $.25 : 33\frac{1}{3} :: \left\{ \begin{array}{l} 5.4 : 10.16 \\ .8 : .375 \end{array} \right\}$ |

384. Ex. If 18 men in 12 days of 8 hours each can build 480 cubic feet of masonry, 6 men in 24 days of 10 hours each can build how many cubic feet?

ANALYSIS.—Since the required term is cubic feet, I write 480 cu. ft. for the first term of the second ratio, or the third term of the proportion.

The required number of cubic feet depends upon three elements, viz.:—(1) the number of men; (2) the number of days; (3) the number of hours per day.

I therefore arrange the numbers representing these elements, in pairs, as the terms of the compound ratio. Thus, (1) 18 men build 480 cu. ft., 6 men will build less,—hence, 18 men : 6 men. (2) Any number of men in 12 days build 480 cu. ft., in 24 days they will build more,—hence, 12 da. : 24 da. (3) Any number of men in any number of days of 8 hours each build 480 cu. ft., in the same number of days of 10 hours each they will build more,—hence, 8 h. : 10 h.

Solving the proportion, I obtain 400 cubic feet, the required result.

385. RULE FOR MAKING STATEMENTS IN COMPOUND PROPORTION.

I. For the third term:—Write the number that is of the same kind as the required term.

II. For the compound ratio:—Write each two of the given numbers that are of the same kind, as a couplet, always considering whether a result depending upon these two terms alone would be greater or less than the third term; and if greater, write the greater of the two for the second term; but if less, write the less of the two for the second term.

a. Apply remarks *a*, *b*, following the rule for making statements in simple proportion (379), to compound proportion.

b. In solving problems in proportion by cancellation, use either process (383).

c. The time necessary to re-write the terms for cancellation after the compound proportion has been written, may be saved by inclosing the *means* in a rectangle, and cancelling any term or any factor of any term in the rectangle by any term or any factor of any term out of it.

PROCESS.

$$\left. \begin{array}{l} 18 \text{ men} : 6 \text{ men} \\ 12 \text{ da.} : 24 \text{ da.} \\ 8 \text{ h.} : 10 \text{ h.} \end{array} \right\} :: 480 \text{ cu. ft.} : \text{— cu. ft.}$$

$$\begin{array}{c|c} \$ 18 & 6 \\ 12 & 24 \\ 8 & 10 \\ \hline & 480 \text{ cu. ft.} \end{array} \quad \begin{array}{l} 600 \text{ cu. ft.} \\ 200 \text{ cu. ft.} \end{array}$$

$$2 \times 10 \times 20 \text{ cu. ft.} = 400 \text{ cu. ft.}$$

$$\begin{array}{l} 12 : \\ 18 : \\ 8 : \end{array} \left\{ \begin{array}{l} \text{—} \\ 45 \\ 10 \end{array} \right\} :: \left\{ \begin{array}{l} 40 \\ 8 \\ 3 \end{array} \right\} : \begin{array}{l} 200 \\ 10 \\ 4 \end{array}$$

PROBLEMS.

80. A bin 16 ft. $\times$ 7 ft. $\times$ 9 ft. contains 810 bushels of wheat. How wide is a bin 24 ft. long and 5 ft. 6 in. deep that contains 1,215 bushels?

81. If 800 yd. of cloth $\frac{1}{4}$ yd. wide make 100 suits, how many similar suits can be made from 1,200 yd. of cloth $\frac{3}{4}$ yd. wide?

82. If 18 men mow 10 acres of grass in 4 days of 10 hours each, in how many days of 8 hours each can 12 men mow 16 acres?

83. If 90 horses eat 640 bushels of oats in 40 days, 24 horses will eat how many bushels in 16 days?

84. If 40 cattle eat 280 bushels of corn in 112 days, 60 cattle will eat 350 bushels in how many days?

85. If 12 men can dig a sewer 40 rd. long, 8 ft. deep, 3 ft. wide, and of 4 deg. of hardness of soil, in 18 da. of 8 h. each; how many men will be required to dig a sewer 200 rd. long, 10 ft. deep, 4 ft. wide, and of 8 deg. of hardness of soil, in 45 da. of 10 h. each?

86. A reservoir 45 yards long and 8 feet deep holds 195,000 gallons. How many gallons will it hold when its length has been increased to 60 yards and its depth to 10 feet?

87. If a block of marble 10 feet long, $2\frac{1}{2}$ feet wide, and 16 inches thick, weighs 900 pounds; what is the weight of a block of granite 12 feet long, 8 feet wide, and 8 inches thick, the specific gravity of the granite being $1\frac{1}{2}$ times that of the marble?

88. 20 miles of railroad were to be finished in 8 months, by 120 men. One half of the time has expired, and only 8 miles have been completed. How many additional men must be employed, to finish the road at the time agreed upon?

89. If \$39.96 is the interest of \$270 for 2 yr. 5 mo. 18 da., \$16.08 is the interest of what sum for 1 yr. 8 mo. 24 da.?

90. If 8 men can lay a stone walk 420 feet long and 6 feet wide in 5 days, 12 men will lay a walk of the same width and 275 feet long in how many days?

91. If \$540 gain \$101.70 interest in 3 yr. 1 mo. 20 da., at 6 per cent, what interest will \$284 gain in 2 yr. 7 mo. 3 da., at 7 per cent?

92. If 325 reams of paper are required for 2,000 copies of an octavo book of 360 pages, how many reams will be required for an edition of 5,600 copies of a 12mo book of 628 pages?

93. If 4 pipes of equal size will fill a reservoir 68 ft. $\times$ 22 ft. $\times$ 12 ft. in 1 da. 9 h., in what time will 3 pipes of the same size fill a reservoir 54 ft. $\times$ 19 ft. $\times$ 10 ft.?

94. If in 24 days of 9 hours each 15 bricklayers can lay a wall 320 feet long, $15\frac{1}{2}$ feet high, and $1\frac{1}{4}$ feet thick; in how many days of 10 hours each can 24 bricklayers lay a wall 400 feet long, $17\frac{1}{2}$ feet high, and 2 feet thick?

III. CAUSE AND EFFECT.

With many teachers, a favorite method of solving problems in proportion is known as *cause and effect*. Cause and effect is not proportion; it is simply a method that in many instances lessens the labor of making statements in proportion. This is especially true of statements in compound proportion.

By the methods of statement already given, the third and fourth terms of a proportion are always simple; by cause and effect these terms may be either simple or compound. Hence, a statement by cause and effect better conforms to the definition of compound proportion (380), than do the methods of statement already given. For purely mathematical reasoning, however, the methods 378 and 384 are to be preferred.

386. A *cause* is that which produces a result or an effect.

387. An *effect* is the result produced by a cause.

a. Men at work are a cause, and the amount of work they do is the effect. Animals eating are a cause, and the quantity they eat is the effect. A sum of money, the time it is loaned, and the rate of interest constitute a cause; and the interest earned, the effect.

b. Causes and effects may be made interchangeable, by varying the statement of a problem. Thus, if I sell 4 barrels of flour for \$22, 4 barrels of flour are the cause and \$22 is the effect—*i. e.*, the amount the 4 barrels produce; but if I buy 4 barrels of flour for \$22, the \$22 is the cause and 4 barrels of flour are the effect—*i. e.*, what the \$22 produce.

388. A cause or an effect may contain one element or more—i.e., it may be either *simple* or *compound*. Thus, in problem 47, page 222, the money paid is the cause, and the number of days of work the money procures is the effect; while in problem 94, page 227, the work done depends for its cause upon the number of men, the number of days they work, and the number of hours per day; and the work, or effect, is measured or computed by its dimensions—i.e., by its length, breadth, and thickness.

- a. Simple causes producing simple effects may be made applications of simple proportion; and either compound causes or compound effects, or both, may be made applications of compound proportion.
- b. Causes must be similar in order to be compared; the same is also true of effects. Hence, causes must be taken for one couplet and effects for the other.
- c. In making a statement by cause and effect, the first cause must produce the first effect, and the second cause the second effect.
- d. When either the two causes or the two effects are the same, each term of the couplet may be made unity.

389. ILLUSTRATIVE EXAMPLES.

Ex. 1. When 5 tons of hay cost \$20, 8 tons will cost how much? (Page 221, Ex. 1.)

EXPLANATION.—Since \$20 buy 5 tons of hay, I write \$20 for the first cause and 5 T. for the first effect.

Since the causes must be like, and the amount required to purchase 8 tons of hay is unknown, I write \$— in the place of the second cause, and 8 T. for the second effect.

Solving the proportion, I obtain \$32, the cost of 8 T.

PROCESS.

1st cause : 2d cause :: 1st effect : 2d effect

\$20 : \$— :: 5 T. : 8 T.

$$\frac{8 \times \$20}{5} = \$32$$

Ex. 2. If a railroad train running 20 miles an hour run from Buffalo to Albany in 15 hours, in how many hours will a train run the same distance at the rate of 30 miles an hour? (Page 221, Ex. 2.)

EXPLANATION.—Since the distance to be run, or the effect, is the same in each instance, I represent the effects each by 1 (388, d).

Since the distance run depends upon two elements—the rate per hour and the number of hours,—I write 20 mi., the rate of speed, and 15 h., the

PROCESS.

1st cause : 2d cause :: 1st effect : 2d effect

20 mi. : 30 mi. } :: 1 : 1
15 h. : — h.

$$\frac{20 \times 15 \text{ h.}}{30} = 10 \text{ h.}$$

time; for the first cause; and since the two causes must contain the same elements, I write 30 mi. and — h. for the second cause.

Solving the proportion, I obtain 10 hours, the required time.

NOTE.—Observe that this problem produces a simple proportion when stated by the ordinary process (378, Ex. 2), and a compound proportion when stated by cause and effect.

Ex. 3. If 18 men in 12 days of 8 hours each can build 480 cubic feet of masonry, how many cubic feet can 6 men build in 24 days of 10 hours each?

PROCESS.

ANALYSIS.—Since the cause that produces 480 cu. ft. combines men, days, and hours per day, I write 18 men, 12 da., and 8 h. for the first cause, and 480 cu. ft. for the first effect.

Since the same elements enter into both causes, I write 6 men, 24 da., and 10 h. for the second cause, and — cu. ft. for the second effect.

Solving the proportion, I obtain 400 cu. ft., the required amount of masonry.

$$\begin{array}{l} 1st\ cause : 2d\ cause :: 1st\ effect : 2d\ effect \\ 18\ men : 6\ men \\ 12\ da. : 24\ da. \\ 8\ h. : 10\ h. \end{array} \left. \vphantom{\begin{array}{l} 18\ men : 6\ men \\ 12\ da. : 24\ da. \\ 8\ h. : 10\ h. \end{array}} \right\} :: 480\ cu.\ ft. : —\ cu.\ ft.$$

$$\frac{6 \times 24 \times 10 \times 480\ cu.\ ft.}{18 \times 12 \times 8} = 400\ cu.\ ft.$$

Ex. 4. If in 24 days of 9 hours each 15 bricklayers can lay a wall 320 feet long, $15\frac{1}{2}$ feet high, and $1\frac{1}{2}$ feet thick; in how many days of 10 hours each can 24 bricklayers lay a wall 400 feet long, $17\frac{1}{2}$ feet high, and 2 feet thick?

PROCESS.

ANALYSIS.—Since the amount of work performed depends upon three elements,—men, days, and hours per day,—I write 15 men, 24 da., 9 h. for the first cause; and since the work done is measured by three elements,—length, height, and thickness,—I write 320 ft., $15\frac{1}{2}$ ft., and $1\frac{1}{2}$ ft. for the first effect.

Since the same elements enter into both causes, I write 24 men, — da., and 10 h. for the second cause; and for a like reason, I write 400 ft., $17\frac{1}{2}$ ft., and 2 ft. for the second effect.

Solving the proportion, I obtain 30 days, the required result.

$$\begin{array}{l} 1st\ cause : 2d\ cause :: 1st\ effect : 2d\ effect \\ 15\ men : 24\ men \\ 24\ da. : —\ da. \\ 9\ h. : 10\ h. \end{array} \left. \vphantom{\begin{array}{l} 15\ men : 24\ men \\ 24\ da. : —\ da. \\ 9\ h. : 10\ h. \end{array}} \right\} :: \left\{ \begin{array}{l} 320\ ft. : 400\ ft. \\ 15\frac{1}{2}\ ft. : 17\frac{1}{2}\ ft. \\ 1\frac{1}{2}\ ft. : 2\ ft. \end{array} \right.$$

$$\frac{4 \times 4 \times 400 \times 35 \times 2 \times 15 \times 9 \times 24\ da.}{2 \times 24 \times 10 \times 320 \times 63 \times 5} = 30\ da.$$

PROBLEMS.

Solve by cause and effect the problems in Proportion, pages 222, 223, 226, 227.

IV. PARTITIVE PROPORTION.

390. *Partitive proportion* is the process of dividing a number into parts that have a given relation to each other and to the number divided.

391. *Partitive ratios* are the numbers that express the relation to one another of the several parts of a whole.

Thus, if it is required to divide a number into three parts that shall be in the ratios of 2, 3, and 5,—the numbers 2, 3, 5 are the *partitive ratios*.

392. The *unit of partition* is the number that, multiplied by the ratio of any required part, will produce that part.

393. Ex. 1. Divide 120 into three parts that shall be in the ratios of 2, 3, and 5.

ANALYSIS.—Assuming unity as the unit of partition, the parts would be represented by the ratios 2, 3, and 5, and the sum of the parts would be 10. But the sum of the parts required is 120, and the true unit of partition must be as many times unity as 120 is times 10. Hence, I divide 120 by 10, the sum of the ratios, and obtain 12, the unit of partition.

I then multiply 12 by 2, 3, and 5 (the partitive ratios), respectively, and obtain 24, 36, and 60, the parts required.

PROCESS.

$$\begin{aligned} 2 + 3 + 5 &= 10 \\ 120 \div 10 &= 12 \\ 2 \times 12 &= 24 \\ 3 \times 12 &= 36 \\ 5 \times 12 &= 60 \end{aligned}$$

The foregoing example may be stated in one equation and three simple proportions:—

ANALYSIS.—Assuming unity as the unit of partition, the parts are 2, 3, and 5, and their sum is 10.

By proportion, 10,—the sum of the parts, with unity as the unit of partition, is to each of the parts based upon the same unit of partition, as 120—the given number,—is to each one of its parts based upon the true unit of partition.

PROCESS.

$$\begin{aligned} 2 + 3 + 5 &= 10 \\ 10 : 2 :: 120 : - \\ 10 : 3 :: 120 : - \\ 10 : 5 :: 120 : - \end{aligned}$$

Solving the several proportions, I have 24, 36, and 60, the parts required.

Multiplying each of the first couplets in the proportions above given, by any number,—as 6 (Prop. I, 377),—the couplets become respectively 60:12, 60:18, and 60:30. But 12, 18, and 30 are respectively the products of 2, 3, and 5, each multiplied by 6; and 60 is the sum of 12, 18, and 30.

Again,—*dividing* both terms of each of the couplets last above obtained by 3 (Prop. II, 377), the couplets become respectively 20:4, 20:6, and 20:10. But 4, 6, and 10 are respectively the quotients of 12, 18, and 30, each divided by 3; and the sum of 4, 6, and 10 is 20. Hence,

PRINCIPLE IV.—*Multiplying or dividing all the ratios in a partitive proportion by the same number does not change the relation of the parts to each other nor to the whole.*

Ex. 2. Divide \$858 among 3 persons—L, M, and N,—in the ratios $\frac{3}{4}$, $\frac{5}{8}$, and $\frac{1}{2}$, respectively.

ANALYSIS.—To simplify the problem, I multiply each of the fractional partitive ratios $\frac{3}{4}$, $\frac{5}{8}$, and $\frac{1}{2}$ by 60, the least common multiple of the denominators,—and obtain the least integral ratios, 45, 50, and 48, respectively.

The sum of these integral ratios is 143. Dividing \$858 by 143, I obtain \$6, the unit of partition of the integral ratios; and multiplying \$6 by the several integral ratios, I obtain \$270 for L, \$300 for M, and \$288 for N.

PROCESS.

$$\begin{aligned}\frac{3}{4} : \frac{5}{8} : \frac{1}{2} &= 45 : 50 : 48 \\ 45 + 50 + 48 &= 143 \\ \$858 \div 143 &= \$6 \\ 45 \times \$6 &= \$270, L \\ 50 \times \$6 &= \$300, M \\ 48 \times \$6 &= \$288, N\end{aligned}$$

Ex. 3. William and Thomas have 85 marbles, and William has 4 times as many as Thomas (*i. e.*, their parts are in the ratio of 4 to 1). How many marbles has each boy?

ANALYTICAL PROCESS.

$$\begin{aligned}4 + 1 &= 5; 85 \text{ marbles} \div 5 = 17 \text{ marbles, the unit of partition;} \\ 4 \times 17 \text{ marbles} &= 68 \text{ marbles, belonging to William;} \\ \text{and } 1 \times 17 \text{ marbles} &= 17 \text{ marbles, belonging to Thomas.}\end{aligned}$$

Ex. 4. Three men—R, S, and T—together own 480 acres of land. S owns 3 times as many acres as R, and T 4 times as many acres as S. How many acres does each man own?

PROCESS.

ANALYSIS.—Assuming R's ratio to be 1 or unity, S's ratio must be 3 times 1 = 3, and T's ratio must be 4 times 3 = 12. The sum of these ratios is 16.

Dividing 480 A. by 16, I obtain 30 A., the unit of partition; and multiplying 30 A. by each of the ratios, I obtain 30 acres for R, 90 acres for S, and 360 acres for T.

$$\begin{aligned}1 + 3 + 12 &= 16 \\ 480 \text{ A.} \div 16 &= 30 \text{ A.} \\ 1 \times 30 \text{ A.} &= 30 \text{ A., R's share;} \\ 3 \times 30 \text{ A.} &= 90 \text{ A., S's " } \\ 12 \times 30 \text{ A.} &= 360 \text{ A., T's " }\end{aligned}$$

Although unity may be assumed for any ratio, it is generally best to use it for the smallest ratio.

PROBLEMS.

- 95.** Divide 30 into two parts that shall have the ratio of 1 to 5.
- 96.** Divide 130 into parts in the ratio of 4 to 9.
- 97.** Divide 272 into parts in the ratios of 2, 4, and 11.
- 98.** Divide 363 into parts in the ratio of $\frac{1}{2}$ to $\frac{2}{3}$.
- 99.** Divide 2,109 into parts in the ratios of $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $1\frac{1}{6}$.
- 100.** Divide 574 into three parts such that the second part shall be 5 times the first, and the third part 7 times the second.
- 101.** Divide 965 into four parts such that the second part shall be $\frac{1}{2}$ of the first, the third $\frac{2}{3}$ of the second, and the fourth $\frac{3}{4}$ of the third.
- 102.** Divide \$3,200 between B and C so that B shall have 7 times as many dollars as C.
- 103.** Divide \$600 between D and E so that E shall have $\frac{2}{3}$ as much as D.
- 104.** Divide \$1,180 between G and H so that G's share shall be to H's as $\frac{3}{8}$ is to $\frac{1}{4}$.
- 105.** Divide 8,700 into parts in the ratios of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{10}$.
- 106.** Divide 11,900 proportionally to $1\frac{1}{2}$, $3\frac{2}{3}$, and $4\frac{1}{2}$.
- 107.** The volume of the oxygen of the atmosphere is to that of the nitrogen as 21 to 79. How many cubic feet of each gas are there in a hall 140 ft. by 90 ft. by $39\frac{1}{2}$ ft.?
- 108.** Water is composed of oxygen and hydrogen in the proportion of 8 of oxygen to 1 of hydrogen, by weight. How many pounds of each element are there in a barrel containing 36 gallons of water?
- NOTE.**—A cubic foot of water weighs $62\frac{1}{2}$ pounds.
- 109.** Japanese bell-metal contains, by weight, 73 parts of copper, 18 of tin, 6 of zinc, 12 of lead, and 3 of iron. How many pounds of each metal are there in a bell weighing 336 pounds?
- 110.** A gold watch-case 18 karats fine, weighing 54 pwt., is alloyed with silver and copper in the proportion of 2 parts of silver to 1 part of copper. The case contains how many pennyweights of each of the three kinds of metal?

111. A man died, leaving \$20,600 to be divided among 2 sons, 3 daughters, 1 grandson, and 4 grand-daughters, as follows:—Each grand-daughter was to receive $\frac{2}{3}$ as much as the grandson, the grandson $\frac{1}{2}$ as much as each daughter, and each daughter $\frac{3}{4}$ as much as each son. How much was the share of each heir?

112. A man's will provided that his estate should be divided among an orphan asylum, an industrial school, and a public library, as follows:—\$13,500 to the asylum, \$6,500 to the school, and \$10,000 to the library. At the final settlement the estate is found to be worth \$19,652.83. How much should each institution receive?

113. D, E, and F are partners in business, to the capital of which they have contributed respectively \$4,500, \$6,000, and \$7,500. Their net gain this year is \$9,000. If this gain is divided among the partners in proportion to their investments, how much is each partner's share?

114. Mr. Jay failed in business, owing \$3,750 to Mr. Brown, \$1,875 to Mr. White, and \$5,000 to Mr. Morris. If he pays to all \$6,000, how much will each creditor receive?

115. Messrs. Day & Ray were partners under an agreement to divide all losses and gains in proportion to the amount of capital invested by each and the time it was employed. Mr. Day contributed \$6,000 for 10 months, and Mr. Ray \$8,000 for 6 months. Their gain was \$2,700. How much was the share of each?

NOTE.—The use of \$6,000 for 10 months equals the use of $10 \times \$6,000 (= \$60,000)$ for 1 month; and the use of \$8,000 for 6 months equals the use of $6 \times \$8,000 (= \$48,000)$ for 1 month. Hence, the gain should be divided in the ratio of \$60,000 to \$48,000.

116. J, K, and L form a partnership,—the gains and losses to be divided in proportion to their several amounts of capital and the time each amount is invested. J contributes \$1,000 for 6 months; K, \$800 for 9 months; and L, \$1,300 for 8 months. How much is each man's share of a gain of \$3,300?

SECTION III. PROGRESSIONS.

394. A *series* is a succession of numbers that vary uniformly according to a common law.

- a.* The *terms* of a series are the numbers that compose it.
- b.* The *extremes* of a series are the first and last terms ; and
The *means* are all the terms except the extremes.
- c.* An *ascending series* is one in which each term after the first is greater than the preceding term ; and
A *descending series* is one in which each term after the first is less than the preceding term.
- d.* The *law of a series* is the rate of increase or decrease of its terms.

I. ARITHMETICAL PROGRESSION.

395. An *arithmetical progression* is a series whose successive terms increase or decrease by a constant difference.

8, 6, 9, 12, 15, 18—and 25, 20, 15, 10, 5 are arithmetical progressions or series,—the former ascending, the latter descending. The constant or common difference in the ascending series is 3 ; and in the descending series, 5.

396. Five elements enter into computations in arithmetical progressions, as follows :—

If any three of these five elements are known, the other two can be found by computation.

	Name	Symbol
(1)	<i>First term</i>	<i>a</i>
(2)	<i>Common difference</i>	<i>d</i>
(3)	<i>Number of terms</i>	<i>n</i>
(4)	<i>Last term</i>	<i>l</i>
(5)	<i>Sum of the terms</i>	<i>S</i>

397. In an ascending arithmetical series, each term except the first is the sum of the preceding term and the common difference.

Thus, if the first term *a* is 3, the common difference *d* is 2, and the number of the terms *n* is 5, the terms of the series are

1st	2d	3d	4th	5th
3,	$3 + 2,$	$3 + (2 + 2),$	$3 + (2 + 2 + 2),$	$3 + (2 + 2 + 2 + 2)$

Each term is equal to the sum of the first term 3, plus the common difference 2 taken a number of times 1 less than the number of the term (*i. e.*, taken once in the second term, twice in the third term, three times in the fourth term ; and so on).

Substituting symbols for the numbers in the foregoing terms, the following *general expression* for the terms of any ascending arithmetical progression is obtained, viz. :—

1st	2d	3d	4th	5th.....nth
a ,	$a + d$,	$a + 2 \times d$,	$a + 3 \times d$,	$a + 4 \times d \dots a + (n - 1) \times d$.

Hence,

PRINCIPLE I.—*The last term of an ascending arithmetical series equals the first term plus the product of the common difference multiplied by the number of terms less 1.*

FORMULA 1.

$$l = a + (n - 1) \times d$$

398. The 5th term of the general series (396) is $a + 4 \times d$. Subtracting $4 \times d$ from the series, the remainder is a , the first term. *In general,*

The n th term of a series is $a + (n - 1) \times d$;

Subtracting $(n - 1) \times d$ the remainder is a , the first term. Hence,

PRINCIPLE II.—*The first term of an ascending arithmetical series equals the last term minus the product of the common difference multiplied by the number of terms less 1.*

FORMULA 2.

$$a = l - (n - 1) \times d$$

399. Assume that an ascending arithmetical series consists of five terms. Then the last term must be $a + 4 \times d$ (397, Prin.).

Subtracting the first term a , the remainder is $4 \times d$, i. e., the common difference taken as many times less one as there are terms in the series. Dividing $4 \times d$ by $(5 - 1) = 4$, the quotient is d , the common difference. *In general,*

The number of terms in any series is n , and the last term is $a + (n - 1) \times d$. Subtracting a and dividing the remainder $(n - 1) \times d$ by $(n - 1)$, the result is d . Hence,

PRINCIPLE III.—*The common difference of an ascending arithmetical series equals the quotient of the difference of the extremes divided by the number of terms less 1.*

FORMULA 3.

$$d = \frac{l - a}{n - 1}$$

400. Assume that an ascending arithmetical series consists of five terms. Then the last term is $a + 4 \times d$. Subtracting the first term a and dividing the remainder $4 \times d$ by d , the quotient is 4; and $4 + 1 = 5$, the number of terms. *In general,*

The number of terms in any series is n , and the last term is $a + (n - 1) \times d$. Subtracting a , dividing the remainder $(n - 1) \times d$ by d , and adding 1 to the quotient $n - 1$, the result is n . Hence,

PRINCIPLE IV.—*The number of terms in an ascending arithmetical series equals the quotient of the difference of the extremes divided by the common difference, plus 1.*

FORMULA 4.

$$n = \frac{l - a}{d} + 1$$

401. The *average term* of an arithmetical series is a number equal to the quotient of the sum of all the terms of the series divided by the number of terms (304).

402. An inspection of the series 3, 5, 7, 9, 11 shows that the middle term equals the average term; also that $\frac{3+11}{2} = 7$, the average term, i. e., in any arithmetical series *the average term is one half of the sum of the extremes*.—And

In the series 3, 5, 7, 9, 11, 13, $-\frac{3+13}{2} = 8$, the average term; i. e.,

In a series of an even number of terms, the average term has no actual existence AS A TERM. In general,

The product of the average term multiplied by the number of terms equals the sum of all the terms. Hence,

PRINCIPLE V.—*The sum of the terms of an arithmetical series equals the product of one half the sum of the extremes multiplied by the number of terms.* FORMULA 5. $S = \frac{a+l}{2} \times n$

a. By proper application of the five principles and formulæ now given, either singly or combined, any problem in arithmetical progression may be solved.

b. The fourteen additional formulæ (see Supplement, page 426) can be developed arithmetically from the formulæ and illustrations already given. The demonstration of their truth will be a valuable mathematical drill to the ambitious student.

c. The principles and formulæ already given may be applied to the solution of problems in *descending series*, by regarding the series as reversed, and consequently ascending.

EXERCISES.

By inspection find each of the five elements (a , d , n , l , S) in each of the following progressions:—

- | | |
|---|--|
| 1. 3, 7, 11, 15, 19, 23, 27, 31, 35. | 4. $5\frac{1}{2}$, $4\frac{1}{2}$, $4\frac{2}{3}$, $4\frac{1}{3}$, $3\frac{1}{2}$, $3\frac{2}{3}$. |
| 2. 217, 206, 195, 184, 173, 162. | 5. 7, 7.8, 8.6, 9.4, 10.2, 11, 12.8. |
| 3. $\frac{1}{2}$, $\frac{1}{12}$, $\frac{5}{6}$, $1\frac{1}{12}$, $1\frac{1}{2}$, $1\frac{7}{12}$, $1\frac{5}{6}$. | 6. 13.1, 10.4, 7.7, 5, 2.3. |

403. ILLUSTRATIVE EXAMPLES.

Ex. 1. Find the 9th term of the series 5, 11, 17, etc.

EXPLANATION.—I find by inspection that the first term a of the given series is 5, the common difference d is 6, and the number of terms n is 9.

Substituting these values for their respective symbols in Formula 1, I have $l = 5 + (9 - 1) \times 6$.

PROCESS.

Formula 1. $l = a + (n - 1) \times d$

$$l = 5 + (9 - 1) \times 6 = 5 + 8 \times 6 = 53$$

Making the computations indicated, I obtain 53, the required term.

Ex. 2. What is the first term of the series in which d is 5, n is 11, and l is 78?

PROCESS.

Formula 2. $a = l - (n - 1) \times d$; substituting given values for symbols,
 $a = 78 - (11 - 1) \times 5 = 78 - 10 \times 5 = 28$, the first term.

Ex. 3. Given $a = 13$, $n = 30$, $l = 158$, to find d .

PROCESS.

$$\text{Formula 3. } d = \frac{l - a}{n - 1}$$

$$\text{Substituting given values for symbols, } \left\{ d = \frac{158 - 13}{30 - 1} = \frac{145}{29} = 5 \right.$$

Ex. 4. Find the number of terms of the series whose first term is 31, common difference 15, and last term 181.

PROCESS.

$$\text{Formula 4. } n = \frac{l - a}{d} + 1$$

$$\text{Substituting given values for symbols, } \left\{ n = \frac{181 - 31}{15} + 1 = \frac{150}{15} + 1 = 11 \right.$$

Ex. 5. Find the sum of the terms of the series in which a is 17, n is 19, and l is 207.

PROCESS.

$$\text{Formula 5. } S = \frac{a + l}{2} \times n$$

$$\text{Substituting given values for symbols, } \left\{ S = \frac{17 + 207}{2} \times 19 = 2,128 \right.$$

Ex. 6. What is the common difference of the series whose first term is 173, number of terms 19, and last term 11?

PROCESS.

$$\text{Formula 6. } d = \frac{l - a}{n - 1}$$

$$\text{Substituting values for symbols [see 399, c], } \left\{ d = \frac{173 - 11}{19 - 1} = \frac{162}{18} = 9 \right.$$

PROBLEMS.

1. Find the 13th term of the series 4, 6, 8, etc.
2. Find the 12th term of the series 2, 5, 8, etc.
3. Find the sum of the series in problem 2.

4. Find the last term of a series whose first term is 8, common difference $3\frac{1}{2}$, and number of terms 7.

5. Find the common difference of a series of 6 terms, the extremes being 1 and 81.

6. How many terms are there in a series whose first term is 5, common difference $2\frac{1}{2}$, and last term 35?

7. What is the sum of a series whose first term is 26, number of terms 12, and last term 4?

8. Find the number of terms of a series the last term being $7\frac{1}{2}$, the common difference $2\frac{1}{2}$, and the first term $33\frac{1}{2}$.

9. How many strokes are made in 12 hours by the hammer of a clock that strikes the hours only?

10. 50 oranges are placed in a line and 3 yards apart, the first orange being 3 yards from a basket. How far will a boy travel who, starting from the basket, gathers the oranges into the basket one by one?

11. How far will a bullet fall in a minute, if it falls 16.1 ft. in the first second, 3 times as far in the second second, 5 times as far in the third second, and so on?

II. GEOMETRICAL PROGRESSION.

404. A *geometrical progression* is a series whose successive terms increase or decrease by a constant multiplier.

a. The *ratio* is the constant or common multiplier.

b. 8, 6, 12, 24—and 81, 27, 9, 3, 1 are geometrical progressions, the former ascending, the latter descending. The ratio in the ascending series is 2, and in the descending series, $\frac{1}{2}$.

c. When the ratio is greater than 1, the series is ascending; when less than 1, it is descending.

405. Five elements enter into computations in geometrical progressions, as follows:—

If any three of these five elements are known, the other two can be found by computation.

	Name	Symbol
(1)	<i>First term</i>	<i>a</i>
(2)	<i>Ratio</i>	<i>r</i>
(3)	<i>Number of terms</i>	<i>n</i>
(4)	<i>Last term</i>	<i>l</i>
(5)	<i>Sum of terms</i>	<i>S</i>

406. In a geometrical series each term except the first is the product of the preceding term by the ratio.

Thus, if the first term a is 3, and the ratio r is 2, the terms of the series are

1st	2d	3d	4th	5th	6th
3,	$3 \times 2,$	$3 \times (2 \times 2),$	$3 \times (2 \times 2 \times 2),$	$3 \times (2 \times 2 \times 2 \times 2),$	$3 \times (2 \times 2 \times 2 \times 2 \times 2),$
3,	$3 \times 2,$	or $3 \times 2^2,$	or $3 \times 2^3,$	or $3 \times 2^4,$	or 3×2^5

Each term is equal to the product of the first term multiplied by that power of the ratio indicated by the number of the term minus 1; i. e., by the first power of the ratio in the second term, the second power in the third term, the third power in the fourth term,—and so on.

407. Substituting symbols for the numbers in the foregoing terms, a general expression for the terms of any geometrical progression is obtained, as follows:—

Terms:—1st	2d	3d	4th	5th	6th.....nth
$a,$	$a \times r,$	$a \times r^2,$	$a \times r^3,$	$a \times r^4,$	$a \times r^5, \dots a \times r^{(n-1)}.$

PRINCIPLE I.—*The last term of a geometrical series equals the product of the first term and that power of the ratio indicated by the number of terms less 1.* **FORMULA 1.**
 $l = a \times r^{(n-1)}$

408. The 5th term (**405**) is $a \times r^4$. Dividing this fifth term by $r^4 (= 5 - 1)$, the quotient is a , the first term. *In general,*

The n th or last term of any geometrical series is $a \times r^{(n-1)}$. Dividing this last term by $r^{(n-1)}$, the quotient is a , the first term. Hence,

PRINCIPLE II.—*The first term of a geometrical series equals the quotient of the last term divided by that power of the ratio indicated by the number of terms less 1.* **FORMULA 2.**
 $a = \frac{l}{r^{(n-1)}}$

409. Dividing any term—as the 5th (**405**), $a \times r^4$ —by a , the quotient is r^4 . Extracting the 4th ($= 5 - 1$) root, the result is r , the ratio. *In general,*

Dividing $a \times r^{(n-1)}$ by a , and extracting the $(n - 1)$ th root of the quotient $r^{(n-1)}$, the result is r , the ratio. Hence,

PRINCIPLE III.—*The ratio of a geometrical series equals that root of the quotient of the last term divided by the first term which is indicated by the number of terms less 1.* **FORMULA 3.**
 $r = \sqrt[n-1]{\frac{l}{a}}$

410. Assume that a geometrical progression consists of six terms. Then the last term must be $a \times r^5$ (**405**). Dividing $a \times r^5$ by a , the quotient is r^5 . But 5, the exponent of r , equals the number of terms less 1 (i. e., $5 = 6 - 1$). Hence,

PRINCIPLE IV.—*The number of terms of a geometrical series equals the exponent plus 1 of that power of the ratio which is equal to the quotient of the last term divided by the first term.* FORMULA 4.

$$r^{(n-1)} = \frac{l}{a}$$

411. In the series 2, 6, 18, 54, the sum of the terms is represented by

$$(1) \quad S = 2 + 6 + 18 + 54$$

Multiplying each term of (1) by the ratio 3, (2) $3 \times S = 6 + 18 + 54 + (3 \times 54)$

Subtracting (1) from (2) (3) $3 \times S - S = (3 \times 54) - 2$

Factoring the first member of (3) (4) $(3 - 1) \times S = (3 \times 54) - 2$

Dividing both members of (4) by (3 - 1) (5) $S = \frac{(3 \times 54 - 2)}{3 - 1}$

Substituting symbols for numbers (6) $S = \frac{r \times l - a}{r - 1}$ Hence,

PRINCIPLE V.—*The sum of an ascending geometrical series equals the quotient obtained by dividing the product of the last term and the ratio, minus the first term, by the ratio minus 1.* FORMULA 5.

$$S = \frac{r \times l - a}{r - 1}$$

412. Assume that an ascending geometrical progression consists of six terms. The last term is $a \times r^5$, i. e., $l = a \times r^5$. Multiplying both terms of the formula by r , we have $r \times l = a \times r^6$, in which the power of the ratio is indicated by the number of terms. Substituting this value of $r \times l$ in Formula 5, we have $\frac{a \times r^6 - a}{r - 1} = \frac{a \times (r^6 - 1)}{r - 1}$. Hence,

PRINCIPLE VI.—*The sum of an ascending geometrical series equals the quotient obtained by dividing the product of the first term and that power of the ratio indicated by the number of terms, minus 1, by the ratio minus 1.* FORMULA 6.

$$S = \frac{a \times (r^n - 1)}{r - 1}$$

a. In finding the sum of the terms when the first term, ratio, and number of terms are known, Principle VI is more direct than Principle V.

b. To find the sum of the terms of a descending series, the product of the last term and ratio is subtracted from the first term, and the remainder is divided by 1 minus the ratio. Hence, under the conditions of a descending series, Formulæ 5 and 6 become respectively

$$\text{FORMULA 7. } S = \frac{a - r \times l}{1 - r}$$

$$\text{and FORMULA 8. } S = \frac{a \times (1 - r^n)}{1 - r}$$

For additional formulæ, see Supplement, page 426.

EXERCISES.

By inspection find each of the five elements (a , r , n , l , S) in each of the following progressions:—

- | | |
|--|--|
| 1. 1, 5, 25, 125, 625, 3,125 | 4. 16, 1.6, .16, .016, .0016 |
| 2. 256, 128, 64, 32, 16, 8, 4, 2 | 5. $1\frac{1}{2}$, $4\frac{2}{3}$, $16\frac{4}{9}$, $57\frac{1}{3}$, $200\frac{1}{4}$, $700\frac{7}{4}$ |
| 3. 2, $1\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{2}$, $\frac{3}{8}$, $\frac{9}{64}$, $\frac{27}{512}$ | 6. 2, 6, 18, 54, 162, 486, 1,458, 4,374 |

413. Using the six progressions given in the foregoing exercises, prove the truth of the following

PROPOSITIONS.

I. Any four consecutive terms of a geometrical series are the terms of a proportion.

II. The second term of any three consecutive terms of a geometrical series is a mean proportional to the other two terms.

414. ILLUSTRATIVE EXAMPLES.

Ex. 1. Find the tenth term of the series 2, 6, 12,—etc. PROCESS.

EXPLANATION.—I find by inspection that the first term a is 2, the ratio r is 3, and the number of terms n is 10.

Formula 1. $l = a \times r^{(n-1)}$
 $l = 2 \times 3^9 = 39,366$

Substituting the given values for their respective symbols in formula 1, I have $l = 2 \times 3^9$.

Making the computations indicated, I obtain 39,366, the required term.

Ex. 2. The last term of a geometrical progression is 15,625, the ratio is 5, and the number of terms is 6. What is the first term?

EXPLANATION.—Substituting the known values for their respective symbols in formula

PROCESS.
 $l = 15,625$ $r = 5$ $n = 6$
Formula 2. $a = \frac{l}{r^{(n-1)}}$

2, I have $a = \frac{15,625}{5^5}$.

Making the computations indicated, I obtain 5, the required term.

$$a = \frac{15,625}{5^5} = \frac{15,625}{3,125} = 5$$

Ex. 3. The first term of a geometrical series is 3, the number of terms 7, and the last term is 12,288. Required the ratio.

EXPLANATION.—Substituting the known values for their respective symbols in formula 3, I have

$$r = \sqrt[6]{\frac{12,288}{3}}.$$

Making the computations indicated, I obtain 4, the required ratio.

PROCESS.

$$\begin{aligned} \text{Formula 3. } r &= \sqrt[n-1]{\frac{l}{a}} \\ r &= \sqrt[6]{\frac{12,288}{3}} = \\ &= \sqrt[6]{4,096} = 4 \end{aligned}$$

Ex. 4. In a geometrical progression, $a = 5$, $r = 3$, and $l = 405$. What is the number of terms?

EXPLANATION.—Substituting the known values for their respective symbols in formula 4, I have

$$3^{(n-1)} = \frac{405}{5} (= 81).$$

I then raise 3, the ratio, to a power that equals 81 (the quotient of the last term divided by the first term), and find that I have the 4th power.

Since $(n-1)$ equals the exponent of the power of 3 that equals 81, $(n-1)$ must equal 4; hence, $n = 5$, the number of terms required.

PROCESS.

$$\begin{aligned} \text{Formula 4. } r^{(n-1)} &= \frac{l}{a} \\ 3^{(n-1)} &= \frac{405}{5} = 81 \end{aligned}$$

$$3 \times 3 = 9, \text{ the 2d power}$$

$$3 \times 9 = 27, \text{ the 3d power}$$

$$3 \times 27 = 81, \text{ the 4th power}$$

$$\text{Hence, } (n-1) = 4, \text{ and } n = 5$$

Ex. 5. Find the sum of the terms of a series whose first term is 2, ratio 5, and last term 6,250.

EXPLANATION.—Substituting the known terms for their respective symbols in formula 5, I have $S = \frac{5 \times 6,250 - 2}{5 - 1}$. Making the computations indicated, I obtain 7,812, the sum of the terms required.

PROCESS.

$$\begin{aligned} \text{Formula 5. } S &= \frac{r \times l - a}{r - 1} \\ S &= \frac{5 \times 6,250 - 2}{5 - 1} = \frac{31,248}{4} = 7,812 \end{aligned}$$

Ex. 6. Find the sum of 7 terms of the series in which $a = 2$ and $r = 3$.

EXPLANATION.—Substituting the known values for their respective symbols in formula 6,

$$\text{I have } S = \frac{2 \times (3^7 - 1)}{3 - 1}.$$

Making the computations indicated, I obtain 2,186, the sum of the series—i. e., the sum of the terms required.

PROCESS.

$$\begin{aligned} \text{Formula 6. } S &= \frac{a(r^n - 1)}{r - 1} \\ S &= \frac{2 \times (3^7 - 1)}{3 - 1} = \frac{2 \times (2,187 - 1)}{2} = 2,186 \end{aligned}$$

Ex. 7. What is the sum of the series in which $l = 9$, $r = \frac{1}{2}$, and $a = 703,125$?

EXPLANATION.—
Substituting known
values for symbols in
formula 7, (412, b,)

PROCESS.

$$\text{Formula 7. } S = \frac{a - r \times l}{1 - r}$$

$$S = \frac{703,125 - \frac{1}{2} \times 9}{1 - \frac{1}{2}} = \frac{703,123\frac{1}{2}}{\frac{1}{2}} = 878,904$$

$$S = \frac{703,125 - \frac{1}{2} \times 9}{1 - \frac{1}{2}}.$$

Making the computations indicated, I obtain 878,904, the sum of the series required.

PROBLEMS.

12. What is the last term of a series whose first term is 3, ratio 4, and number of terms 5?

13. Find the sixth term of the series 4, 8, 16, etc.

14. Find the sum of the first eight terms of the series 4, 8, 16, etc.

15. What is the first term of a series whose last term is 256, ratio 2, and number of terms 12?

16. Required, the ratio of the series in which $a = 4$, $l = 324$, and $n = 5$.

17. Required, the sum of the series, problem 16.

18. A man bought a dozen eggs paying $\frac{1}{2}$ cent for the first egg, $\frac{1}{4}$ cent for the second, 1 cent for the third,—and so on. How much did he pay for the dozen?

19. The first term of a series is 64, the ratio is $\frac{1}{2}$, and the last term 2. Find the sum of the terms.

20. Required, the sum of 6 terms of the series whose first term is 243, and ratio $\frac{1}{3}$.

21. Find the five means of the geometrical series whose extremes are 3 and 2,187.

22. If a man saves 1 cent the first day of April, 4 cents the second day, 16 cents the third, and so on, how much will he have saved at the middle of the month?

23. The seating capacity of 4 halls is in geometrical progression. The smallest hall seats 256, and the largest 6,912. What is the capacity of each of the others?

III. INFINITE SERIES.

415. An *infinite series* is a descending geometrical series the number of whose terms is infinite.

In an infinite series, the last term is regarded as 0. When one of two factors is 0, the product is also 0. Formula 7 (**410**), $S = \frac{a-r \times l}{1-r}$ becomes, in an infinite series, $S = \frac{a-0}{1-r} = \frac{a}{1-r}$. Hence,

PRINCIPLE VII.—*The sum of an infinite series equals the quotient of the first term divided by 1 minus the ratio.* FORMULA 9.
 $S = \frac{a}{1-r}$

PROBLEMS.

- 24.** Find the sum of the infinite series $1, \frac{1}{2}, \frac{1}{4}$, etc.
- 25.** Required, the sum of the terms of the series $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$,—to infinity.
- 26.** The sum of an infinite series is 81, and the ratio is $\frac{2}{3}$. What is the first term?
- 27.** A ball falls to the floor from a height of 10 feet, and rebounds to the height of 5 feet, then falling again, rebounds $2\frac{1}{2}$ feet,—and so on. What distance will it move before coming to rest?
- 28.** A sloop is 20 miles ahead of a steamer that runs 10 miles while the sloop sails but 2 miles. In how many hours will the steamer overtake the sloop?

SECTION IV.

REVIEW.

416. TEST QUESTIONS.

Pupils should now carefully prepare thirty questions, in sets of ten each, for review of Ratio, Proportions, and Progressions.

417. BLACKBOARD OUTLINES.

Pupils will now collect all the terms, principles, formulae, important facts, and rules or steps in processes, and arrange them properly for blackboard outlines of Ratio, Proportions, and Progressions.

CHAPTER VIII.

PERCENTAGE.

In business affairs and in the sciences and the arts, estimates, computations, and comparisons are commonly made in hundredths of the number or thing which is made the basis of the estimate, computation, or comparison.

The term *per cent* means by the hundred.

418. *Percentage* embraces those processes of computation in which one of the elements is always expressed in hundredths.

419. a. The elements that enter into computations in percentage are the *base*, the *per cent*, the *percentage*, the *amount*, and the *difference*.

b. The *base* is the number of which hundredths are to be found.

c. The *per cent* is the number of hundredths of the base to be found.

d. The *percentage* is the required number of hundredths of the base.

e. The *amount* is the base plus the percentage; and

The *difference* is the base minus the percentage.

f. The base, the percentage, the amount, or the difference may be either abstract or concrete; but the per cent is always abstract.

NOTE—For uniformity in arrangement and convenience in reading equations, throughout the subject of percentage the sign $\times$ is read “multiplied by.”

SECTION I.

NOTATIONS AND REDUCTIONS.

420. The *commercial sign*, %, signifies per cent.

18% signifies 18 hundredths, and is read “18 per cent.”

IMPORTANT FACTS.

I. To express per cent decimally, two decimal figures are required.

II. To express 100 per cent or more, an integer or a mixed decimal number is required.

III. To express parts of 1 per cent, either decimal figures or fractions at the right of hundredths are required.

Numbers equivalent to per cents may be expressed by fractions or by mixed numbers, but they are *not* per cents.

Thus, $25\% = .25$; but $.25 = \frac{1}{4} = \frac{1}{100} = \frac{1}{4}$; hence, $\frac{1}{4}$ is an equivalent expression for 25% , and 25% of a number is $\frac{1}{4}$ of it.

421. The per cent may be expressed by words, by the commercial sign, and by decimals. Its equivalent value may be expressed by either a fraction or a mixed number.

By words	By sign	By decimal	Equivalent fraction	By words	By sign	By decimal	Equivalent fraction
100 per cent	= 100%	= 1.00	= 1	$6\frac{3}{4}$ per cent	= $6\frac{3}{4}\%$	= .06 $\frac{3}{4}$	= $\frac{1}{16}$
50 " "	= 50%	= .50	= $\frac{1}{2}$	$\frac{1}{2}$ " "	= $\frac{1}{2}\%$	= .005	= $\frac{1}{200}$
$83\frac{1}{2}$ " "	= $83\frac{1}{2}\%$	= .83 $\frac{1}{2}$	= $\frac{1}{2}$	$\frac{3}{4}$ " "	= $\frac{3}{4}\%$	= .0075	= $\frac{1}{133\frac{1}{3}}$
25 " "	= 25%	= .25	= $\frac{1}{4}$	$\frac{1}{8}$ " "	= $\frac{1}{8}\%$	= .00 $\frac{1}{8}$	= $\frac{1}{125}$
20 " "	= 20%	= .20	= $\frac{1}{5}$	$87\frac{1}{2}$ " "	= $87\frac{1}{2}\%$	= .375	= $\frac{3}{8}$
$16\frac{2}{3}$ " "	= $16\frac{2}{3}\%$	= .16 $\frac{2}{3}$	= $\frac{1}{3}$	$48\frac{1}{2}$ " "	= $48\frac{1}{2}\%$	= .4825	= $\frac{1}{114\frac{1}{2}}$

CASE I. To reduce a fraction to a per cent.

422. Ex. What per cent of a number is $\frac{2}{3}$ of it?

ANALYSIS.—Any number is 100% of itself, and $\frac{2}{3}$ of the number must be 3 times $\frac{1}{3}$ of 100%, or 75% of it. Or

$\frac{2}{3}$ of any number equals $\frac{1}{3}$ of 3 times that number. Hence, $\frac{1}{3}$ of 3 times 100 per cent is $\frac{3.00}{4} = .75 = 75\%$.

PROBLEMS.

Change the following fractions to { 1. $\frac{1}{2}$ | 3. $\frac{2}{3}$ | 5. $\frac{1}{4}$ | 7. $\frac{1}{5}$ | 9. $\frac{1}{6}$
equivalent per cents:— { 2. $\frac{3}{4}$ | 4. $\frac{5}{8}$ | 6. $\frac{1}{3}$ | 8. $\frac{2}{5}$ | 10. $\frac{3}{7}$
11. $\frac{5}{12}$ | 13. $\frac{7}{8}$ | 15. $\frac{9}{16}$ | 17. $\frac{1}{10}$ | 19. $\frac{1}{11}$ | 21. $\frac{1}{12}$ | 23. $\frac{1}{13}$
12. $\frac{7}{10}$ | 14. $\frac{1}{10}$ | 16. $\frac{1}{11}$ | 18. $\frac{1}{14}$ | 20. $\frac{1}{15}$ | 22. $\frac{1}{16}$ | 24. $\frac{1}{17}$

CASE II. To reduce a per cent to a fraction.

423. Ex. Change 18% to an equivalent fraction in its lowest terms.

ANALYSIS.— $18\% = .18 = \frac{18}{100} = \frac{9}{50}$, the fraction required.

PROBLEMS.

Change the following per cents to equivalent fractions in lowest terms:—

25. $66\frac{2}{3}\%$ | 27. $87\frac{1}{2}\%$ | 29. 140% | 31. $57\frac{1}{2}\%$ | 33. $\frac{1}{2}\%$ | 35. $\frac{1}{3}\%$ | 37. $9\frac{1}{16}\%$
26. $83\frac{1}{3}\%$ | 28. $6\frac{2}{3}\%$ | 30. 175% | 32. 35% | 34. $\frac{2}{3}\%$ | 36. $\frac{1}{11}\%$ | 38. $\frac{3}{4}\%$

SECTION II.

THE SIX GENERAL CASES OF PERCENTAGE.

In the solution of each problem in these Six General Cases, the pupil should be required to indicate the elements that enter into the problem.

424. All operations in percentage are based upon this

FUNDAMENTAL TRUTH:—*The percentage is the product of the base multiplied by the per cent.*

CASE I. Base and per cent given, to find the percentage.

PROCESS.

Ex. How much is 18
per cent of \$260?

$$\begin{aligned} \$260 \times .18 &= \$46.80 \\ \text{Or } .18 &= \frac{18}{100} = \frac{9}{50}, \text{ and} \\ 9 \text{ times } \frac{1}{50} \text{ of } \$260 &= \$46.80 \end{aligned}$$

PROBLEMS.

In reading problems arranged as below, read each number in the first column with the numbers of the second column successively.

Thus—(1) Find 6 per cent of 1,250. (2) Of 19,836. (3) Of 478.5. (4) Of $\frac{1}{16}$.

Find		How much is
1- 4. 6 per cent	} of {	1,250.
5- 8. 15 per cent		19,836.
9-12. 4 per cent		478.5.
13-16. $7\frac{1}{2}$ per cent		$\frac{1}{16}$.
		17-20. 25%
		21-24. $2\frac{1}{2}\%$
		25-28. $16\frac{2}{3}\%$
		29-32. $\frac{3}{4}\%$
	} of {	3,742 tons of coal?
		956 $\frac{1}{2}$ acres?
		8,240 miles?
		\$670.28?

33. On an express train were 132 passengers, $31\frac{1}{11}\%$ of whom were in the drawing-room car, $42\frac{1}{4}\%$ in the day coach, and $25\frac{3}{4}\%$ in the smoking car. How many passengers were there in each car?

34. A grain dealer bought 12,650 bushels of grain, of which 34% was wheat, 17% oats, 23% barley, and the remainder rye. What per cent of the grain was rye? How many bushels of each kind of grain did he buy?

35. K had \$3,000 in bank. May 1 he drew out $33\frac{1}{3}$ per cent of it, June 1 15 per cent of the balance, July 1 10 per cent of the balance then in bank, and Aug. 1 a sum equal to 75 per cent of the amount previously drawn. How much had he then in bank?

SHORT METHODS.

425. I. When the given per cent is an aliquot part of 100%:—*Instead of the given per cent, use the aliquot part or parts expressed by a fraction in its lowest terms.*

II. To find 10 per cent of a number:—*Take ($\frac{10}{100}$) = $\frac{1}{10}$ of it, or move the decimal point one place to the left.*

5% is $\frac{1}{2}$ of 10%; $2\frac{1}{2}$ % is $\frac{1}{2}$ of 5% or $\frac{1}{4}$ of 10%; $1\frac{1}{4}$ % is $\frac{1}{2}$ of $2\frac{1}{2}$ %, or $\frac{1}{4}$ of 5%, or $\frac{1}{8}$ of 10%.
30%, 40%, 60%, 70%, and so on, are the products respectively of 10% by the tens digit.

$7\frac{1}{2}$ % is 5% + $2\frac{1}{2}$ %; $12\frac{1}{2}$ % is 10% + $2\frac{1}{2}$ %, and so on.

15% is 10% + 5%, 35% is 30% + 5%, or 25% + 10% = $\frac{1}{4}$ + $\frac{1}{10}$.

45% is 50% - 5%, or $\frac{1}{2}$ - $\frac{1}{10}$ of $\frac{1}{2}$ = $\frac{1}{2}$ - $\frac{1}{20}$ of $\frac{1}{10}$.

55% is 50% + 5%, or $\frac{1}{2}$ + $\frac{1}{10}$ of $\frac{1}{2}$ = $\frac{1}{2}$ + $\frac{1}{20}$ of $\frac{1}{10}$; and so on.

PROBLEMS.

Give short methods—by aliquot parts and by fractional parts—for finding the per cents named in the following problems, and apply those methods in finding the required per cents of the given numbers.

36-39. 20%	} of {	1,200 bu.	84- 87. $11\frac{1}{4}$ %	} of {	.8371
40-43. 25%		1,500 yd.	88- 91. $37\frac{1}{2}$ %		26,400
44-47. 75%		3,650 lb.	92- 95. $6\frac{1}{4}$ %		318,000
48-51. 90%		82.50	96- 99. $16\frac{2}{3}$ %		\$320.44
52-55. 95%	} of {	96.75 bar.	100-103. $40\frac{1}{2}$ %	} of {	\$769.50
56-59. 44%		$1,970\frac{1}{2}$ mi.	104-107. $93\frac{1}{2}$ %		\$975.25
60-63. 110%		2,275 T.	108-111. 1%		\$59,489 $\frac{1}{2}$
64-67. 125%		17,562 gal.	112-115. $\frac{1}{2}$ %		\$725 $\frac{1}{2}$
68-71. 600%	} of {	8,250 A.	116-119. $\frac{1}{4}$ %	} of {	\$10,576 $\frac{1}{2}$
72-75. 100%		318.4 da.	120-123. $\frac{1}{3}$ %		\$8,248.30
76-79. $17\frac{1}{2}$ %		6,280	124-127. $\frac{2}{3}$ %		\$59,876.54
80-83. $22\frac{1}{2}$ %		13.956	128-131. $120\frac{1}{4}$ %		\$107,469.68 $\frac{1}{4}$

NOTE.—Short methods similar to the above may be used in solving problems under the remaining five general cases of percentage.

CASE II. The percentage and base given, to find per cent.

426. Ex. 15 is what per cent of 60 ?

ANALYSIS.—15 is $\frac{1}{4}$ ($= 15$ times $\frac{1}{60}$) of 60. Since 60 is ($\frac{100}{60}$) $= 100\%$ of itself, 15 is 15 times $\frac{1}{60}$ of 100 per cent $= \frac{15}{4}\% = .25$.

Or

15 is $\frac{1}{4}$ ($= \frac{1}{4}$) of 60, and $\frac{1}{4} = \frac{100}{400} = .25 = 25\%$.

IMPORTANT FACT I.—*The per cent is the quotient of the percentage divided by the base (424).*

PROBLEMS.

132-135. 6	} is what per cent of	30 ?	What per cent	} is	17 ?
136-139. 12		48 ?	148-151. Of 8.5		3.36 ?
140-143. 21		126 ?	152-155. Of 12.8		5 ?
144-147. 31		98 ?	156-159. Of $18\frac{1}{2}$		$4\frac{1}{2}$?
			160-163. Of $\frac{1}{15}$		

What per cent

164. Of 5,810 lb. are 830 lb. ? | 166. Of 387 rd. are 81 rd. 9 ft. 2 in. ?

165. Of \$2,565 are \$128.25 ? | 167. Of 10 cu. yd. 19 cu. ft. are 7 cu. ft. ?

168. A newspaper publisher who had 5,600 subscribers, lost 224 of them. What per cent of his subscribers did he lose ?

In the examination of a class of 6 members, 170 questions were submitted, of which A answered 130, B 121, C 90, D 120, E 160, and F 144.

169. What was the standing of each member ?

170. What was the standing of the class ?

What per cent of itself must be added to \$3,265 to amount

171. To \$4,571 ? | 172. To \$5,387 $\frac{1}{2}$? | 173. To \$4,832 $\frac{1}{2}$? | 174. To \$11,862.38 ?

From an invoice of 23 C cocoa-nuts an importer sold 18 C ;—

175. The number sold was what per cent of those remaining ?

176. The number remaining was what part of those sold ?

CASE III. The percentage and per cent given, to find base.

427. Ex. 150 is 60% of what number ?

ANALYSIS.—60% $= \frac{60}{100}$. 150 is $\frac{60}{100}$ of 100 times $\frac{1}{60}$ of 150 $= (\frac{150 \times 100}{60}) = 250$.

Or

60% $= \frac{60}{100} = \frac{3}{5}$; 150 is (60% $=$) $\frac{3}{5}$ of 5 times $\frac{1}{3}$ of 150 $= 250$.

IMPORTANT FACT II.—*The base is the quotient of the percentage divided by the per cent (424).*

PROBLEMS.

177. 264 is 10% of what number? **178.** Is $16\frac{2}{3}$ of what number? **179.** Is 20% of what number? **180.** Is $33\frac{1}{3}$ of what number? **181.** Is $8\frac{1}{3}$ of what number?

182-185. 63	is	$\left[\begin{array}{l} 12\frac{1}{2}\% \\ 73\% \\ \frac{8}{9}\% \\ 1.12\% \end{array} \right]$	$\left\{ \begin{array}{l} \text{of} \\ \text{what} \\ \text{num-} \\ \text{ber?} \end{array} \right.$	198. \$325.60 is 24% of how many dollars?
186-189. 1,315				199. 1,325 lb. is 75% of how many lb.?
190-193. 50.04				200. 3 bu. 1 pk. 5 qt. is $16\frac{2}{3}$ of how many bu.?
194-197. $\frac{4}{11}$				201. 4 $\frac{5}{8}$ 3 $\frac{1}{2}$ 1 gr. is 15% of how many $\frac{3}{4}$?

202. $2\frac{1}{2}$ barrels of apples are $\frac{3}{4}\%$ of the yield of my orchard this year. How many bushels did the orchard produce?

203. I paid James E. Bailey 36 per cent of my indebtedness to him, by a draft on New York for \$4,500. How much do I still owe him?

204. I sold a flouring mill, receiving 45% of the price in cash, and invested 75% of the sum received in a city lot worth \$2,160. For how much did I sell the mill?

CASE IV. Base and per cent given, to find amount or difference.

428. By definition (**419, e**), Amount = base + the percentage; but the percentage = base $\times$ per cent; hence,

Amount = base + base $\times$ per cent = base $\times$ (1 + per cent). And

Difference = base - the percentage; hence,

Difference = base - base $\times$ per cent = base $\times$ (1 - per cent).

IMPORTANT FACT III.—*The amount is the product of the base multiplied by 1 plus the per cent.*

IMPORTANT FACT IV.—*The difference is the product of the base multiplied by 1 minus the per cent.*

PROBLEMS.

Find the amount, and the difference,—

205, 206. When the base is 90 and the per cent 22.

207, 208. The base being 89,725 and the per cent $62\frac{1}{2}$.

209, 210. The per cent being $\frac{1}{10}$ and the base $\frac{4}{11}$.

211, 212. If the base is \$89.73 and the per cent is $6\frac{1}{2}$.

213, 214. If the per cent is 35 and the base is 2 gallons.

215, 216. The base being 88, and the per cent 9.

217, 218. The base being $9\frac{1}{2}$, and the per cent $31\frac{1}{2}$.

219, 220. The base being \$19.25, and the per cent $\frac{1}{3}$.

221. A Section of timber land was bought for \$8,600, and was afterward sold for $16\frac{2}{3}\%$ more than it cost. For how much was it sold?

222. 45 per cent of the 860 pupils in a school are boys. How many of the pupils are girls?

223. A merchant invested \$18,000 in business. The first year he increased his capital 15%, the second year he increased the capital at the close of the first year 30%, and the third year he lost 40% of his total investment. Find his capital at the close of the third year.

224. A Texas ranchman lost 17% of his herd of 13,800 cattle. How many cattle had he left?

CASE V. Amount or difference and per cent given, to find base.

429. Amount = base $\times$ (1 + per cent); hence (158, Ax.), base = $\frac{\text{amount}}{1 + \text{per cent}}$. And

Difference = base $\times$ (1 - per cent); hence (158, Ax.), base = $\frac{\text{difference}}{1 - \text{per cent}}$.

IMPORTANT FACT V.—*The base is the quotient*

1. *Of the amount divided by 1 plus the per cent; or*

2. *Of the difference divided by 1 minus the per cent.*

PROBLEMS.

Given, amount	per cent		Given, difference	per cent	
225-227. { 21,854	{ 145,	{ to find base.	234-236. { 137	{ 32,	{ to find base.
228-230. { \$3,719.25	{ 3.3,		237-239. { 2,165 lb.	{ $12\frac{1}{2}$,	
231-233. { 645 $\frac{1}{2}$ bu.	{ $17\frac{1}{2}$,		240-242. { £38 14s.	{ $\frac{1}{3}$,	

243. $12\frac{1}{2}\%$ per cent of a shipment of strawberries was damaged, and 3,542 quarts reached market in good condition. What was the original shipment?

244. An army, after being twice decimated in battle, contained only 7,695 men. How many men did the army originally muster?

245. At the end of last year the population of a Western town was 4,480, which was $16\frac{2}{3}\%$ more than at the beginning of the year. Required the population at the beginning of the year.

246. I paid 4 times as much for a desk for my study as for a chair. Had I paid 10% more for the desk and 4% less for the chair, the two would have cost me \$134. Find the cost of each.

247. In three successive years a manufacturer invests in his business sums equal to 15%, 10%, and 20%, respectively, of his capital at the time of the investment, and in the fourth year he loses 30% of his capital. At the end of the fourth year his capital is \$23,695.98. How much was his original capital?

CASE VI. Amount or difference and base given, to find per cent.

430. Amount = base $\times$ (1 + per cent); hence (158, Ax. 4, 6), per cent = $\frac{\text{amount}}{\text{base}} - 1$.

Difference = base $\times$ (1 - per cent); hence (158, Ax. 4, 6), per cent = $1 - \frac{\text{difference}}{\text{base}}$.

IMPORTANT FACT VI.—*The per cent is*

1. *The quotient, minus 1, of the amount divided by the base. Or*
2. *1 minus the quotient of the difference divided by the base.*

PROBLEMS.

Given, amount	base		Given, base	difference	
248-250. { 3,165	{ 2,180,	to find	257-259. { \$814.73	{ \$530,	to find
251-253. { 4,182.56	{ 3,017.2,	per	260-262. { \$675.18	{ \$496.50,	per
254-256. { 3,921 $\frac{4}{5}$	{ 406 $\frac{3}{4}$,	cent.	263-265. { \$1,201.90	{ \$655,	cent.

266. An ocean steamship left port with 570 registered passengers, and on reaching her destination the number of passengers was increased by stowaways to 589. The stowaways equalled what per cent of the registered passengers?

One trip an excursion steamboat received \$124 for fares at 10¢ per passenger, and the next trip she carried 1,488 passengers at the same fare:

267. The number of passengers on the second trip was what per cent more than the number on the first trip?

268. The number of passengers on the first trip was what per cent less than the number on the second trip?

269. A flock of 300 sheep increased to 675 sheep in 5 years. What was the average yearly per cent of increase?

431. RULES FOR PERCENTAGE.

I. Base and per cent given, to find the percentage :—

Multiply the base by the per cent.

II. The percentage and base given, to find per cent :—

Divide the percentage by the base.

III. The percentage and per cent given, to find base :—

Divide the percentage by the per cent.

IV. Base and per cent given, to find

1. Amount :—*Multiply the base by 1 plus the per cent.*

2. Difference :—*Multiply the base by 1 minus the per cent.*

V. Amount or difference and per cent given, to find base :—

1. *Divide the amount by 1 plus the per cent; or*

2. *Divide the difference by 1 minus the per cent.*

VI. Amount or difference and base given, to find per cent :—

1. *Divide the amount by the base, and subtract 1 from the quotient; or*

2. *Divide the difference by the base, and subtract the quotient from 1.*

The terms base, per cent, and percentage correspond to the terms multiplicand, multiplier, and product. Also base, 1 plus per cent, and amount; and base, 1 minus per cent, and difference, correspond to the same terms. Hence,

Any two of the five elements of percentage being given, the remaining elements may be found.

432. Formulæ for the Six General Cases may be derived as follows:—

Multiplicand $\times$ multiplier = product. I. *Base $\times$ per cent = the percentage.*

$\frac{\text{Product}}{\text{multiplier}} = \text{multiplicand.}$

II. $\frac{\text{The percentage}}{\text{per cent}} = \text{base.}$

$\frac{\text{Product}}{\text{multiplicand}} = \text{multiplier.}$

III. $\frac{\text{The percentage}}{\text{base}} = \text{per cent.}$

IV. (1) *Base $\times$ (1 + per cent) = amount;* (2) *Base $\times$ (1 - per cent) = difference.*

V. (1) $\frac{\text{Amount}}{1 + \text{per cent}} = \text{base; and}$ (2) $\frac{\text{Difference}}{1 - \text{per cent}} = \text{base.}$

VI. (1) $\frac{\text{Amount}}{\text{base}} - 1 = \text{per cent; and}$ (2) $1 - \frac{\text{difference}}{\text{base}} = \text{per cent.}$

SECTION III.

APPLICATIONS OF THE SIX GENERAL CASES

IN WHICH PER CENT IS A SIMPLE ELEMENT.

Per cent is a *simple element* when it is not affected by time; it is a *compound element* when units of time—as days, months, years—are considered in determining it.

Thus,—In finding 20 per cent of any number, the per cent is a simple element; while in finding the profits of a business for 4 years at the uniform rate of 5 per cent per year, the per cent is a compound element, *i.e.*, 20, the per cent, is determined from the 5 per cent per year and the 4 years.

I. COMMISSION.

433. *Commission* comprehends those computations in which an agent's fee or pay is estimated as a percentage.

- a.* An *agent* is a person authorized to transact business for another.
- b.* A *principal* is the person or party for whom an agent transacts business.
- c.* A *commission-merchant*, *factor*, or *broker* is an agent.
- d.* The *commission* is the compensation allowed to an agent for his services.
- e.* *Guaranty* is an extra compensation allowed to an agent, for assuming the risk of loss on merchandise sold on credit.

Commission is commonly estimated as a per cent of the receipts or expenditures of the agent on account of his principal.

434. The business terms used in commission transactions are—*consignment*, *consignor*, *consignee*, *gross proceeds*, *net proceeds*, *remittance*: *e.g.*, A sends his goods to B to be sold on commission. A is the *principal*, B is the *agent*, the goods sent are a *consignment*, A is the *consignor*, B is the *consignee*, the sum for which the goods are sold is the *gross proceeds*, and the sum due A is the *net proceeds*,—*i.e.*, the gross proceeds less all expenses of sale, transportation, and commission.

435. In computations in commission,

- 1. *Sum expended or collected by agent* = *base*;
- 2. *Commission* = *the percentage*;
- 3. *Remittance to agent* (including commission and sum to be expended) = *amount*;
- 4. *Net proceeds* (*i.e.*, *collection*—*commission*) = *difference*.

PROBLEMS.

Find the missing element:

Sum invested or collected	%	commission	Sum invested plus commission	%	base	Sum collected minus commission	%	base
1. \$6,310	2½	\$——	4. \$1,802.50	3	\$——	7. \$937.50	5½	\$——
2. \$416	—	\$18.72	5. \$7,315	—	\$7000	8. \$2,972.80	—	\$3200
3. \$——	1¼	\$192.75	6. \$——	3½	\$2675½	9. \$——	1⅞	\$42,500

10, 11. If you sell books to the amount of \$85, on a commission of 30%, how much is your commission? What are the net proceeds of the sale?

12, 13. If I send to you \$225, and instruct you to expend it for sheep at \$5 per head after retaining for yourself a commission of 12½%, how many sheep should you buy to fill the order? How much should you retain for your commission?

14, 15. If I remit to you \$142.50 as the proceeds of a collection after retaining my commission of 5%, how much do I retain? What is the amount of the collection?

II. COMMERCIAL DISCOUNTS.

436. Discount is a deduction from, or abatement of, a sum of money that is the basis of a business transaction.

437. Commercial discounts are percentages based upon the list price of goods, the total footing of bills, or the gross amount of an obligation,—per cent being a simple element.

Commercial discounts include *trade discounts*, *time discounts*, and *cash discounts*.

- a. Trade discounts* are percentages deducted from list prices.
- b. Time discounts* are percentages deducted from the amount of a bill of merchandise, as a consideration for payment within a given time.
- c. Cash discounts* are percentages deducted from the amount of a bill of merchandise, sold on credit, as a consideration for immediate payment.
- d.* Trade discounts are made, to obviate the necessity for changing prices in the catalogues and price lists of manufacturers, jobbers, and wholesale dealers, to meet changes in market prices. The list price remains the same, while the trade discount increases as the goods decline in value, and decreases as they advance in value.
- e.* The *net price* is the list price less the trade discount.
- f.* The fluctuations of the market sometimes give rise to two or more trade discounts in succession, in which case the first is deducted from the list price, the second from the remainder,—and so on.

438. M sells a bill of goods to N at certain list prices 16% off. The terms are 60 da., 5% 30 da., or 8% cash. 16% is the *trade discount*, 60 da. from the date of the bill is the *term of credit*; 5% is a *time discount*,—allowed if the goods are paid for in 30 da. from the date of the bill; and 8% is a *cash discount*,—allowed if the goods are paid for at once.

439. In computations in commercial discounts,

1. *List price = base*; also, *Amount of bill or obligation = base*;
2. *Discount = the percentage*; | 3. *Net price = difference*.

Find the missing element :

PROBLEMS.

List price	%	trade discount	Amount of obligation	%	discount	Net price or proceeds	%	list price, or obligation
16. \$12.50	5	\$—	19. \$659.25	2½	\$—	22. \$.12½	20	\$—
17. \$86.25	—	\$17.25	20. \$9,800	—	\$220.50	23. \$675	—	\$810
18. \$—	4½	\$.25⅞	21. \$—	1	\$68.75	24. \$—	15	\$22,950

How much greater is a single discount of 30%

25. Than two successive discounts of 20% and 10%?
26. Than two of 10% and 20% successively?
27. Than two of 15% each?
28. Than three of 15%, 10%, and 5%?

29. If a grocer's purchases of goods for a year amount to \$12,380, how much will he save by cashing all bills at 3% discount?

If you buy a phaeton listed at \$210, 40% off, terms 90 da. or 5% cash,

30. How much ready money will pay the bill?

31. How much is the trade discount? | 32. How much is the cash discount?

III. PROFIT AND LOSS.

440. *Profit and loss* comprehends those computations in which any increase or decrease in values of property is estimated as a percentage.

a. *Prime cost* is the sum actually paid for property.

b. *Gross cost* or *entire cost* is the prime cost increased by expenses,—such as packing, freight, cartage, repairs, commissions, etc.

c. Whether the prime cost or the entire cost is to be taken as the basis of computation, depends upon the conditions of the problem.

441. In computations in profit and loss,

1. *Cost = base*;
2. *Gain or loss = the percentage*;
3. *Selling price at a profit = amount*;
4. *Selling price at a loss = difference*.

PROBLEMS.

Find the missing element :

Cost	%	gain or loss	Selling price at a gain	%	cost	Selling price at a loss	%	cost
33. \$.15	33 $\frac{1}{3}$	\$—	36. \$.11	37 $\frac{1}{2}$	\$—	39. \$.33	12	\$—
34. \$5,825	—	\$843.75	37. \$40	—	\$24	40. \$96	—	\$120
35. \$—	22 $\frac{1}{2}$	\$15.18 $\frac{1}{2}$	38. \$—	16	\$9.37 $\frac{1}{2}$	41. \$—	68 $\frac{1}{2}$	\$6,875

42. A newsboy pays 2¢ for newspapers and sells them for 5¢. What is his gain per cent?

43. A merchant sold goods amounting to \$24.37 $\frac{1}{2}$ at a loss of 6 $\frac{1}{2}$ %. How much did the goods cost him?

44. A fruit-dealer sold bananas that cost him 1 $\frac{1}{2}$ ¢ each for 5¢ each, and oranges that cost him $\frac{3}{4}$ ¢ each for 2¢ each. On which fruit did he gain the greater per cent, and how much the greater?

45. If you sell a horse for \$175 and gain 16 $\frac{2}{3}$ %, how much is your gain?

46. How much did the horse cost you?

47. What per cent of the selling price is gain?

IV. INSURANCE.

442. *Insurance* comprehends those computations arising under contracts for indemnity against loss or damage.

a. The *insured* is the person protected or indemnified.

b. The *insurer* or *under-writer* is the party assuming the risk.

c. A *policy* is a written contract between the parties, that sets forth the liability of the insurer and the conditions imposed upon the insured.

d. *Valuation* is the maximum sum contracted to be paid by the insurer, in case of loss or damage to the insured. It is commonly not more than $\frac{3}{4}$ of the value of the property insured.

e. *Premium* is the sum paid for insurance.

f. The *sum covered* by insurance is the valuation less the premium.

Per cent in insurance is also called *rate*.

g. A *risk* is the property insured.

e.g., If the Continental Insurance Co. insures A's store for 1 year in the sum of \$2,000, for which A pays the company \$28,—the Continental is the *insurer*, A is the *insured*, \$2,000 is the *valuation*, 1 year is the *time*, \$28 is the *premium*, \$1,972 (= \$2,000 - \$28) is the *sum covered* by insurance, and 1 $\frac{4}{5}$ per cent is the *rate*.

443. An *insurance company* is a company or a corporation whose business is to indemnify against loss or damage.

Insurance companies commonly make a specialty of some one kind of insurance from which they take their class name,—as *fire, marine, life, accident, live-stock, guaranty*. (See Supplement, page 426.)

444. *Fire insurance* is indemnity against loss by fire.

445. *Marine insurance* is indemnity against loss by the casualties of navigation.

446. In computations in insurance,

1. *Valuation = base;* | 2. *Premium = percentage;*
3. *Amount covered by insurance = difference.*

PROBLEMS.

Find the missing element:

Valuation	%	premium	sum covered	Valuation	%	premium	sum covered
48. \$3,250	2	\$——	\$——	51. \$——	—	\$253.75	\$14,500
49. \$4,875	—	\$60.93 $\frac{1}{2}$	\$——	52. \$——	$\frac{7}{8}$	\$156.18 $\frac{1}{2}$	\$——
50. \$——	$\frac{5}{8}$	\$——	\$12,453.12 $\frac{1}{2}$	53. \$165,000	—	——	\$164,587.50

54. Find the cost of insuring a house and furniture for \$7,815, at 1 $\frac{1}{4}$ %.

55. What valuation must be written in a policy of insurance at 2%, to cover a loss of \$11,350?

56. What is the valuation on a barn and contents, the premium being \$74, at 1 $\frac{1}{8}$ %?

57. If the premium on a valuation of \$27,000 is \$438.75, what is the per cent or rate?

V. STOCKS.

447. *Stocks*, as a general term, includes Government and State bonds and the stocks and bonds of corporations.

448. A *corporation* is an association of individuals authorized by law to transact some particular business as one party.

449. A *charter* is a law creating a corporation and defining its powers, privileges, duties, and obligations.

450. Capital stock is the money or other property invested in the business of a corporation.

a. A **share** is one of the equal parts into which the capital stock of a corporation is divided. It is commonly \$100.

b. A **stockholder** is the owner of one or more shares.

The stockholders constitute the corporation; and although they may change, and their number increase or decrease, the corporation remains the same.

c. A **certificate of stock** is a document bearing the seal of the corporation, issued to a stockholder, and stating the value of each share, and the number of shares such certificate represents.

d. An **instalment** is a portion of the capital stock paid in by the stockholders, as it is required, to establish or "*float*" the business of the corporation.

451. Securities is a general term used to include all stocks, written evidences of indebtedness, or promises to pay.

452. A bond is a certificate of indebtedness, under seal.

a. The bonds of a business corporation are commonly secured by mortgage upon its property; those of a municipal corporation are a lien upon its taxable property.

b. Bonds are commonly named from the authority that issues them and the rate of interest they bear.

Virginia 6's are bonds bearing 6% interest, issued by the State of Virginia.

Union Pacific 7's are bonds bearing 7% interest, issued by the Union Pacific railroad company, and secured by mortgage upon the railroad.

453. LIST OF THE PRINCIPAL U. S. BONDS OUTSTANDING JAN. 1, 1882.

Names of Bonds.	When Redeemable.	Rates of Interest.	Interest Payable.
Four-and-a-halves of 1891	After Sept. 1, 1891.	4½%	Quarterly.
Fours of 1907	After July 1, 1907.	4%	Quarterly.
Three-and-a-halves	At option of Gov't.	3½%	
U. S. Pacific R.R. cur- }	{ July 1, 1892, and }	6%	Semi-annually.
rency sixes }	{ July 2, 1894. }	6%	Semi-annually.
Currency sixes	1895 to 1899.	6%	Semi-annually.

454. The par value of stocks or bonds is their face value; and

The **market value** is what they will sell for in the market.

a. Stocks and bonds are **above par** or at a **premium**, when their market value is greater than their par value; and they are **below par** or at a **discount**, when their market value is less than their par value.

b. Stock quotations show the market value of stocks in the place where the quotations are made.

Thus, if stock is 8% above par, it is quoted at 108; and if it is 8% below par, it is quoted at 92.

The city of New York is the great centre of stock transactions for the United States, and the quotations are commonly made from its market or exchange.

c. Stocks and bonds $\left\{ \begin{array}{l} \text{at par} \\ \text{above par} \\ \text{below par} \end{array} \right\}$ are quoted $\left\{ \begin{array}{l} \text{at 100.} \\ \text{above 100.} \\ \text{below 100.} \end{array} \right.$

d. The customary rates of brokerage are from $\frac{1}{4}\%$ to $\frac{1}{2}\%$ on the par value of the stocks bought or sold.

e. The brokerage is added to the market price or quotation in computing the cost of stocks, and is subtracted from the market price in computing their proceeds. *e.g.*, If you buy N. Y. 6's at 92, brokerage $\frac{1}{4}\%$, the bonds cost you $92\frac{1}{2}$ after paying brokerage; while, if you sell the same bonds at the same rate and brokerage, you realize only $91\frac{1}{2}$ after paying the brokerage.

f. Stock operations are the buying and selling of stocks, with a view to realize a gain from their rise and fall in the market.

Stock investments are the buying of stocks and bonds, for the purpose of securing the income from them.

g. The income from stocks consists of *dividends*,—*i.e.*, of the profits of the business of the corporation divided among the stockholders in proportion to their several amounts of stock. It depends upon the condition of the business of the corporation.

The income from bonds is fixed; it is not affected by the prosperity of the business of the corporation.

455. In computations in stock operations,

1. *Par value = base.* | 2. *Premium or discount = percentage.*
3. *Quotations at a premium or above par = 1 + per cent.*
4. *Quotations at a discount or below par = 1 - per cent.*
5. *Market value above par = amount.* | 6. *Market value below par = difference.*

456. In computations in stock investments,

1. *Market value = base.* | 2. *Income = percentage.*

PROBLEMS.

Stock operations.—Find the missing element:

Shares	%	Market value	Shares	%	Market value
58. 32	$5\frac{1}{2}$ premium	\$————	61. 54	— discount	\$4,914.75
59. 81	15 discount	\$————	62. —	$16\frac{3}{4}$ above par	\$14,000
60. 17	— premium	\$1,874.87 $\frac{1}{2}$	63. —	$12\frac{1}{2}$ below par	\$5,153.50

Find the cost of

	Shares	Quoted at	Brokerage
64.	15	107 $\frac{1}{8}$	$\frac{1}{8}\%$
65.	230	81 $\frac{1}{4}$	$\frac{1}{4}\%$
66.	75	103 $\frac{1}{2}$	$\frac{3}{16}\%$

Find proceeds from the sale of

	Shares	Quoted at	Brokerage
67.	500	101 $\frac{1}{8}$	$\frac{1}{4}\%$
68.	744	97 $\frac{3}{4}$	$\frac{1}{8}\%$
69.	2,250	61 $\frac{3}{8}$	$\frac{1}{16}\%$

Stock investments.—Find the missing element:

	Quotation	Income on par value	Investment pays		Quotation	Brokerage	Income on par value	Investment pays
70.	102	4%	—%	73.	44 $\frac{1}{8}$	$\frac{1}{4}\%$	5%	—%
71.	—	3 $\frac{1}{2}\%$	5%	74.	—	$\frac{1}{8}\%$	7 $\frac{3}{10}\%$	5 $\frac{70}{100}\%$
72.	81 $\frac{1}{4}$	—%	7 $\frac{1}{8}\%$	75.	86 $\frac{1}{2}$	$\frac{3}{16}\%$	—%	61 $\frac{1}{3}\frac{1}{4}\%$

76. Find the cost of 18 shares of B. & A. RR. stock at 108 $\frac{3}{8}$, brokerage $\frac{1}{8}\%$.77. Find the net proceeds of a sale of 26 shares of C. C. & C. RR. stock at 82, brokerage $\frac{1}{4}\%$?78. If you gain \$256.50 by buying Indiana State bonds at 98 and selling them at 102 $\frac{1}{2}$, what is the par value of the bonds?79. Which is the more profitable investment, and how much per cent, U. Pac. 7's at 136 or Mich. Cent. 6's at 108 $\frac{1}{2}$?80. At what price must U. S. 4 $\frac{1}{2}$'s be bought, to net 4% on the investment?

81. If 23 shares of the Chemical Bank stock were bought for \$3,892.75, what per cent premium did the stock bear?

82. If Georgia 6's are 8% below par, how much money must be invested in them to obtain an annual income of \$870?

83. If Southern Pacific RR. stock is at 30 $\frac{3}{8}\%$ discount, how many shares can be bought for \$2,720.25, brokerage $\frac{1}{8}\%$?84. What per cent on his money will a man receive, by investing in Florida 7's at 92 $\frac{1}{4}$?85. Find the loss on 81 shares of stock purchased for 87 $\frac{1}{4}$ and sold for 84 $\frac{1}{8}$, brokerage $\frac{1}{8}\%$ both for buying and selling.

86. The gross earnings of a manufacturing company this year are \$241,963.75, and the total expenses are \$204,588.75. What per cent dividend can the company declare on its capital stock of \$575,000?

87. A 2% quarterly dividend by a Life Insurance Co. paid a stockholder at the rate of 9 $\frac{7}{8}\%$ annually. At what price did he buy the stock?

VI. TAXES.

457. *Taxes* are sums of money levied and collected by general, State, or local governments, for public purposes.

458. Taxes levied by the General Government are of two kinds,—*duties* or *customs*, and *internal revenue*. (See Supplement, page 438.)

459. Taxes levied by State and local governments are of two kinds,—*property tax* and *poll tax*.

a. Property tax is a tax on property in proportion to its assessed valuation.

b. Poll tax or *capitation tax* is a tax on adult male citizens without regard to property. It is a certain sum per capita, and therefore is not a percentage.

c. Real-estate is fixed or immovable property,—such as lands, buildings, quarries, mines, etc. And

Personal property is movable property,—such as money, notes, furniture, tools, live-stock, etc.

d. Assessors are officers selected to estimate the value of taxable property,—*i. e.*, property liable to taxation.

e. An assessment-roll is a schedule of taxable property, its owners, location, kinds, and *assessed valuation*—or valuation as determined by the assessors.

NOTE.—In some States an assessment-roll is known as “*The Grand List*.”

f. A collector is an officer selected to receive and collect taxes.

g. The rate of taxation is the sum charged to each dollar of the assessed valuation, to raise the required amount of tax.

h. In some States a certain percentage of the whole tax is levied upon the polls, *i. e.*, citizens liable to poll tax; in others the poll tax is a fixed sum for each poll; and in others the whole tax is paid by the property. In some States that levy a poll tax, this tax is applied to some particular purpose, and does not appear upon a general tax-list.

460. In some States the fees of collectors are a per cent of the sum collected; and the amount levied must include the tax to be raised and the collector's commission or fees. In other States the tax-payer pays collector's fees in addition to the tax levied against him.

461. In computations to determine the total sum to be included in a tax-list, to cover both the tax and the collector's fees,—

1. *Amount of tax = difference.* | 2. *Collector's fees = the percentage.*

462. Owing to the diversity of methods of raising taxes in the several States, the only computations shown in this article are those arising from apportioning a property tax.

In computations in property taxes,—

1. *Assessed valuation = base.* | 2. *Tax = the percentage.*

PROBLEMS.

Find the missing element :

Assessed valuation	%	Tax	Assessed valuation	%	Tax
88. \$182,425	$\frac{3}{8}$	\$———	90. \$———	$1\frac{1}{8}$	\$14,186.54
89. \$81,360	—	\$420.56	91. \$2,456,890	—	\$8,491.88

463. In practical tax computations a *Tax Table* is made, by the use of which the only process employed is addition. The construction and use of such a table is shown in

Ex. 1. The total assessment of a school district is \$847,328, and a tax of \$1,187.50 is voted. Construct the requisite tax table.

PROCESS.

$\$1,187.50 \div 847,328 = \$.00140147$ —, the rate of taxation.

Multiplying the rate of taxation by 1, 2, 3, 4, 5, 6, 7, 8, 9, successively, and arranging the results in a column, we have the column at the left below.

TAX TABLE.

	Tax on	Tax on	Tax on	Tax on	Tax on
\$.00140147	\$1 =	\$10 = \$.01	\$100 = \$.14	\$1,000 = \$1.40	\$10,000 = \$14.01
.00280294	2 =	20 = .08	200 = .28	2,000 = 2.80	20,000 = 28.08
.00420441	3 =	30 = .04	300 = .42	3,000 = 4.20	30,000 = 42.04
.00560588	4 = \$.01	40 = .06	400 = .56	4,000 = 5.61	40,000 = 56.06
.00700735	5 = .01	50 = .07	500 = .70	5,000 = 7.01	50,000 = 70.07
.00840882	6 = .01	60 = .08	600 = .84	6,000 = 8.41	60,000 = 84.09
.00981029	7 = .01	70 = .10	700 = .98	7,000 = 9.81	70,000 = 98.10
.01121176	8 = .01	80 = .11	800 = 1.12	8,000 = 11.21	80,000 = 112.12
.01261323	9 = .01	90 = .13	900 = 1.26	9,000 = 12.61	90,000 = 126.13

For explanation of Tax Table, see page 264.

EXPLANATION OF TAX TABLE.

(1) The several taxes are written to the nearest cent,—business custom being followed when fractions of cents are involved.

(2) The taxes shown in the first or left-hand column of the table are taken from the cents and mills of the column obtained by multiplication.

(3) Since the amounts on which taxes are to be found—in the second column of the table—are respectively *ten times* the corresponding amounts in the first column, the decimal points in the column of products are moved one place to the right, to obtain the taxes for the second column.

(4) The taxes in the third column are obtained by moving the decimal points one more place to the right;—and so on for the remaining columns.

The table may be extended another column, or as far as necessary, to meet the largest amount in the assessment roll.

Ex. 2. From the foregoing table, find the amount of tax (1) on \$5,728; (2) on \$16,870.

PROCESSES.

(1) Tax on \$5,000 = \$7.01	(2) Tax on \$10,000 = \$14.01
“ “ 700 = .98	“ “ 6,000 = 8.41
“ “ 20 = .03	“ “ 800 = 1.12
“ “ 8 = .01	“ “ 70 = .10
“ “ <u>\$5,728</u> <u>\$8.03</u>	“ “ <u>\$16,870</u> <u>\$23.64</u>

PROBLEMS.

From the foregoing table find the tax

92. On \$3,185	94. On \$29,817	96. On \$425	98. On \$37,400	100. On \$6,425
93. On \$921	95. On \$11,100	97. On \$760	99. On \$8,250	101. On \$45,280

102. The following sums are the assessments against individuals in an assessment roll from which a tax of \$2,180 is to be apportioned:—
 \$12,272; \$925; \$6,130; \$17,780; \$480; \$19,050; \$7,800; \$5,500;
 \$26,840; \$51,700; \$2,500; \$4,375; \$121,900; \$64,250; \$637; \$288;
 \$18,620; \$82,300; \$14,440; \$1,970; \$3,300; \$2,160; \$11,045; \$520;
 \$7,200; \$110,480; \$4,175; \$25,800; \$8,330; \$72,650; \$9,875; \$4,370.
 Make a tax table, and find the tax to be set in the assessment roll against each of the assessments, in the order in which they are given.

103. Using the assessments given in problem 102, make a table and find each tax in a total levy of \$1,530.86.

FORMULÆ OF PERCENTAGE,—PER CENT A SIMPLE ELEMENT. 265

Pupils should be required to make a copy of this blank, and to write the proper formula for each case in each subject, using the terms corresponding to the elements of the percentage formula.

464. OUTLINES FOR ALL THE FORMULÆ OF PERCENTAGE

	<i>The percentage = base × per cent</i>	<i>Amount = base × (1 + per cent)</i>	<i>Difference = base × (1 - per cent)</i>
Commission	$\frac{\text{---}}{\text{---}} = \frac{\text{---}}{\text{---}} \times \frac{\text{---}}{\text{---}}$	$\text{---} = \text{---} \times \frac{\text{---}}{\text{---}}$	$\text{---} = \text{---} - \frac{\text{---}}{\text{---}} \times \frac{\text{---}}{\text{---}}$
Commercial Discount	$\frac{\text{---}}{\text{---}} = \frac{\text{---}}{\text{---}} \times \frac{\text{---}}{\text{---}}$	$\text{---} = \text{---} \times \frac{\text{---}}{\text{---}}$	$\text{---} = \text{---} - \frac{\text{---}}{\text{---}} \times \frac{\text{---}}{\text{---}}$
Profit and Loss	$\frac{\text{---}}{\text{---}} = \frac{\text{---}}{\text{---}} \times \frac{\text{---}}{\text{---}}$	$\text{---} = \text{---} \times \frac{\text{---}}{\text{---}}$	$\text{---} = \text{---} - \frac{\text{---}}{\text{---}} \times \frac{\text{---}}{\text{---}}$
Insurance	$\frac{\text{---}}{\text{---}} = \frac{\text{---}}{\text{---}} \times \frac{\text{---}}{\text{---}}$	$\text{---} = \text{---} \times \frac{\text{---}}{\text{---}}$	$\text{---} = \text{---} - \frac{\text{---}}{\text{---}} \times \frac{\text{---}}{\text{---}}$
Stocks	$\frac{\text{---}}{\text{---}} = \frac{\text{---}}{\text{---}} \times \frac{\text{---}}{\text{---}}$	$\text{---} = \text{---} \times \frac{\text{---}}{\text{---}}$	$\text{---} = \text{---} - \frac{\text{---}}{\text{---}} \times \frac{\text{---}}{\text{---}}$
Taxes	$\frac{\text{---}}{\text{---}} = \frac{\text{---}}{\text{---}} \times \frac{\text{---}}{\text{---}}$	$\text{---} = \text{---} \times \frac{\text{---}}{\text{---}}$	$\text{---} = \text{---} - \frac{\text{---}}{\text{---}} \times \frac{\text{---}}{\text{---}}$

IN WHICH PER CENT IS A SIMPLE ELEMENT.

<i>Base = the percentage ÷ per cent</i>	<i>Per cent = the percentage ÷ base</i>	<i>Base = $\frac{\text{Amount} + (1 + \text{per cent})}{\text{Difference} + (1 - \text{per cent})}$</i>
$\frac{\text{---}}{\text{---}} = \frac{\text{---}}{\text{---}} \div \frac{\text{---}}{\text{---}}$	$\frac{\text{---}}{\text{---}} = \frac{\text{---}}{\text{---}} \div \frac{\text{---}}{\text{---}}$	$\text{---} = \frac{\text{---} + \frac{\text{---}}{\text{---}} + \frac{\text{---}}{\text{---}}}{\text{---} + \frac{\text{---}}{\text{---}} + \frac{\text{---}}{\text{---}}}$
$\frac{\text{---}}{\text{---}} = \frac{\text{---}}{\text{---}} \div \frac{\text{---}}{\text{---}}$	$\frac{\text{---}}{\text{---}} = \frac{\text{---}}{\text{---}} \div \frac{\text{---}}{\text{---}}$	$\text{---} = \frac{\text{---} + \frac{\text{---}}{\text{---}} + \frac{\text{---}}{\text{---}}}{\text{---} + \frac{\text{---}}{\text{---}} + \frac{\text{---}}{\text{---}}}$
$\frac{\text{---}}{\text{---}} = \frac{\text{---}}{\text{---}} \div \frac{\text{---}}{\text{---}}$	$\frac{\text{---}}{\text{---}} = \frac{\text{---}}{\text{---}} \div \frac{\text{---}}{\text{---}}$	$\text{---} = \frac{\text{---} + \frac{\text{---}}{\text{---}} + \frac{\text{---}}{\text{---}}}{\text{---} + \frac{\text{---}}{\text{---}} + \frac{\text{---}}{\text{---}}}$
$\frac{\text{---}}{\text{---}} = \frac{\text{---}}{\text{---}} \div \frac{\text{---}}{\text{---}}$	$\frac{\text{---}}{\text{---}} = \frac{\text{---}}{\text{---}} \div \frac{\text{---}}{\text{---}}$	$\text{---} = \frac{\text{---} + \frac{\text{---}}{\text{---}} + \frac{\text{---}}{\text{---}}}{\text{---} + \frac{\text{---}}{\text{---}} + \frac{\text{---}}{\text{---}}}$
$\frac{\text{---}}{\text{---}} = \frac{\text{---}}{\text{---}} \div \frac{\text{---}}{\text{---}}$	$\frac{\text{---}}{\text{---}} = \frac{\text{---}}{\text{---}} \div \frac{\text{---}}{\text{---}}$	$\text{---} = \frac{\text{---} + \frac{\text{---}}{\text{---}} + \frac{\text{---}}{\text{---}}}{\text{---} + \frac{\text{---}}{\text{---}} + \frac{\text{---}}{\text{---}}}$

PROBLEMS.

104. I travelled 64% of a journey of 3,250 miles by rail, 53% of the remainder of the distance by steamer, and completed the journey by stage. How many miles did I travel by each conveyance?

105. A produce broker buys 14,000 bushels of corn @ 55¢, commission $\frac{3}{4}$ ¢ per bushel. How much money will cover cost and commission?

106. The net proceeds on a shipment of dried fruit are \$3,137, and the commission is \$94.11. What per cent commission is paid?

107. Novelty castings that cost $4\frac{1}{2}$ ¢ $\varnothing$ lb. to manufacture, must be sold at what price $\varnothing$ lb. to realize $11\frac{1}{3}$ % profit?

What is the commission

108. At $2\frac{1}{4}$ %, on a sale of 3,275 bushels of wheat @ \$1.08?

109. At 2%, on a sale of a farm of 156 acres @ \$62.50?

110. At $3\frac{1}{8}$ %, on a collection the net proceeds of which are \$1,037.50?

111. At $6\frac{1}{4}$ %, on a purchase of wool, cost and commission \$9,607.97?

112. At 5%, on a purchase of beans, cost and commission \$1,460?

113. \$721 $\frac{1}{2}$ for insuring a cargo of 1,250 barrels of beef valued at \$12 $\frac{3}{4}$ per barrel, is at what rate of insurance?

114. I paid a real-estate broker \$2,844.50 to cover cost of Texas lands at \$7.50 per acre, expenses for deeds \$9.50, and his commission of 5%. How much land did he buy for me?

115. If the premium, at $\frac{7}{8}$ %, is \$23, what sum does the policy cover?

116. A horse dealer who asked 40% advance on cost for a horse, accepted an offer of 15% less than his asking price. How much per cent did he gain?

117. I sold $\frac{7}{8}$ of a mill for the sum I paid for $\frac{3}{4}$ of it. How much per cent did I lose?

118. At what price must 170 shares of stocks be sold, to net a premium of \$420, brokerage $\frac{1}{4}$ %?

119. At $5\frac{1}{4}$ ¢ $\varnothing$ yd., how many yards of prints can a broker buy for \$275.25 and retain $2\frac{1}{2}$ % commission on the money invested?

Allowing brokerage at $\frac{1}{8}$ %, what is the par value

120. Of Neptune Silver Lode Co., bought for \$36,733 at 42?

121. Of N. Pac. RR., bought for \$5,525 at $106\frac{1}{2}$?

122. Of L. S. Copper Mining Co., bought for \$2,913 at $121\frac{1}{2}$?

Find the commercial discount

123. On \$842, 22% off.

125. On \$748.60, 20%, 15%, and $2\frac{1}{2}\%$ off.

124. On \$3,684, $33\frac{1}{3}\%$ and 10% off.

126. On \$98.42, 35, 25, and 2% off.

127. The gross amount of a bill of builders' hardware is \$1,432; terms 30, 5, and 5% off, 3 mo., and $2\frac{1}{2}\%$ off cash in 10 da. What is the cash value of the bill?

128. The list price of a piano-forte is \$650, and the trade discounts are 25, 15, and 5% 6 mo., and 10% for cash. What is the net cash cost?

129. When 5 per cents pay 7% income, at what price should they be quoted?

130. In July I bought 7 T. of furnace coal @ \$5.37 $\frac{1}{2}$, $4\frac{1}{2}$ T. of stove coal @ \$5.10, and 3 T. of chestnut coal @ \$4.87 $\frac{1}{2}$. August 1 prices advanced to \$5.50 for furnace coal, \$5.35 for stove, and \$5 for chestnut. What per cent on the cost of my coal did I save by buying in July?

131. A New York commission house remits \$2,332.20 as the proceeds of a sale of 416 barrels of flour @ \$5.75. The commission charges were how much per cent?

132. An annual premium of \$96 is paid for a policy of \$18,000 on a high-school building. What per cent of premium is paid?

133. Find the per cent of loss on merchandise bought for \$462 and sold for \$346.50.

134. A wholesale grocer sold a part of a stock of sugar @ \$43.60 $\text{\textcircled{P}}$ hhd. and realized 8% profit. He sold the remainder @ \$36 $\text{\textcircled{P}}$ hhd. Did he gain or lose on the second sale, and what per cent?

135. At what price must I buy Government $3\frac{1}{2}\%$'s, to secure an income of 5% on my investment?

136. The catalogue price of mowing-machine sundries is 36% above cost, and the trade discount is 15%. The net price is what per cent above cost?

137. A ship valued at \$65,000 and her cargo at \$133,000 are insured—the ship for $\frac{1}{2}$ of her value at $3\frac{3}{8}\%$ premium, and the cargo for $\frac{1}{3}$ of its value at $4\frac{1}{8}\%$ premium. The ship and cargo are a total loss. What is the loss to the owners?

What is the selling price

Of that cost to sell at	Of bought at to sell at
138. Pepper 30¢ p lb. , 15% gain?	141. Prints 6¢ p yd. , 18 $\frac{1}{2}$ % profit?
139. Coffee 16¢ p lb. , 22 $\frac{1}{2}$ % gain?	142. Napkins 65¢ p doz. , 15 $\frac{1}{3}$ % profit?
140. Tea 38¢ p lb. , 15% loss?	143. Gloves \$1.15 p pr. , 40% loss?

144. The net proceeds of a sale of 116 shares of stocks, brokerage $\frac{1}{4}$ %, were \$8,482.50. Find the market quotation.

145. At what per cent of cost must a jobber list barbed wire, so that he can make a trade discount of 25% and still gain 5%?

146. Which gives the greater net price, and how much the greater,—30% off or 20% and 2 fives?

147. Which is the better investment, and how much the better,—8% stocks at 20% discount, or 12% stocks at 15% premium?

Find the per cent of gain on goods

148. Sold at $\frac{1}{2}$ of their cost. | **149.** Bought for \$730 and sold for \$894.25.

Find the net cash values of invoices amounting to

150. \$750, 18 $\frac{1}{2}$ % and 5% off. | **152.** \$1,020, 45, 15, and 7 $\frac{1}{2}$ % off.
151. \$255 on 30 da., 1 $\frac{1}{4}$ % off for cash. | **153.** \$1,615.93, 50, 30, 20, 10, and 5% off.

154. What is the brokerage, at $\frac{3}{8}$ %, and what are the net proceeds, from a sale of 98 shares of Pa. RR. stock at 116 $\frac{1}{2}$?

155. I pay a premium of \$68 for a policy of insurance on my house and contents at 2 $\frac{1}{8}$ % premium. For what sum is the property insured?

156. An auctioneer who charges 10% on sales, sold property for 30% less than its estimated value, and paid the owner \$2,604 as the net proceeds of the sale. What was the estimated value of the property?

157. Goodspeed & Co. bought 2 doz. 2d Readers @ 40¢ ea., 25 and 5% off; 4 doz. 4th Readers @ \$1.20 ea., 25 and 5% off; and 4 doz. Copy-books @ \$1.80, 33 $\frac{1}{3}$ % off. They sold the goods at 25% advance on net cost. How much were their profits?

158. What is the difference between 50, 12 $\frac{1}{2}$, and 10% off, and 60% off, on a bill of \$2,400?

159. The net proceeds are \$2,010.72 and the commission is 4 $\frac{1}{2}$ %. What is the amount of sales?

160. Find the per cent gained by buying goods at $\frac{5}{8}$ of their market value and selling them at $12\frac{1}{2}\%$ above their market value.

161. Which will give the greater income, and how much per cent the greater—5% stocks at 110 or 10% stocks at 105?

Find the cost

Of	marked	to sell at	Of	marked	to sell at
162. Linen	42¢	per yd., 15% profit?	165. Perfumery	\$.75	per bottle, $87\frac{1}{2}\%$ gain?
163. Dress silk	\$1.85	per yd., $31\frac{1}{4}\%$ profit?	166. Hair brushes	\$1.90	each, 15% loss?
164. Cambrics	17¢	per yd., 15% loss?	167. Hand mirrors	\$.87½	each, $48\frac{1}{4}\%$ gain?

168. I bought parlor furniture at 25 and 10% off from list prices, and paid \$303.75 for it. At how much was it listed?

169. I buy Brooklyn City 6% water bonds of 1913 for 127. What per cent does the investment pay me?

170. If you exchange 260 shares of 1st National Bank stock at 13% premium, for Mich. S. RR. stock at 9% discount, how many shares of the RR. stock will you receive?

171. How many 500-dollar Government 3% bonds can I buy for \$28,476.50, at $9\frac{1}{4}\%$ above par, brokerage $\frac{1}{8}\%$?

An invoice of tin ware, bought June 1, amounts to \$660,—terms 40, 10, and 2% off on 3 mo., $2\frac{1}{2}\%$ additional for cash in 10 days. What sum will cancel the bill,—**172.** If it is paid June 6? | **173.** If paid August 20?

174. A real-estate agent sold a city house for \$13,750, and invested the proceeds,—less his commissions of $2\frac{1}{2}\%$ on sales and $1\frac{1}{2}\%$ on investments,—in Western lands. How much did his commissions amount to?

175. A broker sold a consignment of petroleum for \$.96½ per barrel, and invested the proceeds, less his commissions, in Porto Rico sugars @ $4\frac{1}{8}\%$ per lb. His commissions, at 2% on sales and $1\frac{1}{2}\%$ on purchases, amounted to \$334.05½. How many barrels of petroleum did he sell?

176. Required, the premium for insuring a cotton factory 5 years at $1\frac{1}{3}\%$ premium per year, the valuation being \$88,500.

177. A ranchman bought 7,880 sheep at \$4 per head. 15% of the flock died, and he sold the remainder at 23% above cost. Did he make or lose, and how much?

178. A premium of \$843.25, at $2\frac{1}{4}\%$, was paid for insuring a hotel and furniture at one half their value. What was the full value of the property insured?

179. A wholesale dealer sells ladies' gaiters, by the case of 24 pairs, at \$92.16 and gains 20%. If he should sell them at the rate of \$2.80 per pair, would he gain or lose, and how much per cent?

180. A carpet dealer sold a Turkish rug for \$662.50 and gained $32\frac{1}{2}\%$. How much did the rug cost him?

181. A merchant buys Moquette carpeting at \$1.15 per yard, and marks it at a price from which he can make a trade discount of 10% and still gain 20%. At what price does he mark the carpeting?

Find the proceeds from the sale of stocks, as follows:

182. 30 shares Pipe Line at $170\frac{1}{2}$, brokerage $\frac{1}{8}\%$.

183. 17 shares T. R.R. at $8\frac{1}{2}\%$ premium, brokerage $\frac{1}{8}\%$.

184. 35 shares B. & O. Telegraph at $6\frac{1}{2}\%$ discount, brokerage $\frac{3}{8}\%$.

185. Sold to Anson Adams, this day, 3 pc. moire-antique dress silks, 45 yd., 45 yd., and 50 yd., @ $\$2.12\frac{1}{2}$, 10% off; 2 pc. silk-velvet, 95 yd., @ \$3.25, 60 da., 8% off; 4 pc. satin, 29 yd. ea., @ \$1.20, 15% off; and 10 pc. muslin, $41\frac{1}{2}$ yd. ea., @ 15¢, 20 and 10% off. Boxing and cartage 75¢. Make a receipted bill, deducting 5% on the goods for cash payment.

186. A builder bought 10 boxes $15' \times 28'$ * 2 German window-glass @ \$12.50,—terms 40, 25, 10, and 5% off. What was the net amount of the bill?

To what direct discounts are the following trade discounts equal:

187. 50, 10, and 10%?	189. 30, 10, and 10%?	191. 40, $16\frac{2}{3}$, and 10%?
188. 35 and 10%?	190. 50, 5, and $2\frac{1}{2}\%$?	192. $33\frac{1}{3}$, 20, 5, and $2\frac{1}{2}\%$?

193. I buy 560 shares (par value \$100) of Metropolitan Gas stock at 117, and after receiving a 9% dividend I sell it for 8% less than it cost me. Do I gain or lose, and how much?

194. If I sell \$17,500 par value of Kansas 7's at 118, and invest the proceeds in North Carolina 5's at 94, how will my income be affected?

195. The publishers of a cyclopædia supply a subscription agent at $33\frac{1}{3}\%$ below list price, and the agent takes orders at \$6.00 per volume and makes $33\frac{1}{3}\%$. What is the publisher's list price per volume?

196. A shipper paid a premium of \$76.17 for a trip policy for \$5,078 on a cargo of corn from Louisville to New Orleans. What per cent premium did he pay?

197. My agent buys lumber for me @ \$16.25 p M , and his commission, at $2\frac{1}{2}\%$, is \$137.25. How much lumber does he buy?

198. A metal broker received \$633.42 for Scotch pig-iron, profit $3\frac{1}{2}\%$; \$27.54 for spelter, profit 2% ; \$23.76 for American sheet-iron, loss 1% ; and \$13 for solder, profit 4% . What was the cost of the goods sold?

Find the cost of each block of stocks named below, brokerage $\frac{1}{8}\%$:

	Shares	stocks	quoted at		Shares	stocks	quoted at
199.	150	Mich. C. RR.;	118	202.	596	Ætna Ins. Co.;	165
200.	340	Pa. RR.;	$94\frac{1}{8}$	203.	337	Oswego Starch Co.;	$244\frac{3}{8}$
201.	238	R. I. & P. RR.;	$115\frac{1}{4}$	204.	121	Atlantic Cable;	$208\frac{1}{8}$

205. How many shares of Western Union Telegraph stock must a broker sell, to receive \$50 for his services, brokerage $\frac{1}{8}\%$?

206. I pay taxes on an assessed valuation of \$17,350 real-estate and \$3,868 personal property. This year I am taxed $\frac{1}{2}$ -mill for schools, $1\frac{1}{8}\%$ for city purposes, $\frac{5}{8}\%$ for county and State, and a $1\frac{1}{2}$ -mill special tax for street improvements. What is the amount of my taxes?

207. 3 doz. diaries @ \$9, 2 doz. do. @ \$11, 2 unabridged dictionaries @ \$7, 4 rm. legal-cap paper @ \$3.50, 10 rm. commercial note @ \$1.80, and 5 M $\times$ 5 envelopes @ \$2.40 were billed with trade discounts of 15 and 5% on the diaries, 5% on the dictionaries, and 10, 5, and 5% on the stationery. What was the net amount of the bill?

208. A district tax of \$1,365 was laid on an assessed valuation of \$445,000. What was the per cent of taxation?

209. Make a tax table for use in apportioning the tax in problem 208, and by it find the taxes to be paid by the following tax-payers on their assessed valuations named: A, \$12,314; B, \$5,380; C, \$1,075; The Star Shoe Mfg. Co., \$17,500; 1st National Bank, \$50,000; I. C. RR., \$98,600.

210. M owns city real-estate that rents for \$2,480 annually. Should he sell the real-estate for \$32,000 and invest the proceeds in Nebraska School 7's at 95, brokerage $\frac{1}{8}\%$, how would his income be affected?

SECTION IV.

INTEREST,—

THE PER CENT A COMPOUND ELEMENT.

465. *Interest* is a sum paid or allowed for the use of money.

a. Principal is the sum for the use of which interest is paid or allowed.

b. Amount is the sum of principal and interest.

c. Rate of interest is the per cent of the principal allowed for its use one year. It is sometimes called "rate per cent per annum."

d. Per cent of interest is the product of the rate of interest multiplied by the number expressing the time in years.

Per cent of interest is a compound element; it may be equal to, or it may be greater or less than, the rate of interest,—as the time is 1 year, more than 1 year, or less than 1 year.

e. Per cent of amount is $1 +$ per cent of interest; and

Per cent of difference is $1 -$ per cent of interest.

Thus, when .20 is the per cent of interest, 1.20 is the per cent of amount, and .80 is the per cent of difference.

f. Legal interest or *the legal rate* is the rate of interest established by law. It varies in different States and countries.

g. Usury is any rate of interest greater than the legal rate. In most States, a party contracting to receive usury is liable to a penalty.

h. When no rate of interest is mentioned in interest-bearing contracts, legal interest is always understood.

For legal rates of interest, and penalties for usury, see **Supplement**, pages 440, 441.

I. SIMPLE INTEREST.

466. *Simple interest* is a sum paid or allowed for the use of the principal only.

By the common methods of computing interest, a month is regarded as $\frac{1}{12}$ of a year, and a day as $\frac{1}{360}$ of a month; i. e., the year is regarded as consisting of only 360 days.

General Method.

467. Ex. 1. Find
the interest of \$450 for
3 yr. at 7%.

PROCESS.

$.07 \times 3 = .21$, the per cent of interest;
 $\$450 \times .21 = \$94.50$, the interest required.

Ex. 2. Find the interest of \$690 for 4 yr. 8 mo. 18 da. at 5%.

FIRST PROCESS.

$$4 \text{ yr. } 8 \text{ mo. } 18 \text{ da.} = 4.7\frac{1}{2} \text{ yr.};$$

$$.05 \times 4.7\frac{1}{2} = .235\frac{1}{2} = .23\frac{7}{8}, \text{ the per cent of interest};$$

$$\$690 \times .23\frac{7}{8} = \$162.72\frac{1}{2}, \text{ the interest required.}$$

468. Regarding each month as $\frac{1}{12}$ of a year, we have

IMPORTANT FACT I.—*The per cent of interest equals $\frac{1}{12}$ of the product of the rate of interest multiplied by the number expressing the time in months.*

SECOND PROCESS.

$$4 \text{ yr. } 8 \text{ mo. } 18 \text{ da.} = 56.6 \text{ mo.};$$

$$\frac{.05 \times 56.6}{12} = .23\frac{7}{8}, \text{ the per cent of interest};$$

$$\$690 \times .23\frac{7}{8} = \$162.72\frac{1}{2}, \text{ the interest required.}$$

NOTE.—The method shown in the second process is a better general method than that shown in the first process.

EXAMPLE.—Find the interest of \$8,236.50 for 7 mo. 14 da., at 6%.

PROCESS.

$$7 \text{ mo. } 14 \text{ da.} = 7.4\frac{1}{2} \text{ mo.};$$

$$\frac{.06 \times 7.4\frac{1}{2}}{12} = .037\frac{1}{2}, \text{ the per cent of interest};$$

$$\$8,236.50 \times .037\frac{1}{2} = \$307.50, \text{ the interest required.}$$

Six-per-cent Method.

At 6% per annum, the per cent of interest for 1 month is $\frac{1}{12}$ of 6%, equal to $\frac{1}{2}\%$ = $\frac{1}{200}$. Hence,

IMPORTANT FACT II.—*The per cent of interest at 6% equals $\frac{1}{200}$ of the number expressing the time in months.*

Examples 1, 2, Art. 467, and Example, Art. 468, are solved below by the six-per-cent method.

(Ex. 1.) PROCESS.

$$.06 \times 3 = .18, \text{ the per cent of interest at 6\%};$$

The given rate of interest is 7% = 6% $\times 1\frac{1}{6}$; hence,

$$.18 \times 1\frac{1}{6} = .21, \text{ the per cent of interest at 7\%};$$

$$\$450 \times .21 = \$94.50, \text{ the interest required.}$$

(Ex. 2.) PROCESS.

 $4 \text{ yr. } 8 \text{ mo. } 18 \text{ da.} = 56.6 \text{ mo.};$ $\frac{1}{100} \text{ of } 56.6 = .283, \text{ the per cent of interest at } 6\%;$ $.283 \times \frac{4}{1} = .235\frac{1}{2}, \text{ the per cent of interest at } 5\%;$ $\$690 \times .235\frac{1}{2} = \$162.72\frac{1}{2}, \text{ the interest required.}$

(Ex. 3.) PROCESS.

 $7 \text{ mo. } 14 \text{ da.} = 7.4\frac{2}{3} \text{ mo.};$ $\frac{1}{100} \text{ of } 7.4\frac{2}{3} = .037\frac{1}{3}, \text{ the per cent of interest};$ $\$8,236.50 \times .037\frac{1}{3} = \$307.50, \text{ the interest required.}$

PROBLEMS.

Find the per cent of interest, by both methods,

1, 2. For 5 yr.	} at {	6%.		5, 6. For 3 yr. 9 mo. 10 da.	} at {	5%.
3, 4. For 3 yr. 7 mo.		7%.		7, 8. For 1 yr. 1 mo. 2 da.		4½%.

Find the interest, by both methods,

At 6% per annum

				For 1 yr. 9 mo. 2 da.	
9-11. Of \$369.72	} for {	2 yr. 7 mo. 15 da.	18-20. Of \$29.83	} at {	5%.
12-14. Of \$11,450		11 mo. 5 da.	21-23. Of \$407.15		10%.
15-17. Of \$3,119.05		22 da.	24-26. Of \$86.54		7.3%.

Find the amount, by both methods,

27. Of \$373.80 for 1 yr. 7 mo. 6 da. at 6%.	32. Of \$9,170 for 2 yr. 1 mo. 1 da. at 5½%.
28. Of \$21,504 for 11 mo. 25 da. at 5%.	33. Of \$27.88 for 2 mo. 26 da. at 6%.
29. Of \$17.84 for 2 yr. 1 mo. 13 da. at 6%.	34. Of \$1,136,500 for 1 yr. 7 mo. at 3½%.
30. Of \$5,821.85 for 4 yr. 2 da. at 7½%.	35. Of \$7,085.25 for 24 da. at 4%.
31. Of \$191.18 for 5 mo. 29 da. at 6%.	36. Of \$3,175 for 11 da. at 6%.

Exact-interest Method.

469. Exact interest is simple interest for the exact number of days—on the basis of 365 days to the common year and 366 days to the leap-year.

All interest transactions with the General Government are computed by the exact-interest method. This method is the only just method of computing interest, and is constantly growing in favor.

470. IMPORTANT FACT III.—*The per cent of exact interest equals the product of the rate of interest multiplied by the number expressing the time in an exact fraction of the year.*

PROCESS.

Ex. 1. Find the $37 \text{ da.} = \frac{37}{365} \text{ yr.};$
 per cent of interest $.06 \times \frac{37}{365} = .006 \frac{7}{125},$ the per cent of interest required.
 for 37 days at 6%.

PROBLEMS.

Find the per cent of exact interest for

37, 38. 13 da. } at { 7%. | 41, 42. 128 da. } at { 6%. | 45, 46. 29 da. } at { 5%.
 39, 40. 63 da. } at { 4%. | 43, 44. 93 da. } at { 9%. | 47, 48. 33 da. } at { 3½%.

Find the exact interest

49-51. Of \$200 for 63 da. } at { 5%. | 58-60. Of \$84.75 at 4% } for { 37 da.
 52-54. Of \$1,250 for 93 da. } at { 6%. | 61-63. Of \$2,300 at 4½% } for { 89 da.
 55-57. Of \$175.50 for 33 da. } at { 8%. | 64-66. Of \$9,145 at 7% } for { 13 da.

Of	from	to	at the rate in
67. \$250	June 17, 1887,	Sept. 18, 1887,	New York.
68. \$1,356	Jan. 2, 1888,	Apr. 1, 1888,	Virginia.
69. \$64,590	Dec. 19, 1886,	Feb. 4, 1887,	California.
70. \$3,190.35	Nov. 5, 1887,	June 2, 1888,	Michigan.
71. \$406.12	Oct. 3, 1890,	Jan. 6, 1891,	Pennsylvania.
72. \$87.63	Aug. 29, 1889,	Apr. 5, 1890,	Louisiana.

II. ANNUAL INTEREST.

471. By the terms of contracts for the use of money for more than one year, the interest is commonly made due and payable annually or semi-annually. In some States interest not paid at the time specified in the contract draws interest from that time, on the legal principle that any debt should draw interest from the time it is due.

472. Annual interest is simple interest on the principal, plus simple interest on each instalment of interest on the principal not paid when due.

Annual interest is legalized in Michigan at the same rate as secured by the contract; in Ohio, Wisconsin, Vermont, New Hampshire, and Iowa, at 6%; and in Pennsylvania, Georgia, Illinois, and Indiana, by special contract only.

Ex. Find the amount of \$840 for 3 yr. 6 mo., at 6% annual interest.

ANALYSIS.—The simple interest on the principal for 3 yr. 6 mo. is \$176.40.

Since each instalment of interest on the principal draws interest from the time it is due, the first instalment will bear interest 2 yr. 6 mo., the second 1 yr. 6 mo., and the third 6 mo.

The interest on the three equal instalments will equal the interest on one of them for (2 yr. 6 mo. + 1 yr. 6 mo. + 6 mo. =) 4 yr. 6 mo.

Each instalment of interest—*i. e.*, 1 year's interest on \$840—is \$50.40; and the interest on \$50.40 for 4 yr. 6 mo., at 6%, is \$13.608.

The sum of the principal, the interest on the principal, and the interest on unpaid instalments of interest, is \$1,030.01, the amount required.

PROCESS.

$$3 \text{ yr. 6 mo.} = 3\frac{1}{2} \text{ yr.}$$

$$.06 \times 3\frac{1}{2} = .21$$

$$\$840 \times .21 = \$176.40$$

$$2 \text{ yr. 6 mo.} + 1 \text{ yr. 6 mo.} + 6 \text{ mo.} = 4 \text{ yr. 6 mo.}$$

$$4 \text{ yr. 6 mo.} = 4\frac{1}{2} \text{ yr.}$$

$$.06 \times 4\frac{1}{2} = .27$$

$$\$840 \times .06 = \$50.40$$

$$\$50.40 \times .27 = \$13.608$$

$$\$840 + \$176.40 + \$13.61 = \$1,030.01$$

PROBLEMS.

- Find the annual interest $\left\{ \begin{array}{l} 73. \text{ Of } \$8,000 \text{ for 6 yr., at 6\%.} \\ 74. \text{ Of } \$428.60 \text{ for 4 yr. 2 mo. 10 da., at 5\%.} \\ 75. \text{ Of } \$1,156.82 \text{ at 6\%, for 5 yr. 11 mo. 2 da.} \end{array} \right.$

Find the amount with annual interest

76. Of \$568.25 for 2 yr. at 6%, interest payable semi-annually.
 77. Of \$4,120 for 3 yr. 7 mo. 5 da. at 5%, interest payable quarterly.
 78. Of \$48.70 for 5 yr. 19 da. at 7%, interest payable annually.
 79. Of a note for \$620, dated Detroit, Oct. 22, 1886, due Jan. 22, 1889.
 80. Of a note for \$135.64 and semi-annual interest, dated Cedar Rapids, Iowa, June 16, 1885, and payable in 3 yr. 7 mo. 15 da.

III. COMPOUND INTEREST.

473. *Compound interest* is the interest on a principal formed by adding interest to a former principal.

a. Compound interest is illegal in most of the States.

b. In transactions involving compound interest, the interest is commonly compounded—*i. e.*, added to the principal to form a new principal—annually, semi-annually, or quarterly.

c. The amount of the principal for the first interest term is the principal for the second interest term; the amount for the second interest term is the principal of the third interest term; and so on.

Ex. 1. Find the amount of \$500 for 4 yr., at 6%, interest compounded annually.

ANALYTICAL PROCESS.

$\$500 \times 1.06 = \$530,$	amount for 1st interest term of 1 year;
$\$530 \times 1.06 = \$561.80,$	" " 2d " " " "
$\$561.80 \times 1.06 = \$595.508,$	" " 3d " " " "
$\$595.508 \times 1.06 = \$631.24-,$	" " 4th " " " "

ANALYSIS.—Since the amount for each interest term is found by multiplying the preceding amount by 1.06, in computing for four interest terms 1.06 is used four times as a factor, i. e., 1.06^4 , the per cent of amount.

SECOND PROCESS.

$$\$500 \times 1.06^4 = \$631.24 -$$

Hence, the product of the principal multiplied by 1.06^4 is the amount required.

Ex. 2. Find the interest on \$250 for 2 yr. 6 mo., at 6% per annum, interest compounded semi-annually.

PROCESS.

ANALYSIS.—Since the rate of interest is 6%, the per cent of interest for an interest term of 6 mo. is 3%.

$$1.03^5 = 1.159274 +$$

$$1.159274 - 1.00 = .159274$$

The number of interest terms of 6 mo. each in 2 yr. 6 mo. is 5; hence, 1.03^5 ($=1.159274+$) is the per cent of amount.

$$\$250 \times .159274 = \$39.82 -$$

Since the principal is always 100%, the per cent of amount minus 1.00 equals the per cent of compound interest, which is .159274.

Multiplying the principal by the per cent of compound interest produces \$39.82, the compound interest required.

Ex. 3. Find the per cent of compound interest for 3 yr. 4 mo. 6 da,
at 6%, compounded annually.

PROCESS.

$1.06^3 = 1.191016$, the per cent of amount for 3 yr.;

$$4 \text{ mo. } 6 \text{ da.} = 4.2 \text{ mo.};$$

$$\frac{.06 \times 4.2}{12} = .021, \text{ the per cent of interest for 4 mo. 6 da. at } 6\%;$$

$1.00 + .021 = 1.021$, " " " " amount " " " " "

$1.191016 \times 1.021 = 1.216027$, the per cent of amount for 3 yr. 4 mo. 6 da.;

1.216027 - 1.00 = .216027, the per cent of compound interest required.

For Compound Interest Table, see Supplement, pages 442, 448.

PROBLEMS.

Find the per cent of compound interest

- 81-83. For 5 yr. 6 mo. }
 84-86. For 3 yr. 2 mo. 18 da. } at { 5%, compounded annually.
 87-89. For 1 yr. 11 mo. 26 da. } { 6%, compounded semi-annually.
 { 8%, compounded quarterly.

Find the amount

90. Of \$800 for 5 yr., at 5%, compounded annually.
 91. Of \$972.40 for 3 yr. 2 mo., at 6%, compounded annually.
 92. Of \$3,100 for 2 yr. 8 mo. 19 da., at 6%, compounded semi-annually.
 93. Of \$419.15 for 1 yr. 11 mo. 5 da., at 8%, compounded quarterly.
 94. Of \$325 for 10 mo. 27 da., compounded at $2\frac{1}{4}\%$ per quarter.
 95. Of \$5,000 for 3 yr. 22 da., compounded at 2% per half-year.

Find the interest

96. Of \$286.25 for 6 yr., at 5%, compounded semi-annually.
 97. Of \$95.18 for 3 yr. 8 mo., at 4%, compounded annually.
 98. Of \$1,144.06 for 1 yr. 1 mo. 14 da., at 6%, compounded quarterly.
 99. Of \$3,128 for 11 mo. 19 da., at 6%, compounded semi-annually.
 100. Of \$37,185 for 30 yr., at 4%, compounded annually.

IV. THE SIX GENERAL CASES IN INTEREST.

474. The term $\left\{ \begin{array}{l} \text{Principal} \\ \text{Per cent of interest} \\ \text{Interest} \\ \text{Amount} \\ \text{Difference} \\ \text{(principal—interest)} \end{array} \right\}$ corresponds to $\left\{ \begin{array}{l} \text{base;} \\ \text{per cent;} \\ \text{the percentage;} \\ \text{amount;} \\ \text{difference.} \end{array} \right.$

475. CASE I. Principal, rate of interest, and time given, to find interest.

FORMULA.—*Interest* = *principal* $\times$ *per cent of interest*.

CASE II. Interest, rate of interest, and time given, to find principal.

FORMULA.—*Principal* = *interest* $\div$ *per cent of interest*.

CASE III. Principal, rate of interest, and time given, to find amount.

FORMULA.—*Amount* = *principal* $\times$ (*1* + *per cent of interest*).

CASE IV. Amount, rate of interest, and time given, to find principal.

FORMULA.—*Principal* = *amount* ÷ (*1* + *per cent of interest*).

CASE V. Principal, interest, and time given, to find rate of interest.

FORMULA.—*Rate of interest* = *interest* ÷ (*principal* × *time*).

CASE VI. Principal, interest, and rate of interest given, to find time.

FORMULA.—*Time* = *interest* ÷ (*principal* × *rate of interest*).

In the preceding formulæ, *time* is the abstract number expressing the number of years, and may be either integral, fractional, or mixed.

PROBLEMS.

Find the missing elements :

	Principal	Per cent of interest.		Interest	Amount
		Rate of interest	Time		
101, 102.	\$280.70	6%	3 yr. 4 mo.	\$———	\$———
103, 104.	\$1,260.28	5%	———	\$———	\$1,420.75
105, 106.	\$———	———	1 yr. 6 mo. 20 da.	\$81.26	\$951.90
107, 108.	\$———	4½%	2 yr. 8 mo. 14 da.	\$———	\$12,506.48
109, 110.	\$500.86	———	5 yr. 1 mo. 24 da.	\$77.38+	\$———
111, 112.	\$428.40	———	3 mo. 4 da.	\$———	\$434.55+
113, 114.	\$27.08	6%	———	\$.677	\$———
115, 116.	\$———	3½%	11 mo. 28 da.	\$18.43	\$———
117, 118.	\$———	6%	———	\$928.14	\$22,368.12
119, 120.	\$97.05	6%	———	\$———	\$105.18

\$405.26

ROCHESTER, N. Y., July 28, 1887.

On demand, I promise to pay to HARRISON MILLER, or order, four hundred five and 2/10 dollars, with interest, for value received.

HENRY WILDER.

This note was paid May 19, 1888.

121. What was the sum due on the note ?

122. How much longer should the note run, to amount to \$480.10 ?

123. How much would it have amounted to when paid, if the interest were compounded semi-annually ?

124. The interest for 2 yr. 8 mo. 6 da., at 5%, compounded annually, is \$93.88. Find the principal.

125. The amount of a note for 36 da., exact interest at 6%, was \$271.16. Find the principal.

126. \$408.75 at 6%, compounded semi-annually, amounted to \$463.91. Find the time.

127. The exact interest of a debt at 6%, from June 5, 1883, to Sept. 3, 1883, was \$37.59. What was the principal?

128. The interest of \$240 for 3 yr. 2 mo. 12 da. is \$53.76. Find the rate of interest.

129. The interest of \$6,000 for 8 yr., compounded annually, is \$2,864.73. Find the rate of interest.

130. The exact interest of \$96.73 for 93 days is \$1.717-. What is the rate of interest?

At 7%, in what time will \$325 amount to \$340

131. At simple interest? | **133.** Interest compounded annually?

132. At exact interest? | **134.** Interest compounded quarterly?

In what time will any { **135.** At 6% simple interest?
sum double itself { **136.** At 5% compound interest?

V. OTHER METHODS OF COMPUTING INTEREST.

476. The only logical percentage methods of computing interest are those that involve *per cent of interest*; but the same results can be obtained by methods not belonging purely to percentage.

In the following methods, Ex. 2, page 273, is used as the illustrative example, in order that the several methods may be more readily compared.

477. Methods by Analysis.

Ex. Find the interest of \$690 for 4 yr. 8 mo. 18 da., at 5%.

PROCESS.

$\$690 \times .05 = \$34.50$, int. for 1 yr.; $\$34.50 \times 4 = \$138$.—, int. for 4 yr.
 $\$34.50 \div 12 = \$2.875$, " " 1 mo.; $\$2.875 \times 8 = 23$.—, " " 8 mo.
 $\$2.875 \div 30 = \$0.0958\bar{3}$, " " 1 da.; $\$.0958\bar{3} \times 18 = 1.725$, " " 18 da.

Hence, the required interest is \$162.725

478. One of the best analytical business methods of computing interest, and one that gives the best opportunity for the exercise of tact in avoiding all except mental computations, is the

Method by Aliquot Parts.

PROCESS.

ANALYSIS.—	$\$690 \times .05 = \$34.50$,	int. for 1 yr.	} = 4 yr.
3 yr. is 3 times 1 yr.	103.50,	" " 3 yr.	
6 mo. is $\frac{1}{2}$ of 1 yr.	17.25,	" " 6 mo.	} = 8 mo.
2 mo. is $\frac{1}{3}$ of 6 mo.	5.75,	" " 2 mo.	
15 da. is $\frac{1}{4}$ of 1 mo. = $\frac{1}{4}$ of 2 mo.	1.4375,	" " 15 da.	} = 18 da.
3 da. is $\frac{1}{5}$ of 15 da.	.2875,	" " 3 da.	
	<u>\$162.725,</u>	" " 4 yr. 8 mo. 18 da.	

479. *Method by Years and Decimals of a Year.*

PROCESS.

4 yr. 8 mo. 18 da. = 4.716 yr.;

$\$690 \times .05 = \$34.50$, int. for 1 yr.;

$\$34.50 \times 4.716 = \$162.725$, interest required.

480. *Six-per-cent Method by Analysis.*

PROCESS.

The interest on \$1 at 6% | 2. For 1 month is \$.005

1. For 1 year is \$.06 | 3. For 1 day is \$.0004

Hence, $(\$.06 \times 4 =) \$.24$, int. on \$1 for 4 yr.

$(\$.005 \times 8 =) .04$, " " " 8 mo.

$(\$.0004 \times 18 =) .0072$, " " " 18 da.

\$.2832, " " " 4 yr. 8 mo. 18 da.

$\$.2832 \times \frac{1}{2} = \$.2832$, int. on \$1 for 4 yr. 8 mo. 18 da., at 5%.

$\$.2832 \times 690 = \$162.725$, the interest required.

481. *Contracted Six-per-cent Method by Analysis.*

PROCESS.

$\$690$, the principal; $\$690 \div 2 = \$345$, int. for 1 mo. at 6%;

4 yr. 8 mo. 18 da. = 56.6 mo.;

$\$345 \times 56.6 = \$195.27$, int. for 4 yr. 8 mo. 18 da. at 6%;

$\$195.27 \times \frac{1}{2} = \$162.725$, " " " " " 5%.

482. Method by Cancellation.

$$\begin{array}{r} \text{PROCESS.} \\ \begin{array}{r} 23 \qquad \qquad 28.3 \\ \$690 \times 5 \times \$0.6 \\ \hline 100 \times 12 \\ 2 \qquad \qquad 4 \end{array} = \$162.72\frac{1}{2}, \text{ the interest required.} \end{array}$$

483. Method by Proportion.

$$\begin{array}{l} \text{PROCESS.} \\ \$100 \quad \left. \vphantom{\begin{array}{l} \$100 \\ 1 \text{ yr.} \end{array}} \right\} : \left\{ \begin{array}{l} \$690 \\ 4 \text{ yr. 8 mo. 18 da.} \end{array} \right\} :: \$5 : \$\text{---}. \text{ Hence, } \$162.72\frac{1}{2}. \end{array}$$

484. Method by Days,—For Short Time.

Ex. Find the interest of \$350 for
20 days, at 4%. At 5%. At 6%.

At 360% per annum, the per cent of interest for 1 day is .01; and

4% is ($\frac{4}{360}$) = $\frac{1}{90}$ of 360%;

5% is ($\frac{5}{360}$) = $\frac{1}{72}$ of 360%;

6% is ($\frac{6}{360}$) = $\frac{1}{60}$ of 360%.

$$\begin{array}{l} \text{PROCESSES.} \\ \$350 \times .20 \\ \hline 90 = \$.78-, \text{ at } 4\%. \end{array}$$

$$\begin{array}{l} \$350 \times .20 \\ \hline 72 = \$.97+, \text{ at } 5\%. \end{array}$$

$$\begin{array}{l} \$350 \times .20 \\ \hline 60 = \$ 1.17-, \text{ at } 6\%. \end{array}$$

NOTE.—The pupil may be required to write a rule for each of the methods above shown.

Practice in the use of these methods can be secured, by solving the problems on pp. 279, 280, by the method best adapted to each problem.

SECTION V.**COMMERCIAL PAPER.**

485. Commercial paper includes notes, drafts, checks, and bills of exchange.

486. An *indorsement* of any business paper is a writing upon its back that

1. Assigns or transfers ownership;
2. Gives security for the payment of the obligation; or
3. Acknowledges a partial payment.

487. A *promissory note* is a written promise to pay a certain sum of money to a certain party, or to his order, or to the bearer, either on demand or at a specified time.

488. A *draft* is a written order from one party to another to pay to a third party, or to his order, a certain sum of money, either at sight or at a specified time.

489. A *check* is a written order by a party having money in a bank, for the bank to pay a certain sum to a certain party, or to his order, or to bearer.

490. A *bill of exchange* is a draft the parties to which are in different countries.

491. The parties to commercial paper are

1. The *maker*,—the party who signs the paper ;
2. The *payee*,—the party who is to receive the money ;
3. The *drawee*,—the party on whom an order is drawn, *i. e.*, the party to whom an order is addressed.

4. The *indorser*,—the party who indorses the paper.

a. The *original parties* to commercial paper are those necessary to create the paper, *i. e.*, to bring it into existence.

b. The original parties to a promissory note are a maker and a payee ; and To a draft, check, or bill of exchange, are a *drawer* (*i. e.*, a maker), a drawee, and a payee.

c. The *holder* of commercial paper is the owner.

d. The *payer* is the party who is to pay the money to the payee ; *viz.*, the maker of a note or the drawee of a draft or check.

e. The *face* of commercial paper is the sum made payable by the paper.

f. The *buyer* or *remitter* of a draft or bill of exchange is the party who sends or remits it as an equivalent for money.

492. *Negotiable* commercial paper is that which can be bought and sold without regard to unsettled matters of account between original parties, and is worth to the holder its face plus any accrued interest.

493. *Assignable* or *non-negotiable* commercial paper is that which can be bought and sold only as subject to unsettled matters of account between the maker and some or all of the other parties.

a. Commercial paper made payable

1. To the *order* of a certain party or to *bearer* is negotiable.
2. To a certain party only is non-negotiable.

- b.* Negotiable paper becomes non-negotiable after it becomes due.
- c.* To make a note negotiable in Pennsylvania, it must contain the words "without defalcation;" in New Jersey, "without defalcation or discount;" and in Missouri, "negotiable and payable without defalcation or discount."

494. *Days of grace* are three days allowed by law for the payment of a note or draft after the time named in the paper.

In some States days of grace on sight drafts are not allowed.

See Supplement, page 441.

495. The *maturity* of a commercial paper is the day upon which it is due and payable. When grace is allowed, it is the last day of grace.

For laws of the different States relating to maturity, see Supplement, page 441.

496. In computing time, on commercial paper, the day of date is omitted and the day of maturity is included.

- a.* If the time is given in months, calendar months are meant.
Thus,—Two months after Jan. 5 ends with Mar. 5.
- b.* If there are not as many days in the month in which the time ends as were given in the first month, the time never includes more than the last day of the later month.
Thus,—1 month after Jan. 29, 30, or 31 ends with Feb. 28 or 29.
- c.* If the time is given in days, the exact time must be computed.
Thus,—60 days after Dec. 15 ends with Feb. 13.
- d.* The banks of the District of Columbia, Delaware, Maryland, Missouri, Pennsylvania, charge interest for the day on which a note is discounted and for the day on which it matures,—making the interest period practically one day longer than in other States.

497. The *indorsement* of a commercial paper, to transfer ownership, may be either in *blank* or in *full*.

- a.* An *indorsement in blank* is simply the signature of the indorser.
- b.* An *indorsement in full* is an order over the signature of the indorser, to pay to some other party, or to the order of that party.
- c.* An indorsement in blank makes the paper payable to the bearer.
- d.* The indorser of a commercial paper guarantees its payment, and is liable for its payment to all subsequent holders, unless the words "without recourse," or words of similar import, precede his signature.
- e.* Commercial paper made payable *to bearer* need not be indorsed to transfer ownership; but, if
Made payable *to order*, it must be indorsed to transfer ownership.

498. The *protest* of a commercial paper is a notice served upon the maker and the several indorsers, that payment has been demanded at the maturity of the paper and has been refused, and that they are individually held responsible by the holder.

- a.* Protest is usually made by a Notary Public, and is commonly served by depositing in the mail a copy of the protest addressed to the party to be served.
- b.* A commercial paper must be legally protested to hold the indorsers liable for its payment.
- c.* In order to make a protest of a commercial paper legal, the paper must be presented for payment on the day of its maturity, during business hours, at the place where it is payable; and, if no place of payment is specified in the paper, at the residence or place of business of the maker, during reasonable hours.

499. The *acceptance* of a draft or check is the writing, by the drawee, across the face of the paper, of the word "accepted" and the date, followed by his signature.

- a.* If the drawee refuses to accept or to pay a check or draft, the holder must have the paper protested, to hold the drawer and indorsers liable for its payment.
- b.* A *sight* draft is one made payable at sight, or on demand; and
A *time* draft is one made payable at a specified time after sight or after date.
- c.* When no time of payment is mentioned in a commercial paper, it is payable at sight, or on demand.
- d.* To *honor* a draft or check is to accept it or pay it on presentation.

500. Promissory notes are named

- 1. From time of payment,—as *demand notes* and *time notes*.
- 2. From the maker or makers,—as *individual notes*, *joint notes*, and *joint and several notes*.
- 3. From the payee,—as *negotiable notes* and *non-negotiable notes*.
- 4. From the words "with interest" or from their omission,—as *interest-bearing notes* and *non-interest-bearing notes*.

For business forms, see **Supplement**, pages 444-446.

- a.* A *bankable note* is one made payable at a bank.
- b.* A *judgment note* is one having a seal affixed, and containing a power of attorney authorizing the holder to confess judgment for the maker.
- c.* A *chattel note* is one given in payment for property the title to which remains in the vendor until the note is paid.
- d.* All notes bear interest after they become due if not then paid, even though no mention of interest is made in them.

- e. The words "value received" should be in every note, as without them the holder may be compelled to show that value was given to the maker.
- f. Each of the makers of a joint note is liable to the holder, for his proportionate share only of the sum made payable by the note; but the makers of a joint and several note are each liable to the holder for the full amount.
- g. Indorsers are liable for the payment of a note, in the order of their indorsements,—each indorser being liable to all subsequent indorsers, but not liable to preceding indorsers.

501. FACTS TO BE MEMORIZED.

- | | |
|---|--|
| <ul style="list-style-type: none"> 1. If a note is lost or stolen, the loss does not release the maker. He must pay it, but he may demand a bond of indemnity. 2. A note made on Sunday is void. 3. An agreement without consideration is void. 4. A note obtained by fraud can not be collected. | <ul style="list-style-type: none"> 5. A note made by a minor is voidable. 6. Signatures made with a lead-pencil are good in law. 7. Checks or drafts must be presented without unreasonable delay. 8. The time for payment of a note must either be certain or must depend on a certainty. |
|---|--|

I. PARTIAL PAYMENTS.

502. A *partial payment* is a payment of a part of an obligation that is due, or that is drawing interest.

503. The decisions of the United States Courts concerning partial payments made upon notes, bonds, mortgages, and other interest-bearing obligations are based on the following recognized

PRINCIPLES.

- I. *Accrued interest must be paid before the principal can be diminished.*
- II. *Interest must never draw interest.*

504. On these principles is based the

UNITED STATES COURT RULE FOR PARTIAL PAYMENTS.

I. *From the amount of the principal, computed to the time when the payment or the sum of the payments equals or exceeds the interest due, subtract the payment or the sum of the payments.*

II. *With the remainder as a new principal, proceed as before.*

In making the interest computations in partial payments, any one of the legal methods of computing interest may be used.

Ex. On a farm mortgage for \$600, dated Saratoga, N. Y., July 10, 1880, were indorsed payments as follows:—Jan. 25, 1882, \$100; May 20, 1884, \$50; Oct. 2, 1884, \$200. How much was due March 20, 1889?

ANALYTICAL PROCESS.

Dates	Interest Periods	Per cents of Interest	1+ per cent of Interest
1880 yr. 7 mo. 10 da.			
1882 1 25	1 yr. 6 mo. 15 da. = 18.5 mo.	$\frac{1}{100}$ of 18.5 = .0925	1.0925
1884 10 20	2 4 5 = 28 $\frac{1}{2}$ mo.	$\frac{1}{100}$ of 28 $\frac{1}{2}$ = .14 $\frac{1}{2}$	} = 1.16 $\frac{1}{2}$
1884 5 2	4 12 = 4.4 mo.	$\frac{1}{100}$ of 4.4 = .022	
1889 3 22	4 5 20 = 53 $\frac{1}{2}$ mo.	$\frac{1}{100}$ of 53 $\frac{1}{2}$ = .26 $\frac{1}{2}$	1.26 $\frac{1}{2}$
1st principal, \$600 $\times$ 1.0925 = \$655.50, less 1st payment, \$100, leaves \$555.50.			
2d " 555.50 $\times$ 1.14 $\frac{1}{2}$ = 633.73, — 2d payment (\$50) less than int. due.			
" " 555.50 $\times$ 1.16 $\frac{1}{2}$ = 645.95, less 2d and 3d payments, \$250, leaves			
3d " 395.95 $\times$ 1.26 $\frac{1}{2}$ = 502.20, due March 20, 1888. [\$395.95.			

Condensed Form for Written Work.

Dates	Interest Periods	1+ % of Interest	Principals	Amounts of Principals	Payments
1880 yr. 7 mo. 10 da.					
1882 1 25	1 yr. 6 mo. 15 da. = 18.5 mo.	1.0925	\$600.—	\$655.50	\$100
1884 10 20	2 4 5 = 28 $\frac{1}{2}$ mo. }	1.16 $\frac{1}{2}$	555.50	645.95	{ 50 200
1884 5 2	4 12 = 4.4 mo. }				
1889 3 22	4 5 20 = 53 $\frac{1}{2}$ mo.	1.26 $\frac{1}{2}$	395.95	502.20	

PROBLEMS.

1. On a note for \$321.84, dated Lincoln, Neb., Jan. 3, 1887, interest after 3 months, are the following indorsements: June 18, 1887, \$75; Jan. 16, 1888, \$25; July 1, 1888, \$81.75. How much is due Jan. 1, 1889?

The indorsements on page 290 are on the backs of notes 7–10, page 289, and are records of partial payments on the several notes.

- Find the sum due to-day on Note No. 7, page 289.
- How much money would cancel Note No. 8, page 289, Oct. 1, 1888?
- What was the balance due on Note No. 9, April 16, 1889?
- Note No. 10 was taken up Nov. 25, 1890, and a new note was given in place of it. What was the face of the new note?

Find the sums due at the dates of settlement given :

Face and Date	Payment	Settled
6. \$350.— Cleveland, O., Sept. 10, 1886.	June 20, 1887, \$120.	May 17, 1888.
7. \$712 ⁵⁰ New York, June 1, 1886.	Jan. 20, 1887, \$213.50 ; Dec. 10, 1887, \$175 ; June 17, 1888, \$25.	Oct. 30, 1888.
8. \$1,940.—Note. Wilmington, Del., Nov. 11, 1885.	Apr. 29, '86, \$315 ; Dec. 1, '86, \$50 ; Sept. 23, '87, \$496.	Apr. 10, 1888.
9. \$748.— Madison, Wis., Sept. 20, 1884.	Dec. 5, 1884, \$28.75 ; June 25, 1885, \$145 ; Aug. 1, 1885, \$212.50 ; Mar. 10, 1886, \$279.80.	Jan. 4, 1887.
10. \$3,500.—Mortgage. Grand Rapids, Mich., May 28, 1882.	Cash down, \$750 ; May 1, 1883, \$615 ; Nov. 21, 1884, \$708.75 ; Dec. 29, 1885, \$511.20 ; Oct. 31, 1886, \$800.50.	May 31, 1888.

505. When settlements are made within a year after interest commences, the computations of interest are often made by the

MERCANTILE RULE FOR PARTIAL PAYMENTS.

- I. *Compute the interest on the principal for the whole time.*
- II. *Compute the interest on each payment, from its date to the time of settlement.*
- III. *Subtract the amount of the payments from the amount of the principal.*

In making computations by this rule, exact interest for days is commonly considered.

506. SPECIAL STATE LAWS FOR PARTIAL PAYMENTS.

The States of Connecticut, Vermont, and New Hampshire have special laws for computing interest. (See Supplement, page 447.)

Pupils in any of these three States should solve problems 6, 7, 8, 9, 10, above, by the special law for their State.

\$500.-

New York, May 1, 1886.

7

Sixty days after date I promise to pay to
Amos Bradley, or order, five hundred dollars,
value received.

Henry Young.

\$469³/₄

Detroit, Jan. 20, 1887.

8

One year from date I promise to pay to
Edward Miller four hundred sixty nine and
³/₄ dollars, for value received, with interest.

David O Libbey.

\$2,750.-

Philadelphia, Aug. 15, 1887.

9

For value received, I promise to pay to
Milton S. Clark, or bearer, two thousand seven
hundred fifty dollars, with interest, six months
after date, without defalcation.

John J. Miller.

\$1,310⁸/₃

St. Paul, Oct. 8, 1886.

10

On the first day of March, 1889, we promise
to pay to the order of Robert Graham & Co. one
thousand three hundred ten dollars and fifty
cents, with interest after six months, value received.

Eugene S. Knight
Thomas Barrett.

July 10, 1886.
received on the
within note two
hundred dollars

(\$200).

August 25, 1886.
received on the
within note one
hundred fifty
dollars (\$150).

Jan. 20, 1888.
received on drum
stick twenty dol-
lars (\$20)

May 5, 1888.

received ninety
dollars (\$90).

Aug. 23, 1888.

received forty
and 3/4 dollars
(\$40 3/4)

(1880) Eight
hundred fifty
dollars paid.

May 18, 1888.

(1885) Twenty
five dollars paid.
Nov. 1, 1888.

Rec'd forty-one
and 3/4 dollars
(\$41 3/4). April 8, 1887

Rec'd ninety-two
and 3/4 dollars
(\$92 3/4). Apr. 8, 1888

Rec'd five
hundred dollars
(\$500). Mar. 1, 1889.

Rec'd twenty
dollars (\$20) June
1, 1890.

II. BANKING.

507. A *bank* is a chartered institution that receives and loans money, issues bank-notes or bank-bills, or facilitates commercial transactions between distant points by issuing and collecting drafts or bills of exchange.

- a.* A *bank of issue* is one that issues bank-bills that circulate as money.
- b.* A *bank of discount* is one that negotiates and discounts commercial paper.
- c.* A *savings-bank* is one that receives money on deposit—paying interest on the sums deposited—and loans its deposits on real-estate and Government securities.

NOTES.—1. The total amount or money value of bank-bills issued by a bank is known as *its circulation*.

2. Banks of issue and discount also exchange money, receive deposits subject to withdrawal by depositors without notice, and, in some instances, pay depositors low rates of interest on sums that remain on deposit a specified length of time.

3. Savings-banks never issue bank-bills nor discount notes. They loan money on real-estate securities, and on Government, State, city, county, and town bonds.

508. *Bank discount* is simple interest, paid in advance, on a note, from the date of discount to maturity.

C borrows \$300 at a bank for 3 months, and D borrows a like sum of a friend for the same time. Each pays interest on his loan for 3 months 3 days, but C pays his interest on the first day of the time, and D pays his on the last day. C actually has the use of $(\$300 - \$4.65 =) \$295.35$, and the bank has the use of \$4.65 of his loan for the term of discount, while D has the use of \$300 (the whole of his loan) for the same time. Hence,

509. *Bank discount equals legal interest plus interest on the legal interest for the discount period.*

Bank discount is computed

- (1) On the *face* of a note that does not bear interest; and
- (2) On the *amount* of a note that bears interest. Hence,

510. The *sum discounted* is the face of a note that does not bear interest, or the amount of a note that bears interest.

511. The *term of discount* is the time from the date of discount to the maturity of the note.

512. The *proceeds* of a note are the sum discounted minus the bank discount.

a. A note is *nominally due* on the last day of the time named in it; it is *legally due* at maturity. (See Supplement, page 441.)

b. *Due June 1/4*, written on the lower left-hand corner of a note, means that the note is nominally due June 1, and legally due June 4.

513. A *certificate of deposit* is a written statement, signed by the cashier or teller of a bank, setting forth that the party therein named has a specified sum of money on deposit in the bank.

The sum named in the certificate is payable on the surrender of the certificate properly indorsed.

See U. S. Moneys.

514. Brokers and dealers in commercial paper compute interest and discount on the basis of 360 days to the year. Some banks and brokers in large commercial and financial centres compute interest on the same basis, and others on the basis of 365 or 366 days—i. e., exact interest.

Unless otherwise specified, bank discounts in this work are computed on the basis of 360 days to the year.

515. In making computations in bank discount, any interest method may be used.

Ex. 1. Find the proceeds and the bank discount of a sixty-day note for \$750, discounted at 6%.

COMMON PROCESS.	SHORTER PROCESS.
$60 \text{ da.} + 3 \text{ da. (grace)} = 63 \text{ da.} = 2.1 \text{ mo.}$	$\$750$, sum to be discounted.
$\frac{6}{100}$ of 2.1 = .0105, per cent of discount.	$\$750$, disct. for 60 days.
$1 - .0105 = .9895$, " " " proceeds.	.37½, " " 3 "
$\$750 \times .9895 = \$742.12\frac{1}{2}$, proceeds.	$\$7.87\frac{1}{2}$, bank discount.
$\$750 \times .0105$, or }	$\$750$.—
$\$750 - \$742.12\frac{1}{2} \}$ = $\$7.87\frac{1}{2}$, bank discount.	$\$742.12\frac{1}{2}$, proceeds.

Ex. 2, 3. Compute the bank discount and the proceeds of a note for \$254.60 due in 90 days, discounted at 7%. At 4%.

PROCESSES.

<i>At 7%.</i>	<i>At 4%.</i>
<u>\$254.60, sum to be discounted.</u>	<u>\$254.60, sum to be discounted.</u>
\$2.546, interest for 60 da.	\$2.546, interest for 60 da.
1.273, " " 30 "	1.273, " " 30 "
.1273, " " 3 "	.1273, " " 3 "
<u>\$3.9463, discount at 6%</u>	<u>\$3.9463, discount at 6%</u>
.6592, " " 1%	1.3154+, " " 2%
<u>\$4.61, " " 7%</u>	<u>\$2.63+, " " 4%</u>
<u>\$254.60</u>	<u>\$254.60</u>
\$249.99, proceeds.	\$251.97, proceeds.

516. In computations in bank discount,

<i>Sum discounted = principal.</i>	<i>Rate of discount = rate of interest.</i>
<i>Discount = interest.</i>	<i>Proceeds = principal — interest.</i>

PROBLEMS.

Find the dates on which notes are nominally and legally due, if

Dated	time to run	due	Dated	time to run	due
11. Apr. 5,	60 da.	June 4/7	16. Aug. 12,	45 da.	_____
12. May 5,	2 mo.	_____	17. Nov. 29,	30 da.	_____
13. June 20,	90 da.	_____	18. Sept. 15,	4 mo. 10 da.	_____
14. Mar. 1,	6 mo.	_____	19. May 30,	15 da.	_____
15. Jan. 30,	1 mo.	_____	20. July 1,	1 yr.	_____

Find the bank discount on a note

21. For \$627.40, payable in 3 mo. 27 da., at 8%.
22. For \$936.50, nominally due in 30 days, at $4\frac{1}{2}\%$.
23. For \$388.24, that matures in 93 days, at 6%.

Find the proceeds of a note

24. For \$1,000, payable 10 days after to-day, discounted at 7%.
25. For \$221.75, payable in 60 days, discounted at 5%.
26. For \$400, nominally due in 2 months, discounted at 6%.

Required, the bank discount and the proceeds of an interest-bearing note

For	dated	payable in	discounted at	at
27. \$1,500,	Chicago, Apr. 5,	6 mo.,	June 19	8%.
28. \$812 ¹ / ₂ ,	Pittsburgh, Sept. 20,	8 mo.,	Dec. 8	10%.
29. \$18,640,	New Orleans, Mar. 5,	1 yr.,	July 15	7%.
30. \$3,972 ³ / ₄ ,	Atlanta, June 15,	6 mo. 15 da.,	Sept. 9	12 ¹ / ₂ %.
31. \$7,081 ¹ / ₂ ,	San Francisco, Jan. 2,	9 mo. 20 da.,	Apr. 1	15%.
32. \$22,605 ⁰ / ₁₀ ,	Portland, Or., Oct. 10,	4 mo.,	Nov. 25	12%.

For what sum must a note be drawn

33. At 90 days, discount 9%, to obtain \$500 ?

34. At 4 months, discount 7%, to obtain \$1,742³/₄ ?

35. At 30 days, discount 6%, to obtain \$230⁵/₁₀ ?

What must be the face of a bankable note, discounted at a national bank

In	payable in	to obtain proceeds of	In	payable in	to realize
36. New York,	20 da.,	\$8,400 ?	41. Washington,	40 da.,	\$10,000 ?
37. Philadelphia,	4 mo.,	\$3,160 ?	42. Milwaukee,	10 da.,	\$1,492 ?
38. Leavenworth,	100 da.,	\$781 ?	43. Boston,	3 da.,	\$25,000 ?
39. Denver,	3 mo.,	\$2,000 ?	44. Indianapolis,	30 da.,	\$637.50 ?
40. Richmond,	2 mo. 15 da.,	\$952 ?	45. Tallahassee,	90 da.,	\$540.31 ?

III. EXCHANGE.

517. If a party in San Francisco wishes to transmit a sum of money to a party in Chicago, he buys of a San Francisco bank a draft on a Chicago bank, for the required sum, and sends it to the party in Chicago. The Chicago party indorses the draft, presents it at the bank on which it is drawn, and receives its face in money.

518. *Exchange* is a method of making payments between different places, by means of drafts.

519. *Domestic exchange* or *inland exchange* is exchange between places in the same country.

520. *Foreign exchange* is exchange between places in different countries.

- a.* Exchange is *at par* when the cost of a draft equals its face; *above par* or *at a premium* when the cost of a draft exceeds its face; and *below par* or *at a discount* when the cost of a draft is less than its face.
- b.* Time drafts or bills of exchange are subject to bank discount on their face value for the time specified in them plus the days of grace.
- c.* Time drafts are always discounted at the rate of interest in the place where they are issued.
- d.* The acceptance of a draft makes the acceptor liable for its payment at maturity.

521. If St. Louis owes New York \$14,175,000, and New York owes St. Louis \$12,920,000, the balance of trade between the two cities is \$1,255,000 *against* St. Louis and in *favor* of New York.

522. The *balance of trade* between two places or countries is the difference of the amounts due to each place or country from the other.

523. The *course of exchange* between two places or countries is the price paid in one place or country for drafts upon the other.

- a.* Post-office money orders and express money orders are drafts, but are sold at schedule prices not computed as a percentage.
- b.* A party may remit funds from one place to another
 - 1. By absolutely transmitting the money by express or otherwise.
 - 2. By remitting a post-office money order or an express money order.
 - 3. By remitting a draft drawn on the place to which the money is to be sent or on some place that is recognized by all banks as a centre of exchange.
 - 4. By authorizing the party to whom the money is to be sent, to draw a draft on him (the sender) to cover the amount, and sell it to a bank, or leave it with a bank or an express company for collection.
 - 5. By sending his check on a bank where he has money deposited, and the party receiving it placing it in a bank or with an express company, for collection.
- c.* New York is the recognized centre of financial transactions for the whole United States, and drafts on that city are current in any part of the country.

524. In computations in exchange,

- 1. *Face of draft* = *base*. | 2. *Premium or discount* = *the percentage*.
- 3. *Cost of draft* = $\left\{ \begin{array}{l} \text{Amount.} \\ \text{Difference.} \end{array} \right.$ | 4. *Bank discount* = *the percentage*.

I. DOMESTIC EXCHANGE.

525. IMPORTANT FACTS.

I. *The course of exchange on a sight draft is 100% plus the per cent of premium, or minus the per cent of discount.*

II. *The course of exchange on a time draft is the course of exchange on a sight draft minus the per cent of bank discount.*

Ex. 1. Find the cost of a sight draft for \$210, exchange $\frac{1}{4}\%$ premium.

PROCESS.

$$\$210 \times 1.0025 = \$210.52\frac{1}{2}, \text{ the cost required.}$$

Ex. 2. Find the face of a sight draft whose cost is \$1,201.50, exchange $\frac{1}{8}\%$ premium.

PROCESS.

$$\$1,201.50 \div 1.00125 = \$1,200, \text{ the face of draft required.}$$

Ex. 3. Find the cost in New York of a 60-day draft, exchange $\frac{3}{8}\%$ discount.

PROCESS.

$$\begin{aligned} 1.00 - .00375 &= .99625, \text{ the course of exchange on a sight draft;} \\ 60 \text{ da.} + 3 \text{ da. (grace)} &= 63 \text{ da.}; 63 \text{ da.} = 2.1 \text{ mo.}; \\ \frac{3}{8}\% \text{ of } 2.1 &= .0105, \text{ the per cent of bank discount;} \\ .99625 - .0105 &= .98575, \text{ course of exchange on a 60 da. draft;} \\ \$180 \times .98575 &= \$177.43\frac{1}{2}, \text{ cost of draft required.} \end{aligned}$$

Ex. 4. The cost in Philadelphia of a 90-day draft, exchange $\frac{1}{4}\%$ premium, is \$631.68. Find the face.

PROCESS.

$$\begin{aligned} 1.00 + .0025 &= 1.0025, \text{ the course of exchange on a sight draft;} \\ 90 \text{ da.} + 3 \text{ da.} &= 93 \text{ da.}; 93 \text{ da.} = 3.1 \text{ mo.}; \\ \frac{1}{4}\% \text{ of } 3.1 &= .0155, \text{ the per cent of bank discount;} \\ 1.0025 - .0155 &= .987, \text{ the course of exchange on a 90 da. draft;} \\ \$631.68 \div .987 &= \$640, \text{ the face of draft required.} \end{aligned}$$

PROBLEMS.

Find the cost of a sight draft

- | | |
|---|---|
| 46. For \$630, at $\frac{5}{8}\%$ premium. | 49. For \$1,316, at $\frac{1}{4}\%$ discount. |
| 47. For \$168.70, at $\frac{1}{2}\%$ discount. | 50. For \$2,748.32, at $2\frac{1}{2}\%$ premium. |
| 48. For \$913.25, at $1\frac{1}{2}\%$ premium. | 51. For \$3,927, at $1\frac{1}{2}\%$ discount. |

Find the face of a sight draft that cost

- | | |
|---|--|
| 52. \$217.24, at $2\frac{1}{4}\%$ premium. | 55. \$293.40, at $2\frac{1}{2}\%$ discount. |
| 53. \$340, at $\frac{1}{8}\%$ premium. | 56. \$875, at $1\frac{1}{2}\%$ premium. |
| 54. \$316.25, at $\frac{1}{4}\%$ discount. | 57. \$1,478, at $1\frac{1}{2}\%$ discount. |

Find the cost of a draft

- 58.** For \$1,000, at 90 da., exchange $\frac{1}{4}\%$ premium, interest 5%.
- 59.** For \$650, at 60 da., exchange $1\frac{1}{2}\%$ premium, interest 7%.
- 60.** For \$917.60, at 10 da., exchange $\frac{1}{8}\%$ discount, interest 10%.
- 61.** For \$2,138, at 30 da., exchange $\frac{1}{4}\%$ discount, interest 6%.
- 62.** Find the cost in Washington of a draft on Pittsburgh for \$3,385, 30 days sight, exchange $1\frac{1}{4}\%$ premium.
- 63.** What will be the cost in St. Louis of a draft on Philadelphia, at 60 da. sight, for \$739.50, exchange $\frac{1}{8}\%$ discount?
- 64.** Find the cost of a draft on Buffalo for \$3,720, sight, exchange $\frac{1}{8}\%$ premium.
- 65.** Find the face of a draft that costs \$1,000, exchange $\frac{1}{4}\%$ discount.
- 66.** Find the face of a 30 da. draft on Chicago, purchased in St. Paul for \$2,928, exchange $\frac{3}{8}\%$ premium.
- 67.** What is the cost of a draft in Cleveland, O., on Denver, Col., for \$7,828, 60 da. sight, at $1\frac{1}{8}\%$ premium?
- 68.** What is the face of a draft that costs \$2,913.80, payable in 3 mo., interest 8%?
- 69.** What must I pay in San Francisco for a draft on New York for \$2,500, at 60 da. sight, exchange $\frac{1}{4}\%$ discount?

II. FOREIGN EXCHANGE.

526. Bills of exchange are drawn in the currencies of the countries in which they are payable.

Thus, drafts on England, Ireland, or Scotland are drawn in pounds, shillings, and pence; on France, Belgium, or Switzerland, in francs; on Germany in marks; and so on.

- a.* Foreign bills of exchange are usually drawn at sight or sixty days after sight. The former are known as *short exchange*, and the latter as *long exchange*.
- b.* Days of grace are usually allowed on all foreign exchange.
- c.* The course of exchange on time bills is as much less than that on sight bills as the per cent of interest for the time given plus the days of grace.

527. The *par of exchange* is the value of a money unit in one country expressed in the currency of another country.

- a.* The *intrinsic* par of exchange is the value of the coin of one country in the coin of another. It is based upon the relative weight and fineness of the coins.
- b.* The *commercial* par of exchange is the market value of the money unit of one country, in the currency of another. It is commonly called the *rate of exchange*.

For intrinsic par of exchange between the United States and foreign countries,
See Supplement, page 448.

528. A *set of exchange* consists of two or of three copies of the same bill, sent by different carriers or on different dates, to guard against loss or delay.

See Supplement, page 446.

When one of a set of exchange is paid, the others become void or worthless.

529. Exchange with Europe is effected chiefly through the financial agencies of London, Paris, Antwerp, Amsterdam, Hamburg, Frankfort, Bremen, and Berlin.

In exchange with these cities, the money units and the intrinsic par of exchange are as follows:—

Cities	Units	Value in U. S. gold coin
London,	Pound sterling or sovereign,	\$4.8665
Paris, Antwerp, and cities of Switzerland,	Franc,	.193
Amsterdam,	Guilder (Florin),	.407
Hamburg,	Four reichsmarks,	.952

Exchange is quoted 5.18 francs to the dollar. 1 mark = \$.288.

EXERCISES.

Explain the meaning of the following quotations—taken from the New York Journal of Commerce of July 28, 1887.

1. Bankers' sterling, 60 days, actual $\$4.82\frac{1}{2}$ @ $\$4.83$.
2. Bankers' sterling, sight, actual $\$4.84\frac{1}{2}$ @ $\$4.85$.
3. Prime commercial sterling, long, $\$4.81\frac{1}{2}$ @ $\$4.82$.
4. Paris, bankers', 60 days, $5.23\frac{1}{2}$ @ $5.23\frac{1}{2}$.
5. Paris, bankers', sight, $5.21\frac{1}{2}$ @ $5.20\frac{1}{2}$.
6. Swiss, bankers', 60 days, $5.23\frac{1}{2}$ @ $5.23\frac{1}{2}$.
7. Reichsmarks (4), bankers', 60 days, $\$.94\frac{5}{8}$ @ $\$.94\frac{3}{4}$.
8. Guilder, bankers', sight, $\$.40\frac{1}{2}$ @ $\$.40\frac{3}{4}$.

530. In computing the cost of a bill of exchange:—

Reduce the value of the bill expressed in the given currency, to the equivalent value in the required currency.

PROBLEMS.

Find the cost of a bill of exchange

- 70.** On London for £253 7s. 6d., sterling at $\$4.84\frac{1}{2}$.
- 71.** On Paris for 8,450 francs, at 5.19 francs for \$1.
- 72.** On Amsterdam for 6,500 guilders at $\$.412$.
- 73.** On Hamburg for 1,300 marks at $\$.948$ for 4 marks.

Find the face of a set of exchange

- 74.** On Liverpool, that cost \$2,000, sterling $\$4.83\frac{1}{2}$.
- 75.** On Geneva, Switzerland, that cost \$480, at 5.17 francs to \$1.
- 76.** On Berlin, that cost \$910, at $\$.243\frac{1}{2}$ per mark.

Find the cost of a bill

- 77.** On London for £800 12s. 6d., at 60 da., sterling exchange $\$4.82\frac{5}{8}$.
- 78.** On Antwerp for 8,600 francs at $5.19\frac{1}{2}$.
- 79.** On Amsterdam for 3,000 guilders, long, at $\$.401$.
- 80.** On Frankfort for 2,350 marks, short, at $\$.961$ (4).

TRUE DISCOUNT.

531. The *present worth* of a debt due at a future time is the sum of money that, loaned at interest to-day, will amount to the debt at maturity. .

532. *True discount* is the interest on the present worth of a debt, from to-day to maturity. It is the debt minus its present worth.

\$100 put at interest to-day, at 6%, will amount to \$103 in 6 months. Hence, \$100 is the present worth of, and \$3 is the true discount on \$103 due in 6 months, at the rate of 6% discount.

Present worth is also called *true present worth* and *equitable present worth*; and true discount is also called *equitable discount*.

True discount is the interest on the present worth of a debt from a given time to maturity (DEF.), and bank discount is the interest on the debt itself for the same time (508). *e.g.*, The true discount on \$103 due in 6 months, rate of interest 6%, is \$3; and the bank discount is ($\$103 \times .03 =$) \$3.09. Hence,

533. IMPORTANT FACT.—*Bank discount exceeds true discount by the interest on the true discount.*

From the definition (510) it is evident that the sum to be discounted is

- (1) The face of a debt that does not bear interest; or
- (2) The amount of a debt that bears interest.

Commercial discount and bank discount having largely taken the place of true discount, the latter is now but little used.

534. In computations in true discount,

$$\begin{array}{l|l} \text{Present worth} = \text{principal;} & \text{True discount} = \text{interest;} \\ \text{Rate of discount} = \text{rate of interest;} & \text{Sum to be discounted} = \text{amount;} \\ \text{Amount} - \text{present worth} = \text{true discount.} & \end{array}$$

535. The process of finding present worth is an application of Case IV of the Six General Problems of Interest. (See 475.)

Ex. A debt of \$751.40 is due in 8 months. If it is discounted at the rate of 6%, what is the present worth? What is the true discount?

ANALYTICAL PROCESS.

$8 \text{ mo.} = .04$, the per cent of interest.

$1 + .04 = 1.04$, or 1 plus the per cent of interest.

$\$751.40 \div 1.04 = \$722.50$, the required present worth; and

$\$751.40 - \$722.50 = \$28.90$, the required true discount.

PROBLEMS.

What is the present worth			What is the true discount		
Of	due in	discounted	On	due in	discounted
81. \$761	4 mo. 12 da.	at 8%?	84. \$900	20 da.	at 5%?
82. \$83.44	90 da.	at $3\frac{1}{2}\%$?	85. \$1,000	3 mo. 3 da.	at 4%?
83. \$132.25	15 mo.	at 6%?	86. \$47.50	4 mo. 3 da.	at 6%?

Find the present worth of, and the true discount on

Paper	for	dated	due	rate of discount
87. A note	\$684 $\frac{1}{2}$,	July 10, 1886,	in 1 yr. 4 mo.,	6%.
88. A mortgage	\$3,800,	Nov. 1, 1887,	March 25, 1889,	$4\frac{1}{2}\%$.
89. A bond	\$660,	June 15, 1887,	Oct. 19, 1887,	5%.
90. A claim	\$1,125.75,	Sept. 9, 1888,	Jan. 15, 1889,	$7\frac{3}{10}\%$.

SECTION VI.

ACCOUNTS IN WHICH INTEREST IS AN ELEMENT.

I. SAVINGS-BANK ACCOUNTS.

536. Savings-banks pay compound interest to their depositors. They add the interest to the principal either monthly, quarterly, or semi-annually.

537. An *interest term* is the regular interval at the end of which interest on deposits is compounded.

a. Monthly interest terms begin on the 1st day of each month; quarterly terms on the 1st days of Jan., Apr., July, and Oct.; and semi-annual terms on the 1st days of Jan. and July.

b. The interest term of most savings-banks is three months.

c. No interest is allowed on fractional parts of a dollar.

538. Most savings-banks allow interest on only those sums that have been on deposit a full interest term. Hence,

IMPORTANT FACT.—Interest is computed at the end of the interest term, on the smallest balance on deposit at any one time during the term.

EXAMPLE.—The following statement is taken from the books of a savings-bank that pays interest quarterly at 4% per annum, and shows the depositor's balance July 1, 1886.

Statement.

<i>Dates.</i>	<i>Deposits</i>	<i>Drawn out.</i>	<i>Interest</i>	<i>Daily balances.</i>
<i>Jan. 1</i>	<i>98—</i>			<i>98—</i>
<i>" 16</i>		<i>10—</i>		<i>88—</i>
<i>Feb. 21</i>	<i>18 25</i>			<i>106 25</i>
<i>Mar. 7</i>	<i>12 50</i>			<i>118 75</i>
<i>Apr. 1</i>			<i>88</i>	<i>119 63</i>
<i>" 13</i>	<i>20—</i>			<i>139 63</i>
<i>" 29</i>		<i>14—</i>		<i>125 63</i>
<i>May 12</i>		<i>10—</i>		<i>115 63</i>
<i>June 5</i>	<i>15—</i>			<i>130 63</i>
<i>July 1</i>	<i>12—</i>		<i>115</i>	<i>143 78</i>

EXPLANATION OF STATEMENT.—The interest term being quarterly, the interest is added to the daily balance on the first day of each quarter.

The smallest balance during the first quarter is \$88, the interest on which (\$.88) is added to the daily balance April 1.

The smallest balance during the second quarter is \$115.63, and the interest on \$115 (\$.15), together with a deposit of \$12, is added to the daily balance July 1.

PROBLEMS.

1. Make a statement of the following account, and find the balance due the depositor Dec. 1, 1888, interest at 4% per annum.

Monroe County Savings Bank.
in acct. with Henry Wilson.

<i>Dr.</i>				<i>Cr.</i>			
<u>1887</u>				<u>1887</u>			
May 13	To cash	28		July 31	By check	6	54
June 5	" "	30		Sept. 19	" "	10	—
July 20	" "	25		Nov. 1	" "	22	50
Sept. 1	" "	46		<u>1888</u> Jan. 20	" "	15	75
Oct. 1	" interest	—		May 18	" "	85	—
" 1	" cash	50		July 29	" "	31	25
Nov. 18	" "	25		Sept. 5	" "	16	40
Dec. 10	" "	45					
<u>1888</u> Jan. 2	" interest	—					
Feb. 5	" cash	28					
Mar. 1	" "	25					
Apr. 1	" interest	—					
" 1	" cash	40					
July 1	" interest	—					
Oct. 1	" "	—					
Dec. 1	" "	—					

Make statements of the following narratives, and in each find the balance due on the first day of the quarter following the last transaction.

2. East-Side Savings-Bank. Depositor, Martin Godfrey. Deposits,—1885, June 24, \$100; Sept. 20, \$55; Oct. 12, \$36; Nov. 11, \$25; Dec. 28, \$32; 1886, Apr. 15, \$50; May 10, \$20. Checked out,—1885, Aug. 10, \$24.50; Dec. 1, \$33.75; 1886, Mar. 20, \$21.80; Aug. 5, \$18.35. Interest term 3 mo. Rate of interest 4%.

3. Fidelity Savings-Bank. Depositor, Elsie Whipple. Deposits,—1888, Jan. 18, \$10; Feb. 12, \$6.50; Mar. 1, \$5; Mar. 22, \$4.75; Apr. 10, \$3.50; May 18, \$7; June 2, \$4; June 29, \$3.50; July 16, \$5.25. Checked out,—1888, Mar. 29, \$7.65; May 3, \$2.85; May 16, \$5.36; June 13, \$8.25. Interest term 1 mo. Rate of interest 6%.

II. AVERAGE OR EQUATION OF PAYMENTS.

539. *Equation of payments* is the process of finding the equitable time for paying, in one sum, several debts due at different times.

540. The *term of credit* is the time that must elapse before a debt becomes due.

541. The *average term of credit* is the time that must elapse before several debts, due at different times, may be equitably paid in one sum.

542. The *equated time* is the equitable date of payment.

543. A *focal date* is a date from which the terms of credit are computed.

Any date may be assumed as the focal date; but the focal date usually assumed is the earliest date at which any of the debts become due.

544. Equation of payments is based on the following obvious

BUSINESS PRINCIPLES.

I. *Any debt is equitably subject to legal interest from the time it is due.*

II. *The interest of any sum for a given time equals the interest of two times that sum for one half of that time, or of one half of that sum for two times that time; and so on.*

III. *The use of any sum, paid at any time before it is due, equals the use of an equal sum not paid till the same length of time after it is due.*

CASE I. The terms of credit beginning at the same date.

545. Ex. What is the equated time for the payment of \$500 due to-day (Jan. 9), \$300 due in 6 months, and \$400 due in 9 months?

(1) SOLUTION BY THE PRODUCT METHOD.

EXPLANATION.—The interest of any number of dollars for any given number of months equals the interest of \$1 for the number of months expressed by the product of the given dollars and months. The use of \$500 for no time equals the use of \$1 for no time. The use of \$300 for 6

PROCESS.

500
 $300 \times 6 \text{ mo.} = 1,800 \text{ mo.}$
 $400 \times 9 \text{ " } = 3,600 \text{ "}$
 $\underline{1,200 \times ? \text{ " } = 5,400 \text{ mo.}}$
 $5,400 \text{ mo.} \div 1,200 = 4\frac{1}{2} \text{ mo.} = 4 \text{ mo. } 15 \text{ da.}$
 $\text{Jan. 9} + 4 \text{ mo. } 15 \text{ da.} = \text{May 24.}$

months equals the use of \$1 for 1,800 months,—300 times 6 months; and the use of \$400 for 9 months equals the use of \$1 for 3,600 months. The use of \$1 for 5,400 months (the sum of all the times) equals the use of \$1,200 (the sum of all the payments) for $\frac{1}{1200}$ of 5,400 months. I therefore divide 5,400 months by 1,200, and obtain $4\frac{1}{2}$ mo., or 4 mo. 15 da., which is the average term of credit. Then, adding this time to Jan. 9, the date of the purchase, I have May 24, the equated time.

(2) SOLUTION BY THE INTEREST METHOD.

For convenience, interest is computed, in the solutions in this article, at 12% per annum, or 1% per month. Any other rate of interest gives the same result.

PROCESS.

<i>Due</i>	<i>Amounts</i>	<i>Times</i>	<i>Interests</i>
Jan. 9,	\$500		
July 9,	300	6 mo.	\$18
Oct. 9,	400	9 mo.	36
Int. of \$1,200		1 mo. = \$12	\$54
			$4\frac{1}{2} \text{ times} = 4 \text{ mo. } 15 \text{ da.}$

EXPLANATION.—By paying the entire amount—\$1,200—to-day, the debtor will lose the interest of \$300 for 6 mo., and of \$400 for 9 mo.,—or \$18 + \$36, which is \$54. To equitably offset this loss, he may keep the \$1,200 time enough for the interest on it, at 12%, to amount to \$54. \$12 is the interest on \$1,200 for 1 month, at 12%, and \$54 is the interest on the same sum for as many months as the times \$12 are contained in \$54, which is $4\frac{1}{2}$ times, or 4 mo. 15 da., the average term of credit. Then, Jan. 9 + 4 mo. 15 da. = May 24, the equated time.

PROBLEMS.

Find the equated time for the payment of the following accounts:—

4. \$150 due in 30 days, \$200 due in 60 days, and \$250 due in 90 days.
5. The rent of a house at \$300 per annum, payable quarterly.
6. Five notes, all dated June 17,—for \$300 on 3 mo., \$500 on 5 mo., \$150 on 7 mo., \$350 on 9 mo., and \$200 on 1 yr.
7. Four bills of goods, due as follows: Feb. 12, \$300; Feb. 18, \$275; Apr. 6, \$500; July 24, \$100.

CASE II. The terms of credit beginning at different dates.

546. Ex. March 20, I bought a horse for \$175, on a credit of 4 months; May 5, a harness for \$35, on 3 months; and June 15, a carriage for \$225, on 6 months. Find the equated time for the payment of these debts.

EXPLANATION.—To find the dates on which the several payments are due, I add to the date of each payment its term of credit.

Since the first payment is due July 20, I take that date as a focal date.

Subtracting that date from the date on which each of the other payments is due, I have \$175 due at the focal date, \$35 due in 15 days, and \$255 due in 4 months 25 days. Then proceeding by either method given in Case I., I find the average term of credit to be 2 months 21 days, and the equated time Oct. 11.

NOTE.—Most accountants prefer the Interest Method.

PROCESS.

Mar. 20 + 4 mo. = July 20

May 5 + 3 mo. = Aug. 5

June 15 + 6 mo. = Dec. 15

(1) BY PRODUCT METHOD.

$$\begin{array}{rcl}
 175 & & \\
 35 \times & 15 \text{ da.} = & 17 \text{ mo. } 15 \text{ da.} \\
 255 \times 4 \text{ mo. } 25 \text{ da.} = & 1,232 \text{ mo. } 15 \text{ da.} & \\
 465 \times 9 \text{ mo.} & = & 1,250 \text{ mo.} \\
 1,250 \text{ mo.} \div 465 = & 2.68 \text{ mo.} = & 2 \text{ mo. } 21 \text{ da.} \\
 \text{July } 20 + 2 \text{ mo. } 21 \text{ da.} = & & \text{Oct. } 11
 \end{array}$$

(2) BY INTEREST METHOD.

Amounts	Times	Interests
\$175		
35	15 da.	\$.175
255	4 mo. 25 da.	12.325
Int. of \$465 for 1 mo. = \$4.65		\$12.50
		2.68 = 2 mo. 21 da.
July 20 + 2 mo. 21 da. =		Oct. 11

PROBLEMS.

Find the dates for the equitable settlement of the following:—

8. Bill No. 1. Terms cash.

Jan. 10,	\$415.—
Feb. 25,	342.—
Apr. 5,	167.50
“ 30,	90.21

9. Bill No. 2. Terms 2 mo.

Mar. 5,	\$63.25
May 17,	46.91
June 2,	100.—
Aug. 20,	82.70

10. Bill No. 3.

George Adams

to E. A. Harris & Co.,

Dr.

Mar. 14	40 bar. beef. @ \$17.50, m. 3 mo.	—	—
May 1	60 .. pork. 22.— 4 ..	—	—
June 10	150 .. flour. 8 1/2%, cash.	—	—

11. Find the equated time for the payment of three notes, as follows: for \$650, dated July 12, on 90 da.; \$555, dated July 25, on 60 da.; and \$445, dated Aug. 14, on 30 da.

547. RULES FOR EQUATION OF PAYMENTS.

I. When all the terms of credit begin at the same date.

1. For the average term of credit:—

PRODUCT METHOD.—Multiply each term of credit by the number expressing the payment, and divide the sum of the products by the sum of the payments. Or,

INTEREST METHOD.—Find the interest of each debt for its term of credit, and divide the sum of the interests by the interest of the sum of the debts for one period of the time of the same denomination as the terms of credit.

2. For the equated time:—

Add the average term of credit to the date at which the several credits begin.

II. When the terms of credit begin at different dates.

1. *Find the date at which each debt became due, and from the earliest of these dates as a focal date, compute the time to each of the others.*

2. *Find the average term of credit and the equated time, as in Case I.*

a. In finding the average term of credit,—

1. *When the several terms of credit are for months, use months or months and days; and*

2. *When the several terms of credit are for days, use days only.*

b. In finding products or interests,—

If the cents in any item are less than 50, reject them;

If they are 50 or more, reckon them as 1 dollar.

c. *If any result contains a fraction of a day less than $\frac{1}{2}$, reject it; if $\frac{1}{2}$ or more, reckon it as 1 day.*

PROBLEMS.

12. Find the equated time for the payment of five bills of goods, bought May 29, as follows: \$825 on 3 mo., \$675 on 4 mo., \$450 on 2 mo., and \$800 on 1 mo.

13. I make the following advances of money for a friend:—May 19, \$107; May 28, \$35; June 27, \$130; July 3, \$70; Aug. 24, \$80; and Sept. 11, \$175. If I take his note for the whole amount, dated at the equated time, what will be the date of the note?

14. March 8, I sold a farm for \$3,300, payments—\$750 cash, \$800 in 4 mo., \$500 in 8 mo., and the balance in 1 yr. Had I taken a mortgage for the whole amount, to be paid in one payment, when would it have been equitably due?

15. B works for C 6 months from May 15, at \$60 per month, his wages to be paid—one half monthly, and the other half in 3 months. In lieu of receiving his pay according to contract, he takes C's note for the whole amount, bearing date from the average time, with interest. What is the date of the note?

16. Find the equated time for paying the following

Statement of a/c.

New York, Feb. 1, 1888.

B. F. Perkins & Bro.
in a/c with R. & H. Morgan, Dr.

<u>1887</u>				
Oct.	19	To mdsu.	3 mo.	832 67
Nov.	4	" "	30 da.	571 15
Dec.	28	" "	cash.	1,125 —
<u>1888</u>				
Feb.	1	" "	4 mo.	659 —
			<i>Amt. due.</i>	<u>2,187 82</u>

III. AVERAGE OF ACCOUNTS.

548. The *balance of an account* is the difference between the amounts of its debits and credits.

Merchandise balance is the difference between the two sides of an account.

549. *Averaging an account* is the process of finding the equated time for paying the balance of the account.

550. The averaging of accounts is based on the following

BUSINESS PRINCIPLES.

I. *Each item of an account is equitably subject to interest from the time it is due.*

II. *All payments made before the average term of credit expires, are equitably subject to discount from the time they are made.*

III. *All debits not paid till after the average term of credit expires, are equitably subject to interest from the expiration of the average term of credit.*

EXAMPLE.—Find the average time for the payment of the balance of the following account.

<i>Dr.</i>				<i>Henry S. Wilson.</i>				<i>Cr.</i>			
<u>1888</u>								<u>1888</u>			
<i>Jan. 18</i>	<i>To mdse.</i>	<i>250</i>	<i>—</i>	<i>Feb. 11</i>	<i>By cash</i>	<i>188</i>	<i>30</i>				
<i>Mar. 6</i>	<i>" do.</i>	<i>85</i>	<i>19</i>	<i>Apr. 27</i>	<i>" do.</i>	<i>90</i>	<i>31</i>				
<i>May 20</i>	<i>" do.</i>	<i>323</i>	<i>81</i>	<i>June 15</i>	<i>" mdse.</i>	<i>170</i>	<i>—</i>				

PROCESS.

Focal date, Dec. 31, 1887.

<i>Due</i>	<i>Items</i>	<i>Times</i>	<i>Interests</i>	<i>Due</i>	<i>Items</i>	<i>Times</i>	<i>Interests</i>
<i>Jan. 18,</i>	<i>\$250</i>	<i>18 da.</i>	<i>\$ 1.50</i>	<i>Feb. 11,</i>	<i>\$189</i>	<i>1 mo. 11 da.</i>	<i>\$2.58</i>
<i>Mar. 6,</i>	<i>85</i>	<i>2 mo. 6 "</i>	<i>1.87</i>	<i>Apr. 27,</i>	<i>90</i>	<i>3 " 27 "</i>	<i>3.51</i>
<i>May 20,</i>	<i>324</i>	<i>4 " 20 "</i>	<i>15.12</i>	<i>June 15,</i>	<i>170</i>	<i>5 " 15 "</i>	<i>9.35</i>
	<i>\$659</i>		<i>\$18.49</i>		<i>\$449</i>		<i>\$15.44</i>
	<i>449</i>		<i>15.44</i>				
	<i>\$210</i>		<i>\$3.05</i>				

Int. on \$210 for 1 mo. = \$2.10

$\$3.05 \div \$2.10 = 1.452 = 1 \text{ mo. } 1\frac{1}{2} \text{ da., average term of credit.}$

$\text{Dec. 31} + 1 \text{ mo. } 1\frac{1}{2} \text{ da.} = \text{Feb. } 1\frac{1}{2}, \text{ average time of payment.}$

EXPLANATION.—Assuming Dec. 31, 1887, for a focal date, I first find the interest on each item of the account from the focal date to the time it is due.

I next find the respective sums of the debit items, the debit interests, the credit items, and the credit interests; and subtracting \$449, the sum of the credits, from \$659, the sum of the debits, and \$15.44, the sum of the interests on the credits, from \$18.49, the sum of the interests on the debits, I obtain a merchandise balance of \$210, and an interest balance of \$3.05, both in favor of the debit side of the account.

The merchandise balance \$210 is entitled to credit a sufficient time for the interest to amount to \$3.05,—i. e., for as many months as the times the interest on \$210 for 1 month is contained in the interest balance. I therefore then divide \$3.05 of interest

by \$2.10 (the interest on \$210 for 1 month), and obtain 1.452, which equals 1 mo. 14 da., the average term of credit for the balance of the account.

Since both merchandise and interest balances are in favor of the debtor side of the account, I add the 1 mo. 14 da. to the focal date, and obtain Feb. 14, the average time of payment.

In this example the merchandise balance and the interest balance are in favor of the Dr. side of the $\%$,—i. e., the greater interest is on the greater side of the $\%$. If the greater interest were on the less side of the $\%$, the average term of credit would be the same, but the average time of payment would be 1 mo. 14 da. before the focal date.

551. RULE FOR AVERAGING ACCOUNTS.

I. Find the interest on each item, for the time between the focal date and the date the item is due.

II. Find the merchandise balance, and the interest balance; and divide the interest balance by the interest on the merchandise balance for 1 month, expressing the result as months and days.

III. Add this time to the focal date, if the greater interest is on the greater side of the account; but subtract it, if the greater interest is on the less side.

a. If an account is settled after the average time of payment, charge interest on the merchandise balance from the average time.

b. If it is settled before the average time, discount the merchandise balance for the difference of time, at the legal rate.

PROBLEMS.

17. Find the average time for paying the balance of the following $\%$.

Dr.				Edgar Lansing.				Cr.			
1886				1886							
Jan.	4	To mds. & m.	325 50	Feb.	15	By cash.	200				
"	16	" do. 4 "	37 50	Apr.	17	" do.	75				
Mar.	1	" do. 4 "	162 50								
"	26	" do. 4 "	250								

18. Average the following account, allowing a credit of 4 mo. for each debit item.

Dr.				Robt. M. White.				Cr.			
<u>1887</u>				<u>1888</u>							
Nov.	1	To mdse.	131 25	Jan.	27	By cash,	25	—			
"	20	" do.	15	—	Mar.	30	" do.	70	50		
Dec.	19	" do.	68 40	May	1	" produce,	93	28			
<u>1888</u>				June	10	" horse,	137	50			
Jan.	6	" do.	43	84							
Mar.	30	" do.	9	50							
Apr.	15	" do.	56	—							
May	22	" do.	81	75							

19. If a note, drawing interest, be given to balance the following %, for what sum will it be drawn, and what will be its date?

Dr.				Luiswold, Curtis, & Co.				Cr.			
<u>1888</u>				<u>1888</u>							
Mar.	17	To mdse. 2 mo	325	—	July	25	By cash,	125	—		
Apr.	20	" do. 3 "	408	75	Aug.	17	" do.	200	—		
July	18	" do. cash,	164	90	Oct.	24	" draft on N.Y.	550	—		
Aug.	11	" do. 4 mo.	50	—							
Sept.	25	" do. 2 "	225	—							

20. When will the balance of the following % begin to draw interest?

Dr. James S. Andrews.				Cr.			
1887				1887			
Oct. 9	To mds.	3 mo.	300	Nov. 24	By cash,	125	—
Nov. 18	" do.	2 "	529 30	Dec. 4	" do.	500	—
" 27	" do.	3 "	142 63	" 30	" note.	150	—
Dec. 19	" do.	cash,	256 50				

21. Find the balance of the following %, June 1, 1888, allowing interest at 6%.

Dr. E. T. Downing & Co.				Cr.			
1887				1887			
June 11	To mds.	3 mo.	450	Sept. 3	By cash,	400	—
Aug. 25	" do.	3 "	175	Nov. 2	" do.	150	—
Oct. 11	" do.	3 "	425	" 30	" do.	225	—

IV. CASH BALANCE.

552. Define (1) Account current (296); (2) Merchandise balance (548).

553. The *cash balance* of an account is the difference between the two sides, after interest has been added to each item past due at the date of settlement, and deducted from each item not due at that date.

554. Cash balance is based on the following

BUSINESS PRINCIPLES.

I. Each item of an account equitably draws interest from the time it is due.

II. An item paid before it is due is equitably entitled to discount for the time from the date of payment to the date it is due.

EXAMPLE.—Find the cash balance Nov. 18, 1888, on the following account, interest at 6%.

Morton Bros. in acct. current with

Dr.	Chas. T. Amidon.	Cr.	
<u>1888</u>		<u>1888</u>	
Apr. 3 To mdse.	350—	Apr. 15 By cash.	175—
June 1 " "	575 80	June 3 " "	600—
Aug. 11 " " ov 2 mo.	268 50	July 7 " draft mty.	95 30
Sept. 5 " " " 10 da.	725—	Nov. 1 " cash.	800—

PROCESS BY INTEREST METHOD.

Due dates	Items	Time	Interest	Credit dates	Items	Time	Interest
Apr. 3,	\$350.—	7 mo. 15 da.	\$13.13	Apr. 15,	\$175.—	7 mo. 3 da.	\$ 6.21
June 1,	575.80	5 mo. 17 da.	16.03	June 3,	600.—	5 mo. 15 da.	16.50
Oct. 11,	268.50	1 mo. 7 da.	1.66	July 7,	95.30	4 mo. 11 da.	2.07
Sept. 15,	725.—	2 mo. 3 da.	7.61	Nov. 1,	800.—	17 da.	2.27
	<u>\$1,919.30</u>		<u>\$38.43</u>		<u>\$1,670.30</u>		<u>\$27.05</u>
	1,670.30		27.05				
	<u>\$249.—</u>		<u>\$11.38, balance of interest.</u>				
	11.38						
	<u>\$260.38, cash balance required.</u>						

PROCESS BY PRODUCT METHOD.

Due dates	Items	Time	Products	Credit dates	Items	Time	Products
Apr. 3,	\$350.—	225 da.	78,750	Apr. 15,	\$175.—	213 da.	37,275
June 1,	575.80	167 "	96,192	June 3,	600.—	165 "	99,000
Oct. 11,	268.50	37 "	9,953	July 7,	95.30	131 "	12,445
Sept. 5,	725.—	63 "	45,675	Nov. 1,	800.—	17 "	13,600
	<u>\$1,919.30</u>		<u>230,570</u>		<u>\$1,670.30</u>		<u>162,320</u>
	1,670.30		162,320				
	<u>\$249.—</u>		<u>6] 68,250</u>				
	11.38						
			<u>11.38 = \$11.38, balance of interest (468; II).</u>				
			<u>\$260.38, cash balance required. See Notes, page 315.</u>				

NOTES.—1. When the balance of interest is on the same side of the account as the balance of items, it is added to the balance of items; if on the opposite side of the account, it is subtracted from the balance of items.

2. In finding interest or products, fractional parts less than fifty cents of the principal sum are disregarded; and fractional parts fifty cents or greater are computed as one dollar.

PROBLEMS.

22. Find the cash balance of the following account, Nov. 10, 1888, interest at 6%.

<i>Le Roy Avery, in account with</i>				<i>B. F. Bogart & Co.</i>				<i>Cr.</i>			
<u>1888</u>				<u>1888</u>							
May 5	To produce	430	—	May 20	By cash	600	—				
June 11	"	"	619 40	June 28	"	"	350	—			
Aug. 15	"	"	520 80	Sept. 1	"	"	500	—			
Sept. 29	"	"	812 92	Oct. 18	"	"	750	—			

23. Find the cash balance of the following account, Dec. 1, 1886, interest at 7%.

<i>Robert S. Ellis, in account with</i>				<i>Robt. Congdon.</i>				<i>Cr.</i>			
<u>1886</u>				<u>1886</u>							
May 15	To mds.	225	70	June 24	By cash	180	—				
" 29	" " 3 mo.	415	—	July 14	"	"	45 80				
July 8	" " 20 da.	180	40	Sept. 1	"	"	300	—			
Aug. 10	" " 2 mo.	216	20	Oct. 5	" note, 2 mo.	290	—				
Sept. 11	" "	85	65	Nov. 15	" cash	100	—				

24. Find the cash balance of the following account, Nov. 1, 1888, interest at 6%.

<i>Dr.</i>				<i>Cr.</i>			
<i>J. S. Holmes, in account with</i>				<i>Milton T. Barber.</i>			
<u>1888</u>				<u>1888</u>			
<i>Jan. 16</i>	<i>To mds. 2 mo.</i>	<i>975</i>		<i>Mar. 21</i>	<i>By cash,</i>	<i>230</i>	
<i>Feb. 20</i>	<i>" " 2 "</i>	<i>620</i>	<i>50</i>	<i>May 1</i>	<i>" "</i>	<i>350</i>	
<i>Apr. 12</i>	<i>" " 3 "</i>	<i>140</i>	<i>85</i>	<i>" 15</i>	<i>" note, 60 da.</i>	<i>500</i>	
<i>June 1</i>	<i>" " 4 "</i>	<i>116</i>	<i>45</i>	<i>July 20</i>	<i>" draft, 30 "</i>	<i>500</i>	
				<i>Oct. 12</i>	<i>" produce,</i>	<i>650</i>	<i>80</i>

V. AVERAGING ACCOUNTS SALES.

555. An *account sales* is a statement of account made by an agent to a consignor, showing the merchandise sold, expenses paid by him, and his commission.

a. Expenses include freight, cartage, storage, insurance, advertising, etc.

b. The commission is sometimes increased by an extra per cent on credit sales, for guaranty,—i. e., for the agent's assuming all losses from bad debts.

556. Averaging accounts sales is based upon the following

BUSINESS PRINCIPLES.

I. *An agent is entitled to interest on money advanced by him on behalf of his principal.*

II. *An agent may pay himself this interest by retaining the proceeds of the sale according to an average of accounts.*

III. *The date on which an agent's fees are due is that of the average of the sales.*

PROBLEMS.

25. Account sales of 1,840 barrels of flour received June 1, 1886, from Minneapolis, Minn.

Account sales of 1,840 barrels of flour received June 1, 1886, from Kilbir & Morse, Minneapolis, Minn.

<u>1886</u>					
June 10	Sold 440 bar.	@ \$6.10			
" 23	" 375 "	\$6.15			
July 11	" 520 "	\$6.20			
Aug. 1	" 505 "	\$6.18			
<u>Charges</u>					
June 1	To freight, cartage & cooperage: \$558.60				
" 1	" insurance and advertising: 78.25				
Aug. 1	" storage from June 1, @ 2¢ per wk. on 440 bar. 2 wk; ---- wk.				
"	375 "	4 "	----	"	
"	520 "	6 "	----	"	
"	505 "	9 "	----	"	
	----- wk. @ 2¢				
	" commission on \$ ---- @ 2½%				
	" guaranty on \$1,520 @ 2%				
	Balance				

What are the net proceeds of the above account, and when due?

NOTES.—1. All fractions of weeks are counted full weeks in computing storage.

2. The steps in the solution of the foregoing problem are

- (1) Extend all items to commission.
- (2) Average the sales alone to find the date of commission and guaranty.
- (3) Find commission on the total of the sales.
- (4) Make the sales the Dr. side of an account, and the charges the Cr. side, and average this account for the date on which the net proceeds are due.

Put the following narratives in the form of accounts sales, and find in each the net proceeds and the equated time of payment:—

26. J. O. Wilcox of Denver, Col., sold for account and risk of Arnold, Constable and Co. of New York City, the following:—1886, Aug. 8, 7 pc. summer silks, 273 yd., @ \$.68; Aug. 16, 5 pc. blk. silk, 217 yd., @ \$1.26; Sept. 5, 29 pc. French sateens, 1,392 yd., @ 23¢; Sept. 11, 23 pc. alpaca, 874½ yd., @ 48¢; Oct. 3, 17 pc. diagonals, 691 yd., @ 71¢. The charges are,—Aug. 2, freight and cartage, \$53.89; Sept. 1, freight and cartage, \$29.33; Aug. 4, advertising, \$18.75; commission, 3%.

27. Ellwood E. Potter of Phila., Pa., sold for account of Allen Bro's of Battle Creek, Mich., the following:—1888, Nov. 1, 800 bu. of potatoes, @ 54¢; Nov. 13, 550 bu. @ 58¢; Nov. 28, 1,200 bu. @ 57¢; Dec. 12, 1,460 bu. @ 61¢; Dec. 20, 910 bu. @ 63¢; Jan. 14, 1889, 1,025 bu. @ 68¢. The charges are,—1888, Oct. 20, freight, \$98.50; Nov. 22, freight, \$118.43; Jan. 14, 1889, storage, \$82.25; commission, 5%; guaranty on a bill of \$430, at 1½%.

SECTION VII.

PARTNERSHIP.

557. A *partnership* or *company* is an association of persons for the transaction of any business.

A *firm*, *firm-name*, or *house* is the name under which a partnership business is transacted.

558. *Partners* are the persons associated in a partnership or company.

559. Partners are *active* or *silent*, *general* or *special*.

- a.* An *active partner* is one who takes an active part in the management of the business.
- b.* A *silent partner* is one who furnishes capital, and shares in the profits and losses, but takes no active part in the management of the business.
- c.* A *general partner* is one who is responsible for the debts of the company to the amount of his entire property.
- d.* A *special partner* is one whose responsibility is limited to a certain amount specified in the *written articles of partnership*.

Unless specified to the contrary in the written articles of partnership, in common law the partners share equally in the profits and losses of business, without regard to their contributory shares of the capital or stock of the company. Hence, in *drafting articles of partnership*, great care should be taken to state all the details clearly and minutely.

560. Partnerships are *general*, *special*, and *limited*.

- a.* A *general partnership* is one in which the property of the company and the private property of each partner are liable for the debts of the company.
- b.* A *limited partnership* is one having one or more general partners, and one or more special partners. Each special partner is responsible only to the amount of his contributory share of the capital for the debts of the company, in case of failure.
- c.* A *special partnership* is one having one or more general partners, who furnish a stipulated amount of the capital, and are liable for only that amount of the debts of the company *in case of failure*.

The laws relating to limited partnerships, in most of the States, require that

1. The articles of partnership must be in writing, must be signed by all the partners, and must be recorded in a specified public office.
2. There must be at least one general partner.
3. Special partners can take no active part in the management of the business, and their names must not appear in the firm-name.
4. The amount of capital that each special partner contributes must be actually paid in, and notice of the same must be duly advertised.

An omission of any one of these requirements makes the partnership a general partnership.

561. *Capital* or *stock* is the money, or its equivalent, invested in any business.

- a.* *Capital* may be money, real-estate, personal property, time, labor, or skill.
- b.* *Resources* or *assets* are the entire property of a company, including capital and all demands in its favor.
- c.* *Liabilities* or *obligations* are the entire indebtedness of a company, or all demands against it.

d. Net capital or surplus is the excess of resources over liabilities.

e. A deficit is the excess of liabilities over resources.

f. When the resources of a company exceed the liabilities, the company is *solvent*; when the liabilities exceed the resources, the company is *insolvent* or *bankrupt*.

562. *Net gain or loss* is the difference between the total gains and the total losses.

Gains and losses are shared among the partners according to the stipulations and conditions of the articles of partnership. If no stipulations are mentioned, they are divided equally.

563. *Dividends* are the profits divided among the partners of a company.

564. *Assessments* are the sums to be paid by the partners of a company, to meet expenses or cover losses.

565. In partnership business, profits and losses are commonly computed (1) by fractional parts; (2) by percentage; or (3) by proportion,—as shown in the solutions of the three questions in the following

EXAMPLE.—R, T, and W form a partnership for manufacturing window sash, doors, and blinds. In the written articles of partnership it is stipulated that the cash capital shall be \$10,000, of which R shall furnish \$2,500, T \$3,500, and W the remainder; and that the profits and losses shall be shared by the partners severally according to their proportionate shares of the capital. Each partner's share of the whole gain or loss

1. Is what fractional part of the whole gain or loss?
2. Is what per cent of the whole gain or loss?
3. Is what proportional part of the whole gain or loss?

SOLUTIONS.

(1)

R's stock, \$2,500	R's fractional part of stock, $\frac{2500}{10000} = \frac{1}{4}$
T's " 3,500	T's " " " " $\frac{3500}{10000} = \frac{7}{20}$
W's " 4,000	W's " " " " $\frac{4000}{10000} = \frac{2}{5}$
Total " \$10,000	

Hence, of the total gains or losses, R's fractional part is $\frac{1}{4}$, T's fractional part is $\frac{7}{20}$, and W's fractional part is $\frac{2}{5}$.

(2)

R's stock is $\frac{25000}{100000} = \frac{1}{4} = .25 = 25\%$ of entire capital;

T's " " $\frac{35000}{100000} = \frac{7}{20} = .35 = 35\%$ " " "

W's " " $\frac{40000}{100000} = \frac{2}{5} = .40 = 40\%$ " " " Hence,

of the total gains or losses, R's share is 25%, T's share is 35%, and W's share is 40%.

(3)

R's stock is $\frac{25000}{100000} = \frac{1}{4}$ of entire capital;

T's " " $\frac{35000}{100000} = \frac{7}{20}$ " " "

W's " " $\frac{40000}{100000} = \frac{2}{5}$ " " "

Since the entire capital is $(\frac{1}{4} + \frac{7}{20} + \frac{2}{5}) = \frac{11}{10}$, of which R's part is $\frac{1}{4}$, T's $\frac{7}{20}$, and W's $\frac{2}{5}$, the proportionate shares of the total gains or losses of the several partners are as follows:—

R's share of gains or losses : the total gains or losses :: 5 : 20

T's share of gains or losses : the total gains or losses :: 7 : 20

W's share of gains or losses : the total gains or losses :: 8 : 20

566. The methods of computation employed in distributing dividends and apportioning assessments are based upon the following

IMPORTANT FACTS.

Each partner's share of a dividend or assessment

I. Is such a fractional part of } { as his share of the capital is fractional part of the whole capital.

II. Is such a per cent of the } { as his share of the capital is per cent of the whole capital.

III. Is to the whole dividend or } { as his share of the capital is to assessment the whole capital.

PROBLEMS.

1. Brown, Smith, Holt, and Perry were in partnership with a capital as follows: Brown \$41,000, Smith \$58,000, Holt \$93,000, and Perry \$95,000. They gained in one year \$14,568, which they shared in proportion to their shares of the capital. The gain was added to the capital at the end of the year. What was each partner's interest at the beginning of the second year?

2. M, N, P, and Q form a partnership for the manufacture of glass. M's share of the capital is \$58,500, N's share \$62,000, P's \$93,000, and Q's \$98,000. They gain \$14,568. After A receives \$1,800 for managing the business, the profits are divided among the partners in proportion to their shares of the capital, and the net gain is added to the capital. How much is each partner's share of the capital at the beginning of the second year?

3. Jan. 1 three men form a partnership in the milling business. The first invests \$6,500, to which he adds \$800 at the end of 7 months. He withdraws \$650 at the end of 9 months. The second invests \$4,500, and at the end of each 3 months withdraws \$400. The third invests a certain sum, and at the end of 4 months $\frac{2}{3}$ as much more, and from the sum of these investments (\$4,900) he withdraws \$1,800 at the end of the eleventh month. The average monthly profits of the business are \$340, which are shared among the partners at the end of the year according to each one's share of the capital and the length of time it remained in the business. What is each partner's share of the gain?

4. Sept. 1 D, E, and F began the manufacture of wagons with a joint capital of \$45,000, 30% of which was D's, 45% E's, and the remainder F's. If the profits of the business exceed \$12,000 per year, $\frac{1}{3}$ of the excess is to be given to the employes; and if the profits exceed \$15,000 per year, the employes are to receive an additional $\frac{1}{3}$ of the excess over \$15,000. The profits of the business the first year were \$16,570. What was each partner's share in proportion to his investment?

5. A, B, and C owned an iron foundry from which their profits in one year were \$50,464.50. A's capital was \$90,000, $\frac{2}{3}$ of which was equal to $\frac{1}{11}$ of B's and $\frac{1}{3}$ of C's. What were B's and C's shares of the capital? What was each partner's share of the profits divided on the basis of investments?

6. Brooks and Ross began the manufacture of shoes June 1. Brooks's investment was \$9,700 made June 1. Ross's was \$10,300, made July 1. Each man received 5% interest on the sum invested. They gained \$5,640 in one year. How shall the gain be divided between Brooks and Ross, paying Brooks \$750 for services, and Ross \$540?

7. The claims allowed by the administrator of an estate of \$90,000 were, for the widow, $\frac{1}{3}$; for each of two sons, $\frac{1}{3}$; for each of three daughters, $\frac{1}{3}$. After deducting $\frac{1}{4}\%$ of the estate for expenses in settlement, how much should each claimant receive?

8. H, I, and J in company invested \$72,000 in stocks, of which sum J invested \$25,000. H took \$3,400, and I \$2,500 of the profits. How much did H and I invest respectively? and how much is J's gain?

9. August 1 two men entered into partnership in canning and evaporating fruit,—with a paid-up capital of \$13,500. Before beginning business they admitted two other partners, who purchased $\frac{1}{3}$ interest, one paying \$3,500 and the other \$3,000. The firm lost \$1,300, which was shared in proportion to their shares of the capital. How much did each partner lose?

10. Three drovers hired a pasture for a season of six months, for which they paid \$360. They agreed to pay in proportion to the number of animals pastured, allowing 3 sheep to equal 2 calves; 7 calves to equal 2 horses; and 3 cows to equal 5 horses. At the beginning of the time one put in 90 sheep and 17 calves; another 16 horses, 20 calves, and 9 sheep; and the third 25 cows, 4 horses, 18 calves, and 12 sheep. At the end of the 2d month the second man took out 15 calves and 4 sheep, and put in 12 cows. At the end of the 3d month the third man put in 7 calves and took out 19 cows. At the end of the 5th month the first man took out 75 sheep and 13 calves. How much of the rent ought each partner to pay?

11. A, B, and C enter into partnership for five years to manufacture fine shoes, on the following conditions:—Each partner invests \$10,000, and is to receive 6% interest on all sums that he shall have invested from year to year, in excess of the lowest amount remaining in the business to the credit of any one of them; A is to receive \$800 per annum from the profits on account of his superior knowledge of the business; and each partner is entitled to draw annually from the business a sum not to exceed \$1,800; all other profits are to remain in the business and are to be added to the capital. They drew from the business as follows:—

	1st yr.	2d yr.	3d yr.	4th yr.	5th yr.
A	\$1,613.18	\$1,470.10	\$1,775	\$1,581.75	\$1,712.90
B	1,200	1,652.90	1,337.50	1,617.68	1,465.43
C	1,520.85	1,340	1,610.48	1,283.05	1,362.37

Required each partner's interest at the close of the term, the net capital being \$47,214.77.

567. Pupils should now be required to arrange the proper formulæ for all the cases in each subject in Percentage in which per cent is a compound element. The title for these outlines will be **OUTLINES FOR ALL THE FORMULÆ IN PERCENTAGE IN WHICH PER CENT IS A COMPOUND ELEMENT**, and the general form will correspond to that given on page 265.

These two sets of outlines—the one given on page 265, and the one prepared in accordance with the above directions—will form

A COMPLETE CHART OF ALL THE PROCESSES IN PERCENTAGE

SECTION VIII.

REVIEW.

568. PROBLEMS.

1. June 24, 1887, I bought a mill for \$2,240 cash and a 7% mortgage for \$5,760. Jan. 1, 1890, I paid the mortgage. How much was the payment?

2. How much is the interest, in this State, on a mortgage for \$2,750 for 3 yr. 5 mo. 20 da.?

3. An estate that is divided between a widow and her son yields a yearly income, at 5%, to the widow of \$1,325.75, and to the son of \$1,025. What is the value of the estate?

4. The difference between the interest of \$600 and \$750 at 5% is \$18.75. What is the time?

5. A college received, by will, \$50,000 to endow a professorship, the endowment being payable \$12,500 in 6 months, \$20,000 in 1 year, and the remainder in 3 years. How much ready money should a banker pay the college for the endowment, money being worth 4%?

6. Required, the difference between the true and the bank discount on \$200 for 1 yr. 6 mo. 12 da., at 6%.

7. A pension of \$96 per year is 4 years in arrears. Find the amount now due, at 5% compound interest.

8. What investment in Baltimore 6% water bonds bought at 93 $\frac{1}{2}$, brokerage $\frac{1}{2}$ %, will pay the interest of \$5,270 at 7%?

9. A Charleston, S. C., merchant paid \$890, which sum was, the amount of a note for \$825. How long had the note run?

10. A Connecticut farmer mortgaged his farm for \$3,125, and loaned the money, through a Western Loan Association, on real estate in Denver, Colo., for 3 yr. 3 mo. 18 da. How much were his profits?

11. At 5%, what principal will amount to \$1,477.59 from April 8, 1887, to Sept. 5, 1889?

12. Find the equated time for the payment of \$120 due in 8 mo., \$150 due in 1 yr., \$500 due in 1 yr. 3 mo., and \$100 due in 1 yr. 9 mo.

13. 25% of a note that I took in payment for goods was profit, and I sold the note at 10% discount. What per cent net profit did I realize on the goods sold?

14. Memorandum.—Note for \$1,245, dated Springfield, Mass., June 10, 1887. Indorsements: May 10, 1888, \$125; Sept. 22, 1888, \$11.50; July 10, 1889, \$21; Jan. 22, 1890, \$165; Sept. 10, 1891, \$154. Taken up, Jan. 10, 1892. How much was the last payment?

Required,

15. The interest of \$17,650 for 4 yr. 3 mo. 15 da., at 5%.

16. The amount of \$815.87 from May 5, '81, to Apr. 8, '82, at 8%.

17. The rate of interest at which \$800 will gain \$30 in 10 months.

18. The principal that will gain \$42.50 in 3 yr. 4 mo., at 7%.

19. The time in which \$900 will gain \$75 interest in this State.

Find the bank discount on, and the proceeds of

20-23. \$350	} payable in	{ 90 da.	} at	{ 6%.
24-27. \$7,128				

Find the compound interest, at 6%, of

28-31. \$590 } for { 3 yr. 6 mo. } compounded { quarterly.
32-35. \$256 } { 4 yr. 9 mo. } { semi-annually.

36. I have a mortgage for \$5,325 due in 1 yr. 6 mo. 21 da., at 5% simple interest. A banker discounts the mortgage for me at 8%, and I loan the proceeds on real estate security in Kansas, at the highest legal rate of interest. What are my net profits?

37. For a horse, wagon, and harness I am offered \$350 cash or a note for \$370 due in 1 yr. 10 mo. 22 da. Money being worth 7% interest, which is the better offer, and how much the better?

38. What is the present worth of a debt of \$912.48 due in 10½ months, discounted at 8% true discount?

39. At 6% per annum what is the difference between the interest of, and the true discount on \$325.25 for 1 yr. 10 mo. 15 da.?

40. Find the equated time for the payment of \$1,000 cash, \$1,187.50 in 4 mo., \$788.33½ in 9 mo., and \$394.16½ in 1 yr.

41. The proceeds of a note payable in 90 days, at 6%, were \$505. What was the face of the note?

42. B in St. Paul draws on C in Memphis for \$2,500 at 20 days' sight, and a St. Paul bank cashes the draft. Exchange on Memphis being at 2½% discount, what sum does B realize?

43. Find the present worth of a 6% mortgage for \$8,650 due 2 yr. 7 mo. hence, if discounted at 7% true discount.

44. A broker borrows \$500 of a banker for 90 days and pays 1¼% per month. For what sum must the broker give his note?

45. What are the proceeds of a 60-day, bankable note for \$800, discounted at 7%?

46. I owe \$1,000 to-day, and I obtain a discount at a bank for 60 days, at 8%, and pay the debt. For what sum do I draw my note?

47. Find the discount on, and the proceeds of the following note:—

\$340

PROVIDENCE, R. I., Nov. 10, 1889.

Ninety days after date I promise to pay to the order of JAMES R. PARSONS Three Hundred Forty Dollars at the First National Bank in this city, for value received.

WALTER WILSON.

48. What is the cash value of an 8% mortgage for \$616.25 due 5 years hence, at 5% bank discount?

49. When money is worth 10%, I sell a note for \$1,137.22 payable in 8 mo. 9 da., interest after 3 mo. How much do I receive for the note?

50. A banker discounts a claim due in 2 mo. 10 da., and charges 8%. The discount is \$34.73. What is the amount of the claim?

51. In what time will \$500 amount to \$872, at 5% simple interest?

Find the true discount on the following bills of exchange on London, exchange at par:

Face	Drawn	Discounted
52. £250 12s. 6d.	Mar. 6 at 6 mo.,	July 15, at 5%.
53. £47 18s.	Jan. 14 at 4 mo.,	Apr. 1, at 4%.
54. £432 10s. 4d.	Feb. 20 at 90 da.,	Mar. 30, at 3½%.

55. On arriving in New York from a European tour a lady sold to a Nassau St. Banking House the following bill of foreign coins:

4 20-Francis	@ \$3.11	Find the
£6 10s. English coins	@ 4.21	value in U. S.
3 Marks	@ 28¢	money.
1½ Francs	@ 19¢	

56. The books of account of S and W show that

S owes W	W owes S
\$436.70 due Jan. 6;	\$148.37 due Jan. 29;
218.65 " Feb. 2;	173.19 " Feb. 25;
169.28 " Apr. 13;	587.23 " May 7;
321.57 " Aug. 29.	258.— " Sept. 30.

Make a copy of the account as it appears on W's books, average the %, and find the cash balance Nov. 1, at 6% interest.

57. If you buy bills of goods as follows:—Aug. 13, \$123 on 4 mo.; Sept. 23, \$356 on 6 mo.; and Oct. 3, \$270 on 3 mo.;—and give your note dated Nov. 25 for the payment of the sum total at the equated time, and then discount your note at bank at 6%, how much cash will you pay?

Date of note, Oct. 19, 1888; face, \$1,290; rate of interest, 6%. Indorsements,—Jan. 21, 1889, \$25; May 30, 1889, \$250; Sept. 1, 1891, \$575. Required, the sum due, Dec. 5, 1891. (See Supplement, page 447.)

58. By the U. S. Court Rule.

60. By the New Hampshire Rule.

59. By the Vermont Rule.

61. By the Connecticut Rule.

62. Find the annual interest of \$854 for 4 yr. 8 mo. 24 da. at 5%.

63. Find the interest on a U. S. $3\frac{1}{2}\%$ bond of \$5,000 from June 4 to September 28.

64. A clerk deposits \$50 quarterly in a savings-bank, and the rate of interest paid is 4% payable quarterly. How much money has he on deposit at the end of 3 years?

65. What is the exact interest, at 6%, of \$1,275 from Jan. 9 to May 7?

66. Five cents per day is the income of what principal, at 7% per annum?

67-71. In what time will any sum of money double itself at 1% simple interest? At 4%? At 6%? At 7%? At $3\frac{1}{2}\%$?

72. How much longer time will be required for any sum of money to double itself at 5% compound interest than at $7\frac{3}{4}\%$ simple interest?

73. In how much time will I pay a sum equal to the face of a debt, if I pay $\frac{2}{3}\%$ per month?

74. Find the cash balance of the following account, June 12, 1888, interest at 5%:

Darwin Harris, in account with					
Dr.	Sheldon & Sons.				Cr.
1887			1888		
Sept. 12	To mds. 2 mo.	98 75	Jan. 20	By cash	150 —
Dec. 18	" "	110 40	Mar. 12	" "	200 —
1888			May 28	" "	175 —
Jan. 24	" " 1 mo.	206 25			
Apr. 4	" " 2 mo.	184 60			

75. A contractor paints a church for \$700 payable in 7 equal semi-annual instalments, interest at 6% included. How much is each instalment?

76. What is the present value of the claim in problem 50, if discounted at 10% per annum true discount?

77. If from the age of 14 to 18 years you save 20¢ per week, and from the age of 18 to 21 years 45¢ per week, and deposit your money in a savings-bank that allows $3\frac{1}{2}\%$ interest, payable quarterly, how much money will you have on deposit when you are 21 years old?

78. My salary of \$50 per month remains unpaid at the end of the year. How much is due to me, if I receive 6% interest on each month's salary after it is due?

79. A 6% mortgage for \$500, dated Oct. 15, 1887, was sold Apr. 15, 1889, for a sum that netted the seller 8%. How much was paid for the mortgage?

80. I sold a city lot of 25 feet front at \$250 per front foot,—terms, one fourth cash and the remainder in 3 equal annual payments, with interest at 5%, secured by mortgage. The same day I sold the mortgage at 6% true discount. How much cash did I receive for the property?

81. R loans \$2,000 for $2\frac{1}{2}$ years at $8\frac{1}{2}\%$, and W loans an equal sum at 8%, but collects his interest semi-annually, and loans that also at 8%. Which person has the greater amount, principal and interest, at the end of the $2\frac{1}{2}$ years, and how much the greater?

82. I take a lease of a store for 8 years at \$650 per year, and pay the amount in advance at a true discount of 7%. At the same time I deposit, in a savings-bank, a sum the compound interest of which, at 4%, interest payable quarterly, will, at the end of the 8 years, equal the present cash value of the lease. What sum do I deposit?

83. A tradesman whose house-rent is \$360 per year, buys the house for \$3,750, agreeing to pay \$360 at the end of each year, and to allow 5% interest on each unpaid balance. In what time will he pay for the house?

84. Allowing 6% compound interest, what is the present worth of a paid-up lease that has 4 years to run, if the property rents for \$2,000 a year?

85. Memorandum.—Burlington, Iowa, May 25, 1888. Bought city lots for \$3,250, and paid cash, \$800. Nov. 10, paid \$500; Aug. 26, 1889, paid \$750. How much did I owe June 18, 1890?

86. Memorandum.—Fort Wayne, Ind., July 17, 1884, I gave my note for \$1,675 with interest. I paid \$200 Dec. 11; \$50 June 21, 1885; \$50 Sept. 3, 1885; \$175 April 14, 1886; \$175 Nov. 1, 1886; \$30 Feb. 15, 1888; \$125 June 13, 1888; and \$290 Oct. 25, 1888. How much was due March 28, 1889?

87. Memorandum.—Face of mortgage, \$4,500. Dated at Fremont, Neb., Jan. 15, 1885. Indorsement, June 1, 1888, \$2,000. Paid in full, Jan. 2, 1890. How much was the final payment?

88. July 10, 1887. Edward Haynes of Denver, Colo., borrowed of Robt. St. Clair of the same city \$2,000, giving his note "on demand" at $1\frac{1}{2}\%$ per month interest. Aug. 17, '89, H. paid \$700, and Dec. 2 he paid \$75. Write the note, make the indorsements, and compute the amount due May 15, 1890.

89. A public highway 200 rods long and 4 rods wide is laid across a farm, and the land is appraised at \$62.50 per acre. It costs the owner \$125 to fence the road, and the fence must be renewed every 15 years. Money being worth 6% compound interest, what damages should the farmer receive?

569. TEST QUESTIONS.

Pupils should now be required to prepare, in sets of ten each, thirty test questions on that part of Percentage in which the per cent is a simple element, thirty questions in which the per cent is a compound element, and twenty questions covering the contents of the entire Chapter.

570. BLACKBOARD OUTLINES.

Pupils should be required to prepare complete outlines for review of the entire subject of Percentage.

CHAPTER IX.

MEASUREMENTS.

569. Mensuration is the process of determining the number of units of extension in any magnitude, by measurement or by computation.

570. IMPORTANT FACT.—*The mathematical dimensions of surfaces and solids extend at right angles to each other.*

571. In business transactions, quantities and values are associated with certain units suited to the various departments of business.

In estimating materials and labor the following units are used:—

1. **Square foot**,—for stone-cutting, flagging, lumber, and masonry.
2. **Board foot**, *i. e.*, $\frac{1}{12}$ of a cubic foot,—for sawed timber, and for lumber more than 1 inch thick.
3. **Square yard**,—for painting, plastering, ceiling, paving, bricklaying, and masonry.
4. **Square**, *i. e.*, 100 square feet,—for flooring, roofing, and bricklaying.
5. **Bunch** of lath,—100 pieces, each 4 feet long.
 - a. In estimating *matched* lumber $\frac{1}{4}$ is added to the surface measure.
 - b. Shingles are estimated to average 4 inches wide across the butt. A shingle is said to be “laid to the weather” the length of that portion not covered by the next succeeding course of shingles on the roof.
 - c. 1,000 shingles are estimated to cover a *square*.
 - d. In estimating brick walls by surface measure, they are supposed to be $1\frac{1}{2}$ bricks thick.
 - e. A bunch of lath is estimated to cover 5 square yards.
6. **Cubic foot**,—for bricklaying, masonry, and hewn or round timber.
7. **Cubic yard**,—for heavy masonry, excavations, and embankments.
8. **Brick**,—of different sizes, as follows:

Common brick,	$8 \times 4 \times 2$ in.	Philadelphia brick	} $8\frac{1}{4} \times 4\frac{1}{8} \times 2\frac{3}{8}$ in.
Maine “	$7\frac{1}{2} \times 3\frac{3}{8} \times 2\frac{3}{8}$ “	Baltimore “	
North River “	$8 \times 3\frac{1}{2} \times 2\frac{1}{2}$ “	Milwaukee “	

f. On public works, a cubic yard of earth is a *standard load*.

- g.* In computing the number of bricks in a wall, the dimensions of the brick used must be increased by the thickness of the layer of mortar.
- h.* In computing the cubic contents of walls of buildings and of foundations, the length of each wall is measured on the outside. Each corner is thus measured twice.
- i.* In computing the capacity of bins, a cubic foot is either .8 of a bushel even measure, or .64 of a bushel heaped measure.
- j.* In computing the contents of mows and stacks, 450 cubic feet of hay (or if well settled 270 cubic feet) are called a ton.

572. In computing painting, plastering, ceiling, bricklaying, and masonry, the following deductions are usually made:—

- 1. *Materials*,—openings in walls—*i. e.*, doors and windows.
- 2. *Labor*,— $\frac{1}{2}$ the openings and $\frac{1}{2}$ the corners.

Materials are commonly estimated by solid measure, and the work by either solid or surface measure.

SECTION I.

LINES.

573. Define,—1. A line (**103**). 2. A straight line (**103, a**). 3. A curved line, or curve (**103, b**).

All mathematical lines are imaginary lines.

1. A *straight line* never changes its direction; hence, to produce a *straight line* is to extend or continue it in the same direction.

2. A *uniform curve* is one that changes its direction regularly;

A *varying curve* is one that changes its direction irregularly.

a. Parallel lines are lines that are in the same plane, extend in the same direction, and never meet however far produced.

b. Oblique lines are lines that lie in the same plane, extend in different directions, and must meet if sufficiently produced.

c. Horizontal lines are lines parallel to the plane of the horizon.

d. Vertical lines are parts of radii of the earth or of radii produced.

e. A line is perpendicular to another line when its direction is at right angles to the direction of the other line.

f. Lines perpendicular to the plane of the horizon are commonly regarded as vertical lines.

g. The extent of a line taken alone is always measured,—never computed.

EXERCISES.

Draw a representation of

- | | | |
|---------------------|-----------------------|------------------------------|
| 1. A straight line. | 4. Oblique lines. | 7. A uniform curved line. |
| 2. Parallel lines. | 5. A horizontal line. | 8. A varying curved line. |
| 3. A vertical line. | 6. A broken line. | 9. A straight line produced. |
10. A line perpendicular to an oblique line.
11. Three lines parallel to one another and oblique to a horizontal line.
12. Show that vertical lines are never parallel to each other.

574. The units of extension used in measuring lines are the *mile*, *rod*, *yard*, *foot*, *inch*, *chain*, and *link*; also the *meter*, *hand*, *fathom*, and *knot* or *geographic mile*.

SECTION II.

ANGLES.

575. Define,—1. An angle (**114**). 2. A right angle (**116**).

a. An **obtuse angle** is one that is greater than a right angle.

b. An **acute angle** is one that is less than a right angle.

Obtuse and acute angles are also called **oblique angles**.

c. A **plane angle** is one formed by the meeting of two straight lines.

Plane angles only are considered in this work.

d. Angles are measured by **degrees**,—a degree being one of the 360 equal spaces about a point in a plane.

An angle of 90° is $\frac{90}{360}$, or one fourth, of the space about a point in a plane; and an angle of 60° is $\frac{60}{360}$, or $\frac{1}{6}$, of such space.

EXERCISES.

Draw a representation

Show

- | | |
|---------------------------------|--|
| 1. Of a right angle. | 7. That a right angle may be divided into two or more acute angles. |
| 2. Of an acute angle. | 8. That an obtuse angle may be divided into a right angle and an acute angle. |
| 3. Of an obtuse angle. | 9. That an angle of 40° added to an angle of 50° equals an angle of 90° . |
| 4. Of an angle of 45° . | |
| 5. Of an angle of 30° . | |
| 6. Of an angle of 120° . | |

SECTION III.

SURFACES.

576. Define a plane surface, or plane (113).

A surface or area has two dimensions, length and breadth.

A plane surface does not change its line of direction between any of its points.

577. A *curved surface* is one that changes its line of direction constantly between some or all of its points.

- a.* The lateral surfaces of cylinders and cones, and the surfaces of spheres, are curved surfaces.
- b.* Land is commonly treated as portions of a plane surface, in measuring its dimensions or in computing its area.
- c.* The areas of surfaces are computed,—not measured.
- d.* The units of extension applied to area are the *township*, *section*, *acre*, *square chain*, and *square link*; *square mile*, *square rod*, *square yard*, *square foot*, and *square inch*; *square meter*, *ar*, and *hectar*; and the *square*, *t. c.*, 100 sq. ft.

578. A *plane figure* is a plane surface or area bounded by lines or curves, or by a combination of lines and curves.

I. RECTILINEAR FIGURES.

579. A *polygon* is a plane figure bounded by straight lines.

580. A *regular polygon* is one that has all its sides equal.

- a.* All the angles of a regular polygon are equal.
- b.* The *perimeter* of a polygon is the sum of all its sides (*i. e.*, the distance around it).

581. A *triangle* is a polygon of three sides.

- a.* A *right-angled triangle* is one that has one right angle.
- b.* An *obtuse-angled triangle* is one that has one obtuse angle.
- c.* An *acute-angled triangle* is one that has three acute angles.
- d.* An *equiangular triangle* is one that has all its angles equal.
- e.* An *equilateral triangle* is one that has all its sides equal.
- f.* An *isosceles triangle* is one that has two of its sides equal.
- g.* A *scalene triangle* is one that has no two of its sides equal.
- h.* The *base* of a triangle is the side upon which it is supposed to stand.
- i.* The *vertex* of a triangle is the angle opposite the base.

j. The **altitude** of a triangle is the shortest distance between the line of the base and the angle opposite the base. It is always perpendicular to the line of the base.

k. The **perpendicular of a right-angled triangle** is the side that joins the base at the right angle; and

l. The **hypotenuse** is the side opposite the right angle.

In right-angled triangles either side about the right angle may be taken for the base. In all other triangles any side may be regarded as the base.

m. Equiangular triangles are equilateral, and vice versa.

NOTE.—Altitude is a dimension; slant height is not. Distinguish clearly between them.

EXERCISES.

Draw, and indicate the base

- | | |
|----------------------------------|---|
| 1. Of a right-angled triangle. | 5. Of an obtuse-angled triangle having one of its shorter sides for a base. |
| 2. Of an isosceles triangle. | 6. Of an equilateral triangle. |
| 3. Of an obtuse-angled triangle. | 7. Of an acute-angled triangle. |
| 4. Of a scalene triangle. | |

8. To each of these figures, drawn as above required, add a line representing its altitude.

582. A **quadrilateral**, or **quadrangle**, is a polygon of four sides.

a. A **parallelogram** is a quadrilateral whose opposite sides are equal and parallel.

b. A **rectangle** is a right-angled parallelogram.

c. A **square** is an equilateral rectangle.

d. A **rhomboid** is an oblique-angled parallelogram.

e. A **rhombus** is an equilateral rhomboid.

f. A **trapezoid** is a quadrilateral that has two of its sides parallel and the other two oblique.

g. A **trapezium** is a quadrilateral that has no two of its sides parallel.

h. The **diagonal** of a quadrilateral is a straight line joining opposite angles; it divides the quadrilateral into two triangles.

i. The **altitude** of a parallelogram or of a trapezoid is the shortest distance between two parallel sides, one of which is taken for the base. It is always perpendicular to the base. (See 581, Note.)

583. A polygon	is called	A polygon	is called
Of five sides	a pentagon.	Of eight sides	an octagon.
Of six sides	a hexagon.	Of nine sides	a nonagon.
Of seven sides	a heptagon.	Of ten sides	a decagon.

EXERCISES.

Draw, and indicate its altitude,

- | | | |
|---------------------|------------------|----------------------------|
| 9. A parallelogram. | 12. A square. | 15. A rectangle. |
| 10. A trapezoid. | 13. A trapezium. | 16. An irregular pentagon. |
| 11. A rhombus. | 14. A rhomboid. | 17. A regular hexagon. |

CASE I. To compute a required side of a right-angled triangle, the other two sides being given.

All the Important Facts in this chapter are deduced from geometric demonstrations.

584. IMPORTANT FACT I.—*The square of the hypotenuse of a right-angled triangle equals the sum of the squares of the other sides.*

(See Supplement, page 419.)

Ex. 1. Required the hypotenuse of a right-angled triangle whose base is 24 feet and perpendicular 18 feet.

EXPLANATION.—Considering the numbers expressing the lengths of the given sides as abstract, I square 24 the base, and 18 the perpendicular, add the results, and obtain 900, the sum of the squares on the given sides.

I then extract the square root of this sum, affix the denomination common to the base and perpendicular, and obtain 30 ft., the hypotenuse required.

PROCESS.

$$24^2 + 18^2 = 900$$

$$\sqrt{900} = 30$$

Hence, 30 feet.

Ex. 2. The base of a right-angled triangle is 35 rods, and the hypotenuse 37 rods. What is the perpendicular?

EXPLANATION.—Considering the numbers expressing the lengths of the given sides as abstract, I find the difference of the squares of the given sides, which is 144.

I then extract the square root of this difference, affix the denomination common to the given sides, and obtain 12 rods, the perpendicular.

PROCESS.

$$37^2 - 35^2 = 144$$

$$\sqrt{144} = 12$$

Hence, 12 rods.

PROBLEMS.

Given, of a right-angled triangle,

to find

- | | |
|---|--------------------|
| 1. The hypotenuse 30 ft., and the base 18 ft., | the perpendicular. |
| 2. The perpendicular 36 ch., and the base 48 ch., | the hypotenuse. |
| 3. The base 12 ft., and the hypotenuse 20 ft., | the perpendicular. |
| 4. The perpendicular 143 rd., and the hypotenuse 145 rd., | the base. |

5. The base is 27 in., and the perpendicular 33 in.; find the hypotenuse.
6. The hypotenuse is 320 rd., and the perpendicular 200 rd.; find the base.
7. The gable of a building is 28 feet wide, and the ridge of the roof is 9 ft. 4 in. above the sides of the building. If each side of the roof projects 18 inches, what is the width of the roof?
8. From the corner where my front fence is joined by a side fence, I measure along the front fence 12 feet and along the side fence 5 feet; the points thus obtained are 13 feet apart. Is the corner what is known as a square corner?

PROPOSITION.—*Prove that the diagonal of a parallelogram divides the parallelogram into two equal triangles.*

CASE II. Areas and dimensions of parallelograms.

585. Ex. 1. Find the area of a field 48 rods long and 36 rods wide.

ANALYSIS.—Since the field is 48 rods long and 36 rods wide, its area consists of 36 strips of land each 1 rod wide, 48 rods long, and containing 48 sq. rd. Hence, 36 times 48 sq. rd. (equals 1,728 sq. rd. = 10 A. 128 sq. rd.) is the area of the field.

PROCESS.

$$36 \times 48 \text{ sq. rd.} = 1,728 \text{ sq. rd.}$$

$$1,728 \text{ sq. rd.} = 10 \text{ A. } 128 \text{ sq. rd.}$$

Ex. 2. The area of a rectangular field is 5 A. 40 sq. rd., and its length is 35 rods. Required its width.

ANALYSIS.—5 A. 40 sq. rd. equals 840 sq. rd. Since the field consists of strips of land each 1 rod wide, 35 rods long, and containing 35 sq. rd., there are as many rods in the width of the field as there are such strips.

PROCESS.

$$5 \text{ A. } 40 \text{ sq. rd.} = 840 \text{ sq. rd.}$$

$$840 \text{ sq. rd.} \div 35 \text{ sq. rd.} = 40$$

Hence, 40 rods.

Hence, the quotient (40) of the area of the field (840 sq. rd.) divided by the area of each strip (35 sq. rd.) is the number of rods in the width of the field.

Ex. 3. A rhomboid 18 ft. long and 9 ft. 6 in. wide contains how many square feet?

PROCESS.

OUTLINES OF ANALYSIS.—(1) The width (i. e., the altitude) of the rhomboid reduced to feet,—that the dimensions may be expressed in a common unit.

$$(1) \quad 9 \text{ ft. } 6 \text{ in.} = 9\frac{1}{2} \text{ ft.}$$

$$(2) \quad 9\frac{1}{2} \times 18 \text{ sq. ft.} = 171 \text{ sq. ft.}$$

(2) $9\frac{1}{2}$ strips 1 foot wide and 18 feet long,—hence, $9\frac{1}{2}$ times 18 sq. ft.

Result—171 sq. ft., the superficial contents required.

PROBLEMS.

Find the area of a rectangle whose dimensions are

9. 17 rd. and 15 rd. | 12. 63.5 mi. and 27.25 mi. | 15. 38.5×76 ft.
 10. 119 ft. and 68 ft. | 13. 12 ch. and 5 ch. 42 l. | 16. 1.375×14.6 mi.
 11. $7\frac{1}{2}$ in. by $18\frac{1}{2}$ ft. | 14. 37 by 75.7 rd. | 17. 8 by 12 in. | 18. 13 ft. by 11 in.

What is the area of a rhomboid

19. Whose base is 137 rods and altitude 75 rods?
 20. Whose altitude is 93.785 rods, and base 3 rd. 11 ft. 2 in.?
 21. Whose base is $23\frac{1}{2}$ rods, and altitude 19.28 yards?

What is the other dimension

22. Of a rectangle $37\frac{1}{2}$ feet long, and containing 975 sq. ft.?
 23. Of a rectangle 56 rods wide, whose area is 80.4 acres?
 24. Of a rhomboid 415 feet long, and containing 513 sq. yd.
 25. Of a rectangle 115.75 rd. long, whose area is 320 acres?
 26. Of an oblique-angled parallelogram 68.5 feet wide, and containing 72.2 square rods?

CASE III. Areas and dimensions of triangles.

586. IMPORTANT FACT II.—*The area of a triangle equals one half the area of a rectangle of equal base and equal altitude.* (See 584, PROP.)

- Ex. 1. The altitude of a triangle is 29 inches, and the base is 14 inches. Required the area.

PROCESS.

- (1) 29×14 sq. in. = 406 sq. in.
 (2) 406 sq. in. $\div 2$ = 203 sq. in.

- OUTLINES OF ANALYSIS.—(1) Area of rectangle of equal altitude and base.
 (2) One half of area of such rectangle.

- Ex. 2. The area of a triangle is 1 sq. yd. 4 sq. ft. 80 sq. in., and the altitude is 5 ft. 4 in. What is the length of its base?

PROCESS.

- (1) 1 sq. yd. 4 sq. ft. 80 sq. in. = 1,952 sq. in.
 (2) $2 \times 1,952$ sq. in. = 3,904 sq. in.
 (3) 5 ft. 4 in. = 64 in.
 (4) $3,904$ sq. in. $\div 64$ sq. in. = 61
 Hence, 61 in. = 5 ft. 1 in.

OUTLINES OF ANALYSIS.

- (1) Area of triangle reduced to one denomination.
- (2) Area of rectangle of equal base and altitude.
- (3) Length of altitude reduced to units of length corresponding to unit of area.
- (4) Base of rectangle, hence, of the triangle.

Ex. 3. What is the area of a triangle whose three sides are respectively 8 rods, 7 rods, and 5 rods?

NOTE.—The process here given does not admit of arithmetical demonstration.

EXPLANATION.—Considering the given numbers as abstract, I first find one half the sum of the three sides—which is 10.

I next subtract each side separately from one half of the sum of the three sides, and obtain the three remainders 2, 3, and 5.

I then find the continued product of the half-sum 10, and the three remainders 2, 3, and 5.

I now extract the square root of 300, the product found; supply the unit of area corresponding to the common unit of the sides; and I have 17.32 + sq. rd., the area required.

PROCESS.

$$\frac{8 + 7 + 5}{2} = 10$$

$$10 - 8 = 2$$

$$10 - 7 = 3$$

$$10 - 5 = 5$$

$$10 \times 2 \times 3 \times 5 = 300$$

$$\sqrt{300} = 17.32 + ;$$

hence, 17.32 + sq. rd.

PROBLEMS.

Find the area of a triangle

the base being	and the altitude	the altitude being	and the base
27. $48\frac{1}{2}$ feet	60 feet.	30. 6 ch. 40 l.	9 ch. 37 l.
28. 38 inches	4 ft. 4 in.	31. $98\frac{1}{2}$ rods	$116\frac{1}{2}$ rods.
29. 9.25 yards	16 ft. 8 in.	32. 13 rods	5 rd. 6 ft. 9 in.

the three sides being	$\left\{ \begin{array}{l} 33. 24 \text{ ft., } 39 \text{ ft., and } 57 \text{ ft.} \\ 34. 27 \text{ rd., } 63 \text{ rd., and } 11 \text{ ch. } 7 \text{ l.} \\ 35. 8 \text{ yd., } 11 \text{ yd., and } 13 \text{ yd.} \end{array} \right.$
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Find the altitude of a triangle
the base being and the area

36. 62 rods	279 sq. rd.
37. 2 yd. 2 ft. 9 in.	$117\frac{1}{2}$ sq. yd.
38. 19 ch. 5 l.	136 sq. ch.
39. 9.75 feet	86.2 sq. ft.

Find the base of a triangle
the area being and the altitude

40. 126 sq. in.	4 feet.
41. 97.38 sq. ch.	15 ch. 19 l.
42. 47 sq. rd.	19 ft. 6 in.
43. 17 A. 29.8 sq. rd.	12 ch. $44\frac{1}{2}$ l.

44. A triangular field whose base is 50 rods and altitude $48\frac{1}{2}$ rods produced 196 bushels of wheat. What was the yield per acre?

45. The three sides of a triangle are 17 ch., 22 ch., and 31 ch. If the longest side is taken as the base, what is the altitude?

46. A triangular piece of land contains $38\frac{1}{2}$ acres, and the length of one side is 24 chains. What is the shortest distance between the given side and the opposite corner?

CASE IV. Areas and dimensions of trapezoids.

587. The *average length* of a trapezoid is one half the sum of its parallel sides.

IMPORTANT FACT III.—*The area of a trapezoid equals the area of a rectangle whose altitude equals the altitude of the trapezoid, and whose base is the average length of the trapezoid.*

Ex. 1. Find the area of a trapezoid whose parallel sides are 38 ft. and 32 ft., and whose altitude is 26 ft.

PROCESS.

OUTLINES OF ANALYSIS.—(1) One half the sum of the parallel sides equals the average width.

$$(1) \frac{38 \text{ ft.} + 32 \text{ ft.}}{2} = 35 \text{ ft.}$$

(2) Area of rectangle whose altitude is that of the trapezoid, and whose base is its average width.

$$(2) 26 \times 35 \text{ sq. ft.} = 910 \text{ sq. ft.}$$

Ex. 2. Find the altitude of a trapezoid whose area is 12 sq. ft. 72 sq. in., and whose parallel sides are 6 ft. and 4 ft. 6 in.

PROCESS.

$$12 \text{ sq. ft. } 72 \text{ sq. in.} = 1,800 \text{ sq. in.}; 6 \text{ ft.} = 72 \text{ in.}; 4 \text{ ft. } 6 \text{ in.} = 54 \text{ in.}$$

$$\frac{72 \text{ in.} + 54 \text{ in.}}{2} = 60 \text{ in., the average width}; 1,800 \text{ sq. in.} \div 60 \text{ sq. in.} = 30;$$

hence, 30 in. or 2 ft. 6 in., the altitude required.

Ex. 3. The area of a trapezoid is 988 sq. rd., its altitude is 26 rd., and one of its parallel sides is 32 rd. What is the length of the other parallel side?

PROCESS.

$$988 \text{ sq. rd.} \div 26 \text{ sq. rd.} = 38, \text{—hence } 38 \text{ rd. average width};$$

$$38 \text{ rd.} - 32 \text{ rd.} = 6 \text{ rd., the difference between length of each side and average width};$$

$$38 \text{ rd.} + 6 \text{ rd.} = 44 \text{ rd., length of side required.}$$

PROBLEMS.

Find the areas of trapezoids of the following measurements:—

Parallel sides	Altitude	Parallel sides	Altitude
47. 18 in., and 42 in.;	23 inches.	51. $2\frac{1}{2}$ yd., and 16 rd.;	96 rd.
48. $22\frac{1}{2}$ ft., and 19 ft. 3 in.;	16 ft.	52. 2 ft. 7 in., and 13 ft.;	$5\frac{1}{2}$ rd.
49. 28.5 mi., and $7\frac{1}{2}$ mi.;	1 mi. 73 rd.	53. $17\frac{1}{2}$ l., and 46 l.;	21 ft. 11 in.
50. 29 yd., and 17.65 yd.;	35 ft. 7 in.	54. 60.125 rd.; and $30\frac{3}{4}$ rd.;	$\frac{1}{2}$ mi.

55. What is the altitude of a trapezoid whose area is 1,056 sq. rd., the parallel sides being 54 rd. and 42 rd.?

56. How many square feet are there in a wany board 16 ft. long and 11 in. average width?

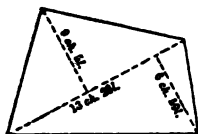
57. The area of a board is 12 sq. ft., its length is 14 ft., and one end is 13 inches wide. How wide is the other end of the board?

58. A field has two parallel sides, one 11 ch. 38 l., the other 14 ch. 29 l., and they are 13 ch. 85 l. apart. What is the value of the land at \$46 per acre?

CASE V. Areas of trapeziums.

588. Ex. 1. Find the area of a trapezium whose diagonal is 13 ch. 28 l. and the altitudes of the triangles, with a diagonal as a base, are 9 ch. 5 l. and 6 ch. 59 l.

PROCESS.



$$(1) \frac{13.28 \times 9.05 \text{ sq. ch.}}{2} = 60.092 \text{ sq. ch.}$$

$$(2) \frac{13.28 \times 6.59 \text{ sq. ch.}}{2} = 43.757 + \text{sq. ch.}$$

$$(3) \text{Sum, } 104.85 - \text{sq. ch.} = 104.85 - A.$$

OUTLINES OF ANALYSIS.

The diagonal divides the trapezium into two triangles (582, A).

(1) The area of one triangle.

(2) The area of the other triangle.

(3) The sum of the triangular areas; hence, the area of the trapezium.

Ex. 2. The four sides of a trapezium in consecutive order are 48 rd., 32 rd., 27 rd., and 21 rd., and the diagonal, separating the first and second sides from the third and fourth, is 36 rd. Required the area.

PROCESS.

(1)	(2)
$\frac{48 + 32 + 36}{2} = 58$	$\frac{27 + 21 + 36}{2} = 42$
$58 - 48 = 10$	$42 - 27 = 15$
$58 - 32 = 26$	$42 - 21 = 21$
$58 - 36 = 22$	$42 - 36 = 6$
$58 \times 10 \times 26 \times 22 = 331,760$	$42 \times 15 \times 21 \times 6 = 79,380$
$\sqrt{331,760} = 575.996 +$	$\sqrt{79,380} = 281.567 +$
(3) $575.996 + 281.567 = 857.56 +$; hence, $857.56 +$ sq. rd.	

OUTLINES OF ANALYSIS.

The diagonal divides the trapezium into two triangles, and forms a side of each.

(1) Area of triangle whose sides are 48 rd., 32 rd., and 36 rd.

(2) Area of triangle whose sides are 27 rd., 21 rd., and 36 rd.

(3) Sum of areas of triangles.

PROBLEMS.

59. The diagonal of a trapezium is 62 rd. 9 ft., and the altitudes of the resulting triangles are 27 rd. 4 ft. and 38 rd. 3 ft. Find the area.

60. The diagonal of a trapezium, and the four sides in consecutive order, beginning at one extremity of the diagonal, are respectively 18 ch. 72 l., 16 ch. 14 l., 12 ch. 2 l., 11 ch. 19 l., and 13 ch. What is the area?

II. CURVILINEAR FIGURES.

589. Define — 1. A circle (122). 2. A circumference (122, *a*).
3. A diameter (122, *c*). 4. A radius (122, *d*).

NOTE.—An *astronomical circle* is a mere circumference, and does not imply area.

a. Astronomical circles are divided into 360 equal parts called degrees, and are used in measuring angles. When so used the point about which the angular space is taken is the centre about which the circumference is drawn.

b. Meridian circles and parallels of latitude, either celestial or terrestrial, are astronomical circles.

590. An *ellipse* is a plane bounded by a regularly varying curved line the sum of the distances from every point of which to two fixed points about which the boundary curve is described, equals the longest distance across the ellipse.

- a. The *major axis* of an ellipse is the longest straight line that can be drawn in it.
- b. The *minor axis* of an ellipse is a straight line perpendicular to the major axis at its middle point, and terminating at both ends in the boundary curve.
- c. The *foci* of an ellipse are the two points in the major axis about which the boundary curve is described.

NOTE.—The boundary curve of an ellipse is readily drawn by fastening a tack or pin securely in each of the foci, and fastening to the tacks or pins thus placed a cord as long as the major axis; a pencil is held firmly against this cord and moved round, thus describing the required curve.

EXERCISES.

- | | | | |
|------|---|------------------------------|-------------------------------------|
| Draw | { | 1. A circle. | 4. An ellipse, and locate the foci. |
| | | 2. A diameter of the circle. | 5. The major axis of the ellipse. |
| | | 3. A radius of the circle. | 6. The minor axis of the ellipse. |

Using a circumference as an astronomical circle, show

- 7. That a right angle is an angle of 90° , and is measured by one fourth of the circle.
- 8. That an acute angle is an angle of less than 90° .
- 9. That an obtuse angle is an angle of more than 90° .
- 10. That an angle of 45° is one half of a right angle.
- 11. The relative circular space occupied by each of the zones of the earth.

CASE I. Radii, diameters, and circumferences of circles.

591. IMPORTANT FACT IV.—*The circumference of a circle is 3.1416 times its diameter.*

For circles less than 100 feet in diameter, it is sufficiently accurate to estimate the circumference as $8\frac{1}{4}$ times the diameter.

Ex. 1. Find the circumference of a circle whose diameter is 46 rods.

EXPLANATION.—I multiply the diameter 46 rd. by 8.1416, the ratio of the circumference to the diameter, and obtain 144.51+ rd.; or

PROCESSES.

1. $8.1416 \times 46 \text{ rd.} = 144.51 + \text{rd.};$ or
2. $46 \text{ rd.} \div .3183 = 144.51 + \text{rd.}$

I divide 46 rd., the diameter, by .3183—the reciprocal of 8.1416 (266, VI)—and obtain 144.51+ rd., the circumference required.

Ex. 2. Required the circumference of a log 4 ft. 8 in. in diameter.

PROCESS.

$$4 \text{ ft. } 8 \text{ in.} = 56 \text{ in.}; \text{ and } 3\frac{1}{4} \times 56 \text{ in.} = 176 \text{ in.} = 14 \text{ ft. } 8 \text{ in.}$$

Ex. 3. Required the diameter of a circle whose circumference is 143.7 rods.

PROCESSES.

1. $143.7 \text{ rd.} \div 8.1416 = 45.73 + \text{rd.};$ or
2. $.3183 \times 143.7 \text{ rd.} = 45.73 + \text{rd.}$

Ex. 4. The circumference of a tree is 9 ft. 3 in. Find the diameter.

PROCESS.

$$9 \text{ ft. } 3 \text{ in.} = 111 \text{ in.}; \text{ and } 111 \text{ in.} \div 3\frac{1}{4} = 35.3 + \text{in.} = 2 \text{ ft. } 11.3 + \text{in.}$$

Ex. 5. Find the radius of a circle 138 ft. in circumference.

PROCESS.

$$138 \text{ ft.} \div 3\frac{1}{4} = 45.272 + \text{ft.}; \text{ and } 45.272 + \text{ft.} \div 2 = 22.636 + \text{ft.} = 22 \text{ ft. } 7.6 + \text{in.}$$

PROBLEMS.

Given

to find

- | | |
|--|--------------------|
| 61. The diameter of a circle, 20 feet, | the circumference. |
| 62. The radius of a circle, 15 inches, | the circumference. |
| 63. The circumference of a circle, $75\frac{1}{2}$ rods, | the diameter. |
| 64. The radius of a circle, 13.73 feet, | the diameter. |
| 65. The circumference, 100 miles, | the radius. |
| 66. The diameter, 43 ch. 76 l., | the circumference. |
| 67. The radius, 27 rd. 1 ft. $7\frac{1}{2}$ in., | the circumference. |
| 68. The circumference, 107 yd. 1 ft. 11 in., | the diameter. |

69. Find the diameter of a circular cistern whose circumference is 15 ft. 9 in.

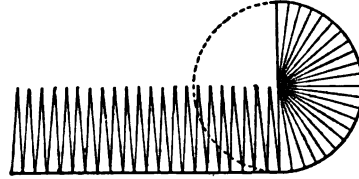
70. What is the circumference of a wheel 3 ft. 8 in. in diameter?

71. Find the radius of a globe whose circumference is 88 inches.

72. What is the distance around a circular race track 840 ft. 4 in. in diameter?

CASE II. Areas of circles.

592. Any circle may be supposed to consist of a great number of triangles whose common altitude is the radius of the circle, and the sum of whose bases is the circumference of the circle.



593. The total area of any number of triangles having a common altitude may be computed by multiplying the sum of their bases by the common altitude and dividing the product by 2. Hence,

IMPORTANT FACT V.—*The number of units in the area of a circle equals one half the product of the numbers expressing the radius and the circumference.*

594. And since the radius is one half the diameter,

IMPORTANT FACT VI.—*The number of units in the area of a circle equals one fourth the product of the numbers expressing the diameter and the circumference.*

595. Since the circumference equals 3.1416 times the diameter,

$$\frac{\text{diam.} \times \text{circum.}}{4} = \frac{\text{diam.} \times (\text{diam.} \times 3.1416)}{4} = \frac{\text{diam.}^2 \times 3.1416}{4} = \text{diam.}^2 \times .7854$$

IMPORTANT FACT VII.—*The number of units in the area of a circle equals .7854 of the square of the number expressing its diameter.*

596. Since the diameter equals .3183 times the circumference,

$$\frac{\text{circum.} \times \text{diam.}}{4} = \frac{\text{circum.} \times (\text{circum.} \times .3183)}{4} = \frac{\text{circum.}^2 \times .3183}{4} = \text{circum.}^2 \times .07968$$

IMPORTANT FACT VIII.—*The number of units in the area of a circle equals .07958 of the square of the number expressing its circumference.*

Ex. 1. Find the area of a circle whose radius is 8 feet.

STEPS IN PROCESSES.

1st Process, — (1) Circumference computed from radius.

(2) One half of product of radius and circumference.

2d Process, — (1) Diameter computed from radius.

(2) .7854 time square of diameter.

The difference of the results by the two processes is about $\frac{1}{12}$ of a square foot, the second result being more nearly accurate.

PROCESSES.

$$\text{1st, — (1) } 3\frac{1}{2} \times 2 \times 8 \text{ ft.} = 50\frac{1}{2} \text{ ft.}$$

$$(2) \quad \frac{8 \times 50\frac{1}{2}}{2} = 201\frac{1}{2}$$

Hence, $201\frac{1}{2}$ sq. ft. Or

$$\text{2d, — (1) } 2 \times 8 \text{ ft.} = 16 \text{ ft.}$$

$$(2) \quad .7854 \times 16^2 = 201.06 +$$

Hence, $201.06 +$ sq. ft.

Ex. 2. What is the area of a circle 75 feet in circumference?

PROCESS.

$$.07958 \times 75^2 = 449.64 -; \text{ hence, } 449.64 - \text{ sq. ft.}$$

Ex. 3. Find the diameter of a circle whose area is 846.3 sq. rd.

ANALYSIS. — Since the area of a circle is .7854 time the square of the diameter, the first step in the process of finding the diameter from the area, must be to find the quotient of the area, 846.3 sq. rd., divided by .7854. This quotient is 1,077.5401 sq. rd., and it must be the square of the diameter.

PROCESS.

$$846.3 \text{ sq. rd.} \div .7854 = 1,077.5401 \text{ sq. rd.}$$

$$\sqrt{1,077.5401} = 32.8263 +$$

Hence, $32.8263 +$ rd. = 32 rd. $13.6 +$ ft.

The square root of 1,077.5401 is 32.8263 + ; hence, the required diameter is $32.8263 +$ rd. = 32 rd. $13.6 +$ ft.

PROBLEMS.

Find the area of a circle, having given

73. Radius 12 in.

76. Diameter 15 ft. 4 in.

79. Radius .6462 mi.

74. Circumference 1 mi.

77. Circum. 8 yd. 2 ft. 9 in.

80. Circum. 62 ft. 10 in.

75. Diameter 26 rd.

78. Radius 17 ch. 2 l.

81. Diameter 1.1 yd.

82. Circum. 25.1328 rd., diameter 132 ft. | **83.** Radius 10 ft., circum. 62 ft. 10 in.

Find the diameter of a circle

Find the circumference of a circle

84. Whose area is 1 sq. mi.

87. Whose area is 37.013 sq. yd.

85. Whose area is 29 sq. in.

88. Whose area is 209 sq. ft. 87 sq. in.

86. Whose area is 16 A. 23 sq. rd.

89. Whose area is 1 sq. ft.

CASE III. Areas of ellipses.

597. IMPORTANT FACT IX.—*The number of units in the area of an ellipse equals .7854 of the product of the numbers expressing its two axes.*

Ex. The major axis of an ellipse is 24 feet and the minor axis is 18 feet. Required the area.

PROCESS.

$.7854 \times 18 \times 24 = 339.29+$; hence, $339.29+$ sq. ft., the area.

PROBLEMS.

90. The major axis of an ellipse is 20 inches, and the minor axis is 14 inches. What is its area?

91. Find the area of an ellipse whose axes are 18 rd. and 12 rd.

92. The area of an ellipse is 192 sq. in., and its major axis is $18\frac{1}{2}$ in. Find its minor axis.

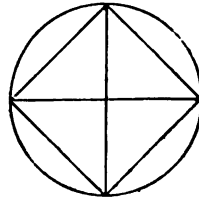
93. What is the major axis of an ellipse whose minor axis is 3 ft. $9\frac{1}{2}$ in., and whose area is 14 sq. ft. 76.75 sq. in.?

94. The area of an ellipse is 86 square rods and the ratio of its axes is as 7:4. Find the axes.

CASE IV. Inscribed squares.

598. A square is *inscribed* in a circle, or conversely,—a circle is *circumscribed* about a square,—if the vertex of each angle of the square is in the circumference of the circle.

The diameter of a circle is the hypotenuse of a right-angled triangle whose base and perpendicular are each a side of the inscribed square.



Ex. 1. Find one side of a square inscribed in a circle 15 ft. in diameter.

OUTLINES OF ANALYSIS.

(1) The square of the hypotenuse.

Since the square of the hypotenuse equals the sum of the squares of the other two sides, and since these two sides are equal, their squares must be equal.

(2) The square of the base or of the perpendicular.

(3) The base or the perpendicular.

PROCESS.

$$(1) \quad 15^2 = 225$$

$$(2) \quad \frac{1}{2} \text{ of } 225 = 112.5$$

$$(3) \quad \sqrt{112.5} = 10.6066 +$$

Hence, 10.6066 + ft.

Ex. 2. Find the diameter of a circle circumscribing a square 8 inches on each side.

PROCESS.

$2 \times 8^2 = 128$; $\sqrt{128} = 11.31 +$; hence, 11.31 + in., the diameter required.

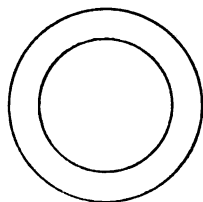
PROBLEMS.

95. What is one side of the largest square stick of timber that can be cut out of a log 2 ft. 9 in. in diameter?

96. I wish to cut a tree that will exactly square 1 ft. 9 in. at the butt. What must be the circumference of the tree?

97. What are the lateral dimensions of the largest square stick that can be passed through a ring whose inside circumference is 3 ft. 1.7 in.?

CASE V. Areas of rings.



599. The area of a ring bounded by two concentric circles equals the difference of the areas of the two circles; but $.7854 \times \text{the greater diam.}^2$ (i. e., the area of the greater circle) minus $.7854 \times \text{the less diam.}^2$ (the area of the smaller circle) $= .7854 \times (\text{greater diam.}^2 - \text{less diam.}^2)$; the difference of the squares of two numbers equals the product of their sum and difference (584),—hence, the area of a ring bounded by two concentric circles equals $.7854 \times (\text{greater diam.} + \text{less diam.}) \times (\text{greater diam.} - \text{less diam.})$.

PROBLEMS.

98. What is the area of a ring whose inside diameter is 325 feet, and outside diameter 375 feet?

99. The inside diameter of a ring 4 rods wide is 64 rods. Required its area.

100. A circular race-track $\frac{1}{4}$ mile inside circumference is 8 rods wide. How many acres are there in the track?

101. An elliptical race-course has for its major and minor axes—outside measurements—18 ch. 5 l. and 13 ch. 77 l., and the track is 12 rods wide. Required the area of the track.

600. FORMULÆ

For finding

I. Hypotenuse of right-angled triangle :— $\sqrt{\text{base}^2 + \text{perpendicular}^2}$.

II. Perpendicular of right-angled triangle :— $\sqrt{\text{hypotenuse}^2 - \text{base}^2}$.

III. Base of right-angled triangle :— $\sqrt{\text{hypotenuse}^2 - \text{perpendicular}^2}$.

IV. Areas of parallelograms :— $\text{Altitude} \times \text{base}$.

V. Dimensions of parallelograms :— $\text{Area} \div \text{given dimension}$.

VI. Areas of triangles :— $\frac{\text{Altitude} \times \text{base}}{2}$.

VII. Dimensions of triangles :— $\frac{2 \times \text{area}}{\text{given dimension}}$.

VIII. Areas of trapezoids :— $\text{Altitude} \times \text{average length}$.

IX. Altitudes of trapezoids :— $\text{Area} \div \text{average length}$.

X. Longer parallel sides of trapezoids :— $\text{Area} \div \text{altitude} + (\text{average length} - \text{shorter parallel side})$.

XI. Shorter parallel sides of trapezoids :— $\text{Area} \div \text{altitude} - (\text{longer parallel side} - \text{average length})$.

XII. Circumferences of circles :— $3.1416 \times \text{diameter}$; or $\text{diam.} \div .3183$.

XIII. Diameters of circles :— $\text{circum.} \div 3.1416$; or $.3183 \times \text{circum.}$

XIV. Areas of circles :— $\frac{\text{Diameter} \times \text{circumference}}{4}$; or $.7854 \times \text{diameter}^2$; or $.07958 \times \text{circumference}^2$.

- XV. Areas of ellipses:— $.7854 \times \text{minor axis} \times \text{major axis}$.
- XVI. Sides of inscribed squares:— $\sqrt{\frac{\text{diameter}^2}{2}}$.
- XVII. Diameters of circumscribing circles:— $\sqrt{2 \times \text{area of inscribed sq.}}$
- XVIII. Areas of rings:— $(\text{greater diameter} + \text{less diameter}) \times (\text{greater diameter} - \text{less diameter}) \times .7854$.
- XIX. Areas of regular polygons:— $\frac{1}{2}$ of distance from centre to middle of any side $\times$ perimeter of polygon.
- XX. Areas of trapeziums and of irregular polygons of more than four sides:—*Separate into triangles, compute the area of each, and find the sum of these areas.*

PROBLEMS.

- 102.** An assembly hall is 67 ft. 8 in. long and 40 ft. 3 in. wide. How much will it cost to ceil the hall at \$9.38 per square?
- 103.** The area of a floor 26 ft. 9 in. wide is 902½ sq. ft. Find its length.
- 104.** What will be the cost of slating a roof 32 feet long and 26 feet wide at \$1.86 per square yard?
- 105.** A board 9½ inches wide contains 12½ square feet. How long is it?
- 106.** A ladder 28 feet long standing with its foot 6 feet from a building, reaches how high on the building?
- 107.** Find the diagonal of this school-room,—i.e., the distance from one of its upper corners to the opposite lower corner.
- 108.** The length of one of the parallel sides of a trapezoid is 35 ft., its area is 468 sq. ft., and its altitude is 18 ft. Required the length of the other parallel side.
- 109.** A tinsmith wishes to make a lot of 6-inch pipe. If he allows ½ of an inch for the seam, how wide shall he cut the material?
- 110.** A trapezoid has two right angles, its parallel sides are 24 feet and 32 feet, and its slant height is 40 feet. Find its area.
- 111.** Find the cost of lathing and plastering the walls and ceiling of a room 16 feet square and 8 ft. 6 in. high, at 32¢ per square yard, no deductions being made for doors nor windows.

112. The posts of a grape trellis are 8 feet high and 12 feet apart. How long a scantling will be required for a brace extending from the top of one post to the foot of the next?

113. What is the length of the diagonal of a mile square?

114. Find the number of acres in a triangular field whose sides are 25 rods, 32 rods, and 47 rods.

III. GOVERNMENT LANDS.

601. *Government lands* are commonly divided by parallels and meridians into rectangular tracts known as Townships, Sections, and Quarter-Sections.

a. A **township** is a tract of land 6 miles square.

b. A **section** is $\frac{1}{36}$ of a township, and is 1 mile square.

c. A **quarter-section** is $\frac{1}{4}$ of a section, and is $\frac{1}{4}$ mile square.

d. A **Range** is a line of townships extending north and south.

e. A **base line** is the parallel of latitude from which townships are surveyed, and numbered north and south.

f. A **principal meridian** is the meridian from which ranges are surveyed, and numbered east and west.

Tp. 22 N. of R. 8 E. is the 22d township north of a base line and in the 8th range east of a principal meridian.

g. In transfers and by private surveys sections are also subdivided into Half-Sections, Half-Quarter-Sections, or 80's, and Quarter-Quarter-Sections, or 40's, as shown in Diagram 2.

h. Sections are numbered from the north-east corner of a township, west and east consecutively, as shown by Diagram 1; and parts of sections are described by their location in the section and by their size, as shown by Diagram 2.

DIAGRAM 1.

N					
6	5	4	3	2	1
7	8	9	10	11	12
18	17	16	15	14	13
19	20	21	22	23	24
30	29	28	27	26	25
31	32	33	34	35	36
S					

A TOWNSHIP. 6 miles square.

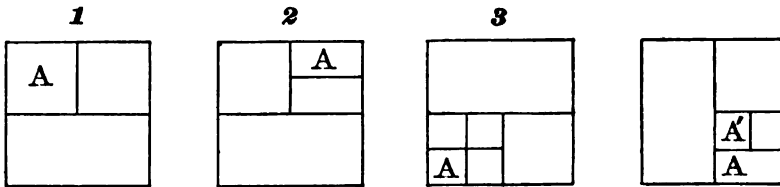
DIAGRAM 2.

N		
N. $\frac{1}{4}$ Section. 320 ACRES.		
W	S. W. $\frac{1}{4}$ 160 ACRES.	N. E. $\frac{1}{4}$ of S. E. $\frac{1}{4}$
		S. E. $\frac{1}{4}$ of S. E. $\frac{1}{4}$
S		

A SECTION. 1 mile square.

PROBLEMS.

Write a description of each tract marked A in the following diagrams of sections, and find the entire number of acres of each section included in the tracts so marked.



Draw a diagram to show the location of each part of a section, explain the location of the section and township, and find the number of acres conveyed, in the following descriptions:—

5. The N. $\frac{1}{4}$ of the S. E. $\frac{1}{4}$ of Sec. 12, Tp. 13 N. of Range 5 W.

6. The N. W. $\frac{1}{4}$ of the N. W. $\frac{1}{4}$, and the E. $\frac{1}{4}$ of the N. W. $\frac{1}{4}$ of Sec. 23, Tp. 6 N. of Range 11 E.

7. The S. $\frac{1}{4}$ of the N. E. $\frac{1}{4}$, and the W. $\frac{1}{4}$ of the S. E. $\frac{1}{4}$, and the N. E. $\frac{1}{4}$ of S. W. $\frac{1}{4}$,—of Sec. 30, Tp. 28 N. of Range 6 W.

SECTION IV.

SOLIDS OR BODIES.

602. Define (1) A solid (108); (2) Volume (109); (3) Cubic contents (111); (4) Solidity (111, a); (5) Capacity (111, b); (6) A rectangular solid (120); (7) A cube (121).

603. The *convex surface* of a solid is its surface exclusive of its ends or bases.

The term *convex* is thus applied to the sum of plane surfaces as well as to curved surfaces. The plane surfaces composing a convex surface are known as *lateral surfaces*.

I. PRISMS.

604. A *prism* is a solid whose ends or bases are equal parallel polygons, and whose sides are parallelograms.

a. The *altitude* of a prism is the perpendicular distance between its bases or the planes of its bases.

b. A *right prism* is one whose sides are rectangles.

c. The *entire surface* of a prism is the total surface of its sides and ends.

d. Prisms are named from the plane figures that form their bases; as triangular prisms, rectangular prisms, pentagonal prisms.

CASE I. Surfaces of prisms.

605. Ex. Required, the entire surface of a rectangular solid 16 feet long, 4 feet wide, and 3 feet thick.

OUTLINES OF ANALYSIS.

- (1) Area of the two ends or bases.
 (2) Area of the two sides 8 ft. wide.
 (3) Area of the two sides 4 ft. wide.
 (4) Sum of areas in (1), (2), (3);
 or entire surface.

PROCESS.

$$\begin{array}{ll}
 (1) & 2 \times (3 \times 4 \text{ sq. ft.}) = 24 \text{ sq. ft.} \\
 (2) & 2 \times (3 \times 16 \text{ sq. ft.}) = 96 \text{ sq. ft.} \\
 (3) & 2 \times (4 \times 16 \text{ sq. ft.}) = 128 \text{ sq. ft.} \\
 (4) & \text{Hence, } 248 \text{ sq. ft.}
 \end{array}$$

PROBLEMS.

1. The base of a triangular prism is 1 ft. 8 in. on each side, and the altitude of the prism is 13 ft. 2 in. Required the convex surface.
2. Find the entire surface of a cube 8 ft. 5 in. on each edge.
3. A prism 11 ft. 5 in. long has for its base an isosceles triangle the base and altitude of which are 3 ft. 2 in. and 4 ft. 1 in. respectively. What is the entire surface of the prism?
4. The entire surface of a cube is 294 sq. in. Find one of its edges.
5. The base of a regular hexagonal prism is 2 inches on each side, and the altitude of the prism is 1 ft. 3 in. Required its entire surface.

CASE II. Volumes of prisms.

606. Ex. 1. A rectangular prism is 8 feet long, 9 inches wide, and 6 inches thick. Required its cubic contents.

ANALYSIS.—The area of the base (6 in. by 9 in.) is 54 sq. in., or $\frac{3}{4}$ of a square foot.

Since the area of the base is $\frac{3}{4}$ of a square foot, each foot in length of the prism will contain $\frac{3}{4}$ of a cubic foot; hence, $8 \times \frac{3}{4}$ cu. ft. equals 8 cu. ft., the cubic contents required.

EXPLANATION.—I first reduce all the dimensions to the common denomination feet, and, considering the results as abstract numbers, find their continued product which is 8.

Supplying the unit of volume corresponding to feet, I have 8 cu. ft., the solid contents required.

ANALYTICAL PROCESS.

$$\begin{aligned}
 6 \times 9 \text{ sq. in.} &= 54 \text{ sq. in.} = \frac{3}{4} \text{ sq. ft.} = \frac{3}{4} \text{ sq. ft.} \\
 8 \times \frac{3}{4} \text{ cu. ft.} &= 6 \text{ cu. ft.}
 \end{aligned}$$

COMMON PROCESS.

$$\begin{aligned}
 \frac{3}{4} \times \frac{3}{4} \times 8 &= 6 \\
 \text{Hence, } 6 \text{ cu. ft.}
 \end{aligned}$$

Ex. 2. The base of a triangular prism is a right-angled triangle whose base and perpendicular are 8 inches and 1 foot, and the altitude of the prism is 10 feet. Required its cubic contents.

OUTLINES OF ANALYSIS.

- (1) Area of triangular base.
 (2) Cubic contents of the prism.

PROCESS.

- (1) $\frac{1}{2}$ of $\frac{8}{12}$ of 1 sq. ft. = $\frac{1}{3}$ sq. ft.
 (2) $10 \times \frac{1}{3}$ cu. ft. = $3\frac{1}{3}$ cu. ft.

Ex. 3. The cubic or solid contents of a rectangular solid are 3 cu. ft., its length is 18 ft., and its width 6 in. Required its thickness.

OUTLINES OF ANALYSIS.

- (1) Area of surface 18 ft. long and 6 in. wide.
 Since the area of one side of the solid is 9 sq. ft.,
 each foot in thickness contains 9 cu. ft.; hence,
 (2) Part of a foot thick.
 (3) $\frac{1}{3}$ ft. reduced to inches.

PROCESS.

- (1) $18 \times \frac{6}{12}$ sq. ft. = 9 sq. ft.
 (2) $3 \text{ cu. ft.} \div 9 \text{ cu. ft.} = \frac{1}{3}$
 (3) Hence, $\frac{1}{3}$ ft., or 4 in.

607. IMPORTANT FACT I. — *The dimensions of the plane figures bounding a prism are also dimensions of the prism.*

PROBLEMS.

6. How many cubic feet of timber are there in a stick 26 ft. long by 1 ft. 9 in., by 1 ft. 9 in.?

7. How much is the cost of a cubic block of marble 4 feet on each edge, at 65¢ per cubic foot?

8. D purchased 35 joists each 16 ft. long, 8 in. wide, and $2\frac{1}{2}$ in. thick, at \$13.80 per M board measure. What was the cost?

9. A stone wall 1 ft. 4 in. thick surrounds a lot 100 ft. by 240 ft., and is capped by triangular coping-stones 9 in. high and as wide as the wall. Required their cost at 56¢ per cu. ft.

10. What is the capacity in gallons of a water tank 8 ft. $\times$ 6 ft. $\times$ 4 ft.?

11. A wood-rack is $10\frac{1}{2}$ ft. long, and 3 ft. 4 in. wide. How high must it be to hold 1 cord?

A box is 18 in. $\times$ 14 in., inside measurement. How deep must it be,

12. To hold a bushel of beans? | 13. To hold a bushel of peaches?

II. CYLINDERS.

608. A *cylinder* is a solid of uniform diameter whose bases or ends are equal parallel circles.

The *axis of a cylinder* is a line that joins the centres of its bases.

CASE I. Surfaces of cylinders.

609. Ex. 1. Find the convex surface of a cylinder 2 feet in diameter and 12 feet long.

ANALYSIS.—The convex surface of a cylinder may be supposed to be cut by a line perpendicular to its bases, and spread out on a plane, thus forming a rectangle whose base is the circumference of the cylinder and whose altitude is the altitude of the cylinder. Hence,

I first find the circumference of a circle 2 ft. in diameter, which is $6\frac{2}{3}$ ft.

I then find the area of a rectangle 12 ft. by $6\frac{2}{3}$ ft., which is $75\frac{2}{3}$ sq. ft., the convex surface required.

PROCESS.

$$3\frac{1}{2} \times 2 \text{ ft.} = 6\frac{2}{3} \text{ ft.}$$

$$12 \times 6\frac{2}{3} \text{ sq. ft.} = 75\frac{2}{3} \text{ sq. ft.}$$

Ex. 2. Find the entire surface of a 4-foot cylinder 16 feet long.

OUTLINES OF ANALYSIS.

PROCESS.

(1) Circumference found from diameter.

$$(1) \quad 3\frac{1}{2} \times 4 \text{ ft.} = 12\frac{1}{2} \text{ ft.}$$

(2) Convex surface.

$$(2) \quad 16 \times 12\frac{1}{2} \text{ sq. ft.} = 201\frac{1}{2} \text{ sq. ft.}$$

(3) Surface of ends or bases.

$$(3) \quad 2 \times \frac{1}{2} \times 4 \times 12\frac{1}{2} \text{ sq. ft.} = 25\frac{1}{2} \text{ sq. ft.}$$

(4) Sum of surfaces.

$$(4) \quad \text{Hence,} \quad 226\frac{3}{4} \text{ sq. ft.}$$

PROBLEMS.

14. How much will be the cost of painting a cylindrical water-tank 3 ft. 10 in. in diameter and 12 ft. 4 in. long, at \$1.18 per square?

15. How many feet of lumber are there in a cistern $7\frac{1}{2}$ feet in diameter, 8 feet high, open at one end, and made of material $1\frac{1}{2}$ inches thick?

16. Find the inside surface of a standard Winchester bushel measure.

17. How many square feet of boiler iron are there in a petroleum-car tank 33 ft. long and $4\frac{1}{2}$ ft. in diameter, allowing $\frac{1}{8}$ for laps?

18. How many square feet of iron are there in a length of 6-in. Russia stove-pipe 2 ft. 4 in. long, allowing $\frac{1}{2}$ inch for seam?

19. The entire surface of a cylinder 3 ft. 2 in. in diameter is 182.6 sq. ft. Required its length.

CASE II. Volumes of cylinders.

610. Ex. 1. Find the volume of a cylinder 11 in. in diameter and 3 ft. 4 in. long.

PROCESS.

OUTLINES OF ANALYSIS.

(1) Area of circular base.

$$(1) \quad .7854 \times 11 \times 11 \text{ sq. in.} = 95.0334 \text{ sq. in.}$$

(2) Solid contents,—from

$$(2) \quad 40 \times 95.0334 \text{ cu. in.} = 3,801.336 \text{ cu. in.}$$

linear unit—(inches).

$$(3) \quad = 2 \text{ cu. ft. } 345.3+ \text{ cu. in.}$$

(3) Volume reduced to higher denominations.

Ex. 2. The volume of a cylinder 10 ft. long is 49 cu. ft. 151.2 cu. in. Find the diameter.

PROCESS.

OUTLINES OF ANALYSIS.

- | | |
|---|--|
| (1) Contents reduced to cubic inches. | (1) $49 \text{ cu. ft. } 151.2 \text{ cu. in.} = 84,823.2 \text{ cu. in.}$ |
| (2) Area 1 in. long that equals the same volume, divided by the number expressing inches in length. | (2) $84,823.2 \text{ sq. in.} \div 120 = 706.86 \text{ sq. in.}$ |
| | (3) $706.86 \text{ sq. in.} \div .7854 = 900 \text{ sq. in.}$ |
| | (4) $\sqrt{900} = 30$; hence, 30 in. = 2 ft. 6 in. |

- (3) Area of circumscribed square. (4) One side of circumscribed square.

PROBLEMS.

20. Prove that the capacity of the standard Winchester bushel measure is 2,150.42+ cubic inches.

21. A pail 6 inches in diameter and $6\frac{1}{4}$ inches deep contains how many quarts?

22. What is the capacity, in barrels, of the petroleum car-tank in prob. 17?

23. The tank of a street sprinkler is 12 ft. long and 12 ft. 6.2 in. in circumference, outside measurements. If constructed of $1\frac{1}{4}$ -inch lumber, with heads 2 inches from the ends of the staves, what is its capacity in gallons?

24. The capacity of a cylindrical cistern is 18.572 barrels, and its diameter is 5 ft. 4 in. Required its depth.

III. PYRAMIDS AND CONES.

611. A *pyramid* is a solid whose base is a polygon, and whose sides are triangles terminating in a common point or vertex.

612. A *cone* is a solid whose base is a circle, whose top terminates in a point or vertex, and any straight line connecting its vertex and the circumference of its base is wholly in its surface.

a. The *axis of a pyramid* or of a *cone* is the line connecting the vertex with the centre of the base.

b. A *right pyramid* or a *right cone* is one whose axis is perpendicular to its base.

- c. The *altitude of a pyramid or of a cone* is the perpendicular height of its vertex above its base or the plane of its base.
- d. The *slant height of a pyramid* is the distance from its vertex to the middle of a side of its base. And
The *slant height of a cone* is the distance from its vertex to the circumference of its base.
- e. The *convex surface of a pyramid or of a cone* is its total surface exclusive of its base.
- f. Pyramids, like prisms, are named from their bases.

NOTE.—Do not fail to distinguish between *altitude* and *slant height*.

613. IMPORTANT FACTS.

II. The *slant height of a pyramid or of a cone* is a dimension of surfaces that bound it, but is not a dimension of the solid.

III. The *volume of a pyramid* equals one third of the volume of a prism having the same base and altitude.

IV. The *volume of a cone* equals one third of the volume of a cylinder having the same base and altitude.

CASE I. Surfaces of pyramids and cones.

614. Ex. 1. Find the convex surface of a triangular pyramid whose slant height is 5 ft., and the sides of whose base are 4 ft., 5 ft., and 7 ft.

OUTLINES OF ANALYSIS.

- | | |
|--|---|
| (1) Area of triangular side 4 ft. at base. | (1) $\frac{1}{2}$ of 5×4 sq. ft. = 10 sq. ft. |
| (2) Area of triangular side 5 ft. at base. | (2) $\frac{1}{2}$ of 5×5 sq. ft. = $12\frac{1}{2}$ sq. ft. |
| (3) Area of triangular side 7 ft. at base. | (3) $\frac{1}{2}$ of 5×7 sq. ft. = $17\frac{1}{2}$ sq. ft. |
| (4) Convex surface. | (4) Hence, $\frac{40}{40}$ sq. ft. |

PROCESS.

COMMON PROCESS.

$$\frac{1}{2} \text{ of } 5 \times (4 + 5 + 7) \text{ sq. ft.} = 40 \text{ sq. ft.}$$

Ex. 2. Find the entire surface of a cone whose altitude is 4 feet and whose base is 6 feet in diameter.

PROCESS.

OUTLINES OF ANALYSIS.

Since the axis and the slant height of a cone are respectively the perpendicular and the hypotenuse of a right-angled triangle, and the radius of the base of the cone is the base of the triangle,—

- | |
|---|
| (1) $\sqrt{4^2 + 3^2} = 5$; hence, 5 ft. |
| (2) $3\frac{1}{2} \times 6$ ft. = $18\frac{1}{2}$ ft. |
| (3) $\frac{1}{2}$ of $5 \times 18\frac{1}{2}$ sq. ft. = $47\frac{1}{2}$ sq. ft. |
| (4) $.7854 \times 6 \times 6$ sq. ft. = 28.27 sq. ft. |
| (5) Hence, $\frac{76.41 + \text{sq. ft.}}$ |

- | | | |
|----------------------------|----------------------|---------------------|
| (1) Slant height. | (3) Convex surface. | |
| (2) Circumference of base. | (4) Surface of base. | (5) Entire surface. |

PROBLEMS.

25. Find the entire surface of a cone whose base is 5 ft. 8 in. in diameter, and whose slant height is 7 ft. 2 in.

26. Find the entire surface of a square pyramid whose base is 5 ft. on each side, and whose slant height is 9.65 ft.

27. Required the cost of slating a church spire whose base is an octagon $4\frac{1}{2}$ ft. on each side, and whose slant height is 75 ft., at \$9.63 per square.

CASE II. Volumes of pyramids and cones.

615. Ex. 1. Find the volume of a rectangular pyramid whose base is 5 ft. on each side, and whose altitude is 14 ft.

OUTLINES OF ANALYSIS.

- (1) Area of base of pyramid.
 (2) One third of volume of a prism of the same base and altitude as the pyramid (613, III).

PROCESS.

- | | |
|---|---|
| (1) | $5 \times 5 \text{ sq. ft.} = 25 \text{ sq. ft.}$ |
| (2) $\frac{1}{3}$ of $14 \times 25 \text{ cu. ft.}$ | $= 116\frac{2}{3} \text{ cu. ft.}$ |

Ex. 2. Required the solid contents of a cone 3 feet in diameter at the base, and 4 ft. 6 in. high.

OUTLINES OF ANALYSIS.

- | | | |
|---------------------------------------|-----|---|
| (1) Area of circular base. | (1) | $.7854 \times 3 \times 3 \text{ sq. ft.} = 7.0686 \text{ sq. ft.}$ |
| (2) Solid contents of cone (613, IV). | (2) | $\frac{1}{3}$ of $4\frac{1}{2} \times 7.0686 \text{ cu. ft.} = 10.6 + \text{cu. ft.}$ |

PROCESS.

PROBLEMS.

28. A lye vat 4 feet deep is 4 feet square at the top and tapers to a point at the bottom. How many bushels of ashes will it contain?

29. The cup of a funnel is 8 inches in diameter and $6\frac{1}{2}$ inches deep. Required its capacity.

30. The sides of the base of a triangular pyramid are 6 ft., 7 ft., and 8 ft., and the altitude of the pyramid is 21 ft. Find its volume.

31. A conical pile of grain is 48 ft. 4 in. in circumference, and 3 ft. 10 in. deep at the apex. How many bushels are there in the pile?

IV. FRUSTUMS.

616. A *frustum* of a pyramid or of a cone is the bottom part of the pyramid or cone cut off by a plane parallel to the base.

617. A *segment* of a pyramid or of a cone is the top part of the pyramid or cone cut off by a plane parallel to the base.

a. The *altitude of a frustum* is the perpendicular distance between its ends or bases.

b. The *slant height of a frustum* is the shortest distance between the perimeters of its bases.

618. IMPORTANT FACTS.

V. *The volume of a frustum of a pyramid equals the total volume of three pyramids each of equal altitude with the frustum, and having for their bases respectively,—(1) the top of the frustum, (2) the base of the frustum, and (3) a mean proportional between the top and base.*

VI. *The volume of a frustum of a cone equals the total volumes of three cones each of equal altitude with the frustum, and having for their bases respectively,—(1) the top of the frustum, (2) the base of the frustum, and (3) a mean proportional between the top and base.*

619. Prove

PROPOSITION I. *The side of a square that is a mean proportional to two other squares is a mean proportional to their sides.*

PROPOSITION II. *The diameter of a circle that is a mean proportional to two other circles is a mean proportional to their diameters.*

CASE I. Surfaces of frustums.

620. Ex. What is the entire surface of a frustum of a quadrangular pyramid, whose top is 4 ft. square, base 6 ft. square, and slant height 5 ft.?

STEPS IN PROCESS.—Find,—

- (1) Average width of each side.
- (2) Area of four sides, or convex surface.
- (3) Area of top of frustum.
- (4) Area of base of frustum.
- (5) Total areas, or entire surface.

PROCESS.

- (1) $\frac{1}{2}$ of $(4 \text{ ft.} + 6 \text{ ft.}) = 5 \text{ ft.}$
- (2) $4 \times 5 \times 5 \text{ sq. ft.} = 100 \text{ sq. ft.}$
- (3) $4 \times 4 \text{ sq. ft.} = 16 \text{ sq. ft.}$
- (4) $6 \times 6 \text{ sq. ft.} = 36 \text{ sq. ft.}$
- (5) Hence, 152 sq. ft.

PROBLEMS.

32. How many square yards of surface are there on the outside of a sap reservoir or store-tub, open at the top, 3 ft. 9½ in. square at the top, 2 ft. 11 in. square at the bottom, and 4 ft. deep?

33. How many pounds of steel $\frac{1}{8}$ in. thick will be required for a petroleum tank 12 ft. in diameter at the top, 15 ft. in diameter at the bottom, and 14 ft. slant height, allowing $\frac{1}{16}$ for laps and waste, the steel weighing 7.56 pounds to the square foot?

CASE II. Volumes of frustums.

621. Ex. 1. Find the solid contents of the frustum of a pyramid that is 4 feet square on the top, 6 feet square on the base, and 5 feet high.

OUTLINES OF ANALYSIS.

PROCESS.

- | | | |
|--|-----|--|
| (1) Area of base equal to top of frustum (618, V). | (1) | $4 \times 4 \text{ sq. ft.} = 16 \text{ sq. ft.}$ |
| (2) Area of base equal to base of frustum (618, V). | (2) | $6 \times 6 \text{ sq. ft.} = 36 \text{ sq. ft.}$ |
| (3) Area of base that is a mean proportional to the other two (618, V). | (3) | $\sqrt{16 \times 36} \text{ sq. ft.} = 24 \text{ sq. ft.}$ |
| (4) Sum of areas of bases. | (4) | 76 sq. ft. |
| (5) Volume of the three pyramids having a common altitude of 5 ft. (618, V). | (5) | $\frac{1}{3} \text{ of } 5 \times 76 \text{ cu. ft.} = 126\frac{2}{3} \text{ cu. ft.}$ |

COMMON PROCESS.

$$(4^2 + 6^2 + 4 \times 6) = 76; \frac{1}{3} \text{ of } 5 \times 76 = 126\frac{2}{3}; \text{ hence, } 126\frac{2}{3} \text{ cu. ft.}$$

Ex. 2. Find the volume of the frustum of a cone whose top diameter is 3 feet, bottom diameter 5 feet, and altitude 4 feet.

OUTLINES OF ANALYSIS.

PROCESS.

- | | | |
|--|-----|--|
| (1) Area of base equal to top of frustum (618, VI). | (1) | $.7854 \times 3 \times 3 \text{ sq. ft.} = 7.0686 \text{ sq. ft.}$ |
| (2) Area of base equal to base of frustum (618, VI). | (2) | $.7854 \times 5 \times 5 \text{ sq. ft.} = 19.635 \text{ sq. ft.}$ |
| (3) Area of base that is a mean proportional to the other two (618, VI). | (3) | $\sqrt{7.0686 \times 19.635} \text{ sq. ft.} = 11.771 \text{ sq. ft.}$ |
| (4) Sum of areas of bases. | (4) | $38.474 + \text{sq. ft.}$ |
| (5) Volume of the three cones having a common altitude of 4 ft. (618, VI). | (5) | $\frac{1}{3} \text{ of } 4 \times 38.474 + \text{cu. ft.} = 51.298 + \text{cu. ft.}$ |

COMMON PROCESS.

$.7854 \times (3^2 + 5^2 + 3 \times 5) = 38.474$; $\frac{1}{3}$ of $4 \times 38.474 = 51.298$;
hence, 51.298 cu. ft.

PROBLEMS.

34. Required the capacity of a cistern whose top diameter is 7 ft. 6 in., bottom diameter 9 ft., and height 6 ft.

35. A berry basket is $5\frac{1}{2}$ inches square at the top, $4\frac{1}{2}$ inches square at the bottom, and $2\frac{1}{2}$ inches deep. Will it contain a quart of berries?

36. A tin pail is 13 inches in diameter at the top, 9 inches in diameter at the bottom, and 8 inches deep. Required its capacity.

Required the number of cubic feet of timber	{	37. In a stick 1 ft. 8 in. square at one end, 2 ft. 2 in. square at the other, and 34 feet long. 38. In a log 28 feet long, 2 ft. 1 in. in diameter at one end, and 2 ft. 9 in. at the other.
---	---	---

V. SPHERES.

622. A *sphere* or *globe* is a solid bounded by a uniform curved surface.

- a.* A *great circle* of a sphere is any circle on its surface whose plane includes the centre of the sphere.
- b.* A *circumference* of a sphere is any great circle of the sphere.
- c.* A *diameter* of a sphere is a straight line passing through its centre, and terminating at both ends in its surface.
- d.* Every point in the surface of a sphere is equally distant from the centre.

623. IMPORTANT FACTS.

VII. *The surface of a sphere equals four times the area of a circle of the same diameter as the sphere.*

VIII. *The volume of a sphere equals two thirds of the volume of a cylinder whose diameter and altitude are each equal to the diameter of the sphere.*

624. According to **623**, **VII**, the surface of a sphere equals

$$4 \times \frac{\text{diameter} \times \text{circumference}}{4} = \text{diameter} \times \text{circumference}.$$

625. According to **623**, VIII, the volume of a sphere equals $\frac{1}{6}$ of diameter $\times$ $(.7854 \times \text{diameter}^2) = \frac{1}{6}$ of $.7854 \times \text{diameter}^3 = .5236 \times \text{diameter}^3$.

PROBLEMS.

Find the surface and the volume of a sphere
39, 40. 16 ft. 5 in. in diameter. | **41, 42.** 56 inches in circumference.

626. FORMULÆ

For finding

- I. The convex surfaces of prisms or of cylinders:—*Altitude $\times$ perimeter of base.*
- II. The entire surfaces of prisms or of cylinders:—*Convex surface + areas of bases.*
- III. The volumes of prisms or of cylinders:—*Altitude $\times$ area of base.*
- IV. The convex surfaces of pyramids or of cones:— $\frac{1}{2}$ of *slant height $\times$ perimeter of base.*
- V. The volumes of pyramids or of cones:— $\frac{1}{3}$ of *altitude $\times$ area of base.*
- VI. The convex surfaces of frustums:—*Slant height $\times$ $\frac{1}{2}$ of sum of perimeters of bases.*
- VII. The volumes of frustums:— $\frac{1}{3}$ of *altitude $\times$ (area of top + area of base + mean proportional to areas of top and base).*
- VIII. The surfaces of spheres:—*Diameter $\times$ circumference.*
- IX. The volumes of spheres:— $.5236 \times \text{diameter}^3$.

PROBLEMS.

43. A cubical bin holds 200 bushels of potatoes. What are its dimensions?

44. What is the entire inner surface of a cubical box whose capacity is 110,592 cu. in.?

45. How many feet of lumber are there in a board 15 feet long, 16 inches wide, and $1\frac{1}{4}$ inches thick?

46. Find the cost of 2-inch lumber for a platform 64 feet long and 28 feet wide, at \$22.50 per M.

47. What is the value of a pile of 15-inch wood, 58 feet long and 6 feet high, at \$1.50 per face cord?

48. How many cords in a pile of 4-ft. wood, 18 ft. long, 8 ft. high at one end, and 6 ft. high at the other?

49. The pressure of the atmosphere at the level of the sea is 15 pounds to the square inch. What is its pressure on a solid 4 feet long on the edge, 1 foot square at one end, and 2 feet square at the other?

50. How many feet of matched lumber 1 inch thick will be required for a cubical box that will contain 100 bushels of grain?

51. A bunch of shingles 1 ft. 8 in. wide consists of 25 double layers. How many shingles are there in the bunch?

52. A teamster drew at one load 4 stones each 4 ft. long, 2 ft. 6 in. wide, and 8 in. thick. What was the weight of the load, if a cubic foot of the stone weighs 152.5 lb.?

53. An iron 25-lb. weight is $4\frac{1}{2}$ in. square on the bottom and $3\frac{1}{2}$ in. square on the top. How high is the weight, iron weighing 7,790 oz. to the cubic foot?

54. Find the number of Philadelphia brick in the walls of a building 32 ft. by 56 ft., and 28 ft. high, the gables being 10 ft. high, the walls $1\frac{1}{2}$ bricks thick, and the mortar $\frac{1}{4}$ in. thick; allowing the doors and windows to occupy 242 sq. ft.

55. At 22¢ per bunch for lath and 30¢ per square yard for lathing and plastering, how much will it cost to lath and plaster this room?

Lead weighs 709.375 lb. to the cubic foot,

56. What is the diameter of a ball of lead that weighs 50 lb.?

57. Find the surface of a leaden ball weighing 80 pounds.

The "Monument to the Confederate Dead" in Hollywood Cemetery, Richmond, Va., is a pyramid of granite, 40 ft. square at the base, and 90 ft. high. Find

58. Its convex surface.

60. Its slant height.

59. Its cubic contents.

61. The length of one slant edge.

62. Its weight in gross tons, allowing a cubic yard to weigh 4,050 lb.

In a school-house 40 ft. long, 26 ft. wide, and having side walls 14 ft. high, are 10 windows 7 ft. 4 in. by 3 ft. 2 in., 3 outside doors 7 ft. 10 in. by 3 ft. 4 in., 3 inside doors 7 ft. 2 in. by 3 ft., baseboards 8 in. wide, and a partition across one end. The roof projects 1 ft. 8 in., and its ridge is 8 ft. 8 in. above the side walls. The foundation walls are 4 ft. high from bottom of trench and 2 ft. thick. Find

63. The amount of matched lumber necessary to sheathe the building.
64. The amount of siding, allowing $\frac{1}{4}$ for laps and waste.
65. The cost of labor for sheathing and siding at \$1.62 $\frac{1}{4}$ per square.
66. The quantity of shingles required for the roof.
67. The amount of 1 $\frac{1}{4}$ in. matched flooring for the floor.
68. The cost of lathing and plastering at 35¢ per square yard.
69. The number of rolls of paper 16 yd. long and 18 in. wide required to paper the walls.
70. The cost of eave troughs at 9¢ per running foot.
71. The number of perch of stone in the foundation walls.

See Supplement, pp. 368, 376.

SECTION V.

REVIEW.

627. BLACKBOARD OUTLINES.

Prepare full blackboard outlines for review of this Chapter on Measurements.

628. TEST QUESTIONS.

Prepare forty fair, thorough, test questions, in sets of ten each, for the review of Measurements.

SUPPLEMENT.

ROMAN NOTATION.

Page 15; Art. 43.

§ 1. The symbols of number used in the Roman method of notation and the values they express are as follows :—

Letters;	I	V	X	L	C	D	M
Values;	one	five	ten	fifty	one hundred	five hundred	one thousand

§ 2. The symbols I, X, C, M, represent values corresponding, respectively, to the first, second, third, and fourth orders of integral units in the Arabic method; and the intermediate symbols V, L, D, to one half the value of the second, third, and fourth orders. They may therefore be classified as follows :

- a. *Unit* letters,—I, X, C, M;
- b. *Half-unit* letters,—V, L, D.

NOTE.—The use of half-unit letters prevents much repetition.

§ 3. The letters of the Roman notation, like the figures of the Arabic, express two values;

- a. A *fixed* (or simple) value when standing alone, and
- b. A *related* (or local) value when combined with other letters.

§ 4. The law of related values expressed by letters in Roman notation is, *increase by addition or decrease by subtraction.*

PRINCIPLES.

§ 5. *Affirmative*,—

Governing values.

I. *The value expressed by a unit letter is repeated with every repetition of the letter.*

II. *The value expressed by any letter placed after one expressing greater value is added to that greater value.*

III. *The value expressed by a unit letter placed before a letter expressing greater value is subtracted from that greater value.*

IV. *The value expressed by a letter with a bar over it is one thousand times the fixed value of the letter.*

§ 6. *Restrictive*,—

Governing arrangement.

I. *A letter should not be used more than three times in expressing the integral units of any order.*

II. *Half-unit letters must never be repeated, nor placed before letters expressing greater value.*

III. *Unit letters are placed before letters of the next higher unit or half-unit value only.*

*e.g. IV and IX, or XL and XC;
but never IL, IC, or XM.*

IV. *A bar must never be placed over the letter I.*

NOTE.—Principle IV of Values is not used in writing numbers less than four thousand. Hence, owing to the present limited application of the Roman notation, this principle is practically obsolete.

RULE FOR ROMAN NOTATION.

Write the several orders of units of a given integer in succession, beginning with the highest order, and omitting all orders of units not named in the integer, except ones.

EXERCISES.

Write in Arabic notation

1. XX	9. LIV	17. CDXLIV	25. MCXI
2. CCC	10. MCD	18. M $\overline{\text{X}}$ DCCLIII	26. M $\overline{\text{X}}$ CMXCIX
3. LX	11. $\overline{\text{V}}$	19. XCIX	27. DCXXXV
4. CLXI	12. $\overline{\text{C}\overline{\text{C}}}$	20. $\overline{\text{C}}\text{M}\overline{\text{X}}$	28. $\overline{\text{X}}\text{LM}\overline{\text{V}}$ CDXLIV
5. XC	13. MMM	21. CXVIII	29. $\overline{\text{L}}$ MLIX
6. CM	14. M $\overline{\text{V}}$	22. MXL	30. $\overline{\text{V}}$ MMMDCCCLXXXVIII
7. XLIII	15. XXIX	23. LXXVII	31. $\overline{\text{C}}\overline{\text{C}}$ XXXMMCCXXII
8. XIX	16. DV	24. CCXXII	32. M $\overline{\text{C}}$ $\overline{\text{X}}$ MCDXVII

Write in Roman notation

33. 14	38. 68	43. 257	48. 1,776	53. 2,876	58. 17,874
34. 7	39. 25	44. 821	49. 1,492	54. 4,892	59. 62,587
35. 93	40. 96	45. 555	50. 1,720	55. 7,486	60. 119,856
36. 80	41. 88	46. 718	51. 1,848	56. 5,555	61. 724,211
37. 44	42. 79	47. 987	52. 1,521	57. 8,871	62. 999,999
63. This day of the month.	65. The number of this Christian year.				
64. This month of the year.	66. This century.				

Write in Roman notation, and then change into Arabic, five examples

67-71. To illustrate Principle I of values. | 77-81. To illustrate Principle III of values.
 72-76. " " " II " " | 82-86. " " " IV " "
 87-91. To illustrate Principle V of values.

NOTATION OF COMPOUND NUMBERS.

TABLES AND SCALES.

Page 35; Art. 126.

1. **Arrangement.**—In the upper line of each table are the units of the table, in order, the lowest unit at the right; and the lower line is the scale of the table, ascending from right to left.

2. **Readings.**—Uniformity should be observed in reading and reciting the tables. For example, the first table in measures of extension should be recited as follows :

The units of the table of linear measure are the inch, foot, yard, rod, mile. Of these units 12 inches are 1 foot, 3 feet are 1 yard, $5\frac{1}{2}$ yards are 1 rod, 320 rods are 1 mile; and the scale (ascending) is 12, 3, $5\frac{1}{2}$, 320.

Or, The units of the table of linear measure are the mile, rod, yard, foot, inch. Of these units 1 mile is 320 rods, 1 rod is $5\frac{1}{2}$ yards, 1 yard is 3 feet, 1 foot is 12 inches; and the scale (descending) is 320, $5\frac{1}{2}$, 3, 12.

a. In any written compound number, the highest denomination expressed stands at the left; and the other denominations, from highest to lowest stand, in their order, to the right.

b. The proper abbreviations for the names of the denominations are given in the tables. Pupils should not be allowed to use any other abbreviations for denominations.

126. MEASURES OF EXTENSION.

i. Linear.	{	1 mi.	1 rd.	1 yd.	1 ft.	1 in.
			320	5 $\frac{1}{2}$	3	12
ii. Cloth.	{	1 yd.	1 qr.	1 eighth	1 sixteenth	1 in.
			4	2	2	2 $\frac{1}{2}$
iii. Mariners'.	{	1 league	1 knot	1 cable	1 fathom	1 span
			3	8.45	120	8
						9
iv. Square.	{	1 sq. mi.	1 A.	1 sq. rd.	1 sq. yd.	1 sq. ft.
			640	160	80 $\frac{1}{2}$	9
						144
	{	1. Linear.	{	1 mi.	1 ch.	1 l.
						1 in.
					80	100
						7.92
v. Surveyors'.	{	2. Square.	{	1 Tp.	1 sec.	1 A.
						1 sq. ch.
						1 P.
						1 sq. l.
					86	640
						10
						16
						625
vi. Solid or Cubic.	{	1 cu. yd.	1 cu. ft.	1 cu. in.		
			27	1,728		
vii. Dry.	{	1 bu.	1 pk.	1 qt.	1 pt.	
			4	8	2	
	{	1. Commercial.	{	1 gal.	1 qt.	1 pt.
						1 gi.
					4	2
						4
viii. Liquid.	{	2. Pharmacists'.	{	Cong. 1	O. 1	f $\frac{3}{4}$ 1
						f $\frac{3}{4}$ 1
						m 1
					8	16
						8
						60

NOTES.—1. Cong. (congius) is gal., O. (octarius) is pt., f $\frac{3}{4}$ is fluid oz., f $\frac{3}{4}$ is fluid dr., m is minim or drop.

2. The old road measure—*furlong* (= 40 rd.)—is now but little used.

3. A *Gunter's Chain*, used in land surveying, is 4 rods, 66 feet, or 100 links long. Hence, 25 links are 1 rod. The denomination *rod* is seldom used in linear chain measure.

4. a. An *Engineers' Chain*, used by topographical, civil, and military engineers, is 100 feet long, and consists of 100 links. But engineers express short distances in yards and feet.

b. Since 100 links are 1 chain, 1 link is .01 of a chain, 67 links are .67 of a chain, and so on. Hence, links may be written as hundredths of a chain. They are commonly so written in computations; thus, 45 ch. 14 l. = 45.14 ch.

5. In measuring clock pendulums, *12 points are 1 line*, and *12 lines are 1 inch*.
6. The yard in use at the U. S. Custom-Houses is divided into tenths and hundredths.
7. The *geographical mile*, *nautical mile*, or *knot*, is 6,086.7 ft. or 1.152 statute miles. These terms are applied as follows:—*geographic mile* to longitude; *nautical mile* to sea distances; *knot* to rate of speed of vessels.
8. A *marine league* is 3 nautical miles or 3.458 statute miles.
9. A *degree of latitude* as adopted by the U. S. Coast Survey is .69.16 statute miles.
10. Formerly 40 square rods were called a *rood*, and 4 *roods* an *acre*. The denomination *rood*, common in old deeds, mortgages, and surveys, is now but little used.
11. A pile of wood, earth, or rough stone 8 feet long, 4 feet wide, and 4 feet high, is a *cord*. Hence, *128 cu. ft. are 1 cd.* In many places wood is sold by *face measure*, i. e., 8 ft. long and 4 ft. high—or 32 feet of face of pile—are 1 cd.
12. There is no legal standard, neither State nor National, for a *perch of masonry*.
13. A *peach basket* should hold 21 qt. stricken measure, and a *potato basket*, 3 pk.
14. By many retailers, dry measure is called *wooden measure*, and liquid measure is called *tin measure*.

ix. Troy. { 1 lb. 1 oz. 1 pwt. 1 gr.
12 20 24

x. Apothecaries'. { lb. i oz. i or ʒ i dr. i or ʒ i sc. i or ʒ i gr. i
xii viii iii xx

xi. Assayers'. { 1 lb. 1 carat 1 carat gr. 1 Troy gr.
24 4 60

xii. Lapidaries'. { 1 lb. 1 carat 1 diamond gr. 1 part 1 Troy gr.
18 4 16 .05

xiii. Avoirdupois. { 1. Net ton. { { 1 T. 1 lb. 1 oz.
2,000 16
Or { 1 T. 1 cwt. 1 lb. 1 oz.
20 100 16
2. Gross ton. { 1 T. 1 cwt. 1 qr. 1 lb. 1 oz.
20 4 28 16

2 A

129. MEASURES OF DURATION.

xvi. Time.	{ 1 C.	1 yr.	1 da.	1 h.	1 min.	1 sec.
		100	{ 365 or 366	24	60	60

Denominations and units not in the scale: { 7 da. are 1 wk.;
28 to 31 da. are 1 mo.;
12 mo. are 1 yr.

	Seasons	Months	Abbreviations	No. of month in civil year	No. of days in each month	No. of days in Seasons
§ 1. The year is divided into	Spring;	March,	Mar.	3rd	31	92
		April,	Apr.	4th	30	
		May,		5th	31	
	Summer;	June,		6th	30	92
		July,		7th	31	
		August,	Aug.	8th	31	
	Autumn;	September,	Sept.	9th	30	91
		October,	Oct.	10th	31	
		November,	Nov.	11th	30	
	Winter.	December,	Dec.	12th	31	90 or 91
		January,	Jan.	1st	31	
		February,	Feb.	2nd	28 or 29	

NOTES.—1. The Society of Friends use the numbers of the months instead of the names; as, 1st mo., 2d mo., 3d mo., etc. They also use the numbers of the days of the week instead of the names; as, 1st day, 2d day, 3d day, etc.

2. In most business transactions 30 days are considered 1 month.

3. The *civil day* begins and ends at 12 o'clock, midnight. A.M. denotes time between midnight and noon; M., noon; and P.M., between noon and midnight.

§ 2. Bissextile or Leap-year.

1. The *civil year* is the year adopted by governments for the computation of time, and is of two kinds— { Common year, 365 da.
Leap-year, 366 da.

2. A *tropical or solar year* is 365 da. 5 h. 48 min. 49.7 sec.

NOTE.—The tropical or solar year is the *true year*, and is the time of the earth's revolution from the vernal equinox to the same equinox again.

3. If every year were reckoned as a common year, civil time would gain on solar time

In 1 year,	5 h. 48 min. 49.7 sec. or	11 min. 10.8 sec. less than	½ da.
4 “	23 “ 15 “ 18.8 “ or	44 “ 41.2 “ “ “	1 “
400 “	96 da. 21 “ 31 “ 20 “ or	3 da. 2 h. 28 “ 40 “ “ “	100 “
4,000 “	968 “ 23 “ 18 “ 20 “ or	31 “ 0 “ 46 “ 40 “ “	1,000 “

4. If every fourth year were reckoned as leap-year,

In 4 years solar time would gain on civil time 44 min. 41.2 sec.
 400 " " " " " " " " 8 da. 2 h. 28 " 40 "

NOTE.—If three centennial years of every four be reckoned as common years, the error of 8 da. in time gained in 400 yr., as shown above, will be corrected.

5. If every fourth year not centennial and every fourth centennial year were reckoned as leap-years,

In 400 years solar time would gain on civil time 2 h. 28 min. 40 sec.
 4,000 " " " " " " " " 1 da. 0 " 46 " 40 "

6. If every fourth year not centennial, and every fourth centennial year excepting each 4,000th year be reckoned as a leap-year, the error of 1 day in the time gained, as shown above, will be corrected, and civil time will vary from solar time only 46 min. 40 sec. in 4,000 years. Hence,

I. *Every fourth year, centennial years excepted, is a leap-year.*

II. *Every fourth centennial year, the years exactly divisible by 4,000 excepted, is a leap-year.*

§ 3. Old Style and New Style.

1. The Julian Calendar, decreed by Julius Cæsar B.C. 46, assumed the civil year to be 365 da. 6 h., one entire day being added every four years, and the 6 hours rejected.

2. A.D. 1582, the civil year having gained on the solar year 10 days, Pope Gregory XIII decreed that the day following the 3d of October in that year should be called the 14th, and published the Gregorian Calendar for all Catholic countries.

3. This calendar has since been adopted or is used by all civilized nations except Russia, whose dates are now 12 days behind ours.

4. The manner of reckoning time by the Julian Calendar is called Old Style (O. S.), and by the Gregorian Calendar, New Style (N. S.).

In O. S. the year began March 25; in N. S. it begins January 1.

5. England adopted the Gregorian Calendar A.D. 1752, the dates of the civil year then being 11 days behind those of the solar year.

130. MEASURES OF MONEY VALUE.

DECIMAL.	xvii. U. S.	{ 1 e. 1 dol. 1 dime 1 ct. 1 m.
		10 10 10 10
	xviii. Canada.	{ 1 dol. 1 ct.
		100
	xix. France.	{ 1 fr. 1 dc. 1 ct. 1 m.
		10 10 10
	xx. Sterling.	{ £1 1s. 1d. 1 far.
		20 12 4
	xxi. German.	{ 1 mark 1 pfennig
		100

Foreign Currencies.

English Money.			French Money.			German Money.		
Denomination	Symbol		Denomination	Symbol		Denomination	Symbol	Value in U. S.
Pound,	£		Franc,	fr.		Metal Coin		
Shilling,	s.		Decime,	dc.		{ Silver		
Penny,	d.		Centime,	ct.		{ 50-cent,		\$.462
Farthing,	far.		Millime,	m.		{ 25-cent,		.281
						{ 10-cent,		.092
						{ 5-cent,		.046
						{ Bronze, 1-cent,		.01
Metal Coin	Value in U. S.		Metal Coin	Value in U. S.				
{ Gold			{ Gold			{ Denomination	Symbol	
{ Sovereign,	\$4.8665		{ 100-franc,	\$19.80		{ Reichsmarken,	mark	
{ Half-sovereign,	2.433		{ 40-franc,	7.72		{ Pfennig,	pfennig	
{ Guinea,	5.11		{ 20-franc,	3.86				
{ Half-guinea,	2.555		{ 10-franc,	1.93		{ Metal Coin	Val. in U. S.	
			{ 5-franc,	.965		{ 20-mark,	\$4.77	
{ Silver			{ 5-franc,	.965		{ 10-mark,	2.885	
{ Crown,	1.216		{ 2-franc,	.886		{ 5-mark,	1.192	
{ Half-crown,	.608		{ 1-franc,	.193		{ 2-mark,	.477	
{ Florin,	.486		{ 50-centime,	.096		{ 1-mark,	.2885	
{ Shilling,	.243		{ 25-centime,	.048		{ 20-pfennig,	.0477	
{ Sixpence,	.12					{ 10-pfennig,	.02385	
{ Fourpence,	.08					{ 5-pfennig,	.012	
{ Threepence,	.06					{ 2-pfennig,	.0048	
{ Penny,	.02					{ 1-pfennig,	.0024	
{ Half-penny,	.01							
{ Farthing,	.005							

NOTES.—1. The following-named coins, though still in circulation, are no longer coined :—

English,—guinea and half-guinea; **Canada**,—20-cent, or shilling (silver);

German,—silver thaler (\$.746), and groschen (\$.025).

2. The gold coins used in Canada are the English sovereign and half-sovereign.

3. French money is read in francs and centimes, as U. S. money is read in dollars and cents.

4. The French 20-franc piece is equal in value to the Napoleon—still in circulation.

131. ENUMERATION, COUNT, OR TALE.

xxii. Counting.	{	1 gt. gr.	1 gro.	1 doz.	1 unit
		12	12	12	
	{	Also { 1 C	1 score	1 unit	{ 2 things of a kind are a pair, and 6 things of a kind are a set.
		5	20	In some cases	
xxiii. Paper.	{	1 B.	1 bun.	1 rm.	1 quire
		5	2	20	1 sheet

Writing Paper—Flat.

Names	Dimensions	Names	Dimensions	Names	Dimensions
Letter,	10 × 16 in.	Royal,	19 × 24 in.	Colombier,	23 × 34 in.
Law blank,	18 × 16 "	Super royal,	20 × 28 "	Double royal,	{ 24 × 38 "
Flat cap,	14 × 17 "	Elephant,	23 × 28 "		{ 19 × 48 "
Crown,	15 × 19 "	Imperial,	23 × 31 "	Atlas,	26 × 33 "
Demy,	16 × 21 "	Double demy,	{ 21 × 32 "	Double elephant,	27 × 40 "
Double cap,	17 × 28 "		{ 16 × 42 "	Antiquarian,	31 × 53 "
Medium,	18 × 23 "	Double medium,	{ 23 × 36 "	Emperor,	48 × 72 "
			{ 18 × 46 "		

Writing Paper—Folded.

Names	Dimensions
Billet note,	6 × 8 in.
Octavo note,	7 × 9 "
Commercial note,	8 × 10 "
Packet note,	9 × 11 "
Bath note,	8½ × 14 "
Letter,	10 × 16 "
Commercial letter,	11 × 17 "
Packet post,	11½ × 18 "
Extra packet post,	11½ × 18½ "
Foolscap,	12½ × 16 "
Legal cap,	12½ × 16 "

Print Paper—Used for newspapers and books.

Names	Dimensions
Medium,	19 × 24 in.
Royal,	20 × 25 "
Super royal,	22 × 28 "
Imperial,	22 × 32 "
Medium and half,	24 × 30 "
Small double medium,	24 × 36 "
Double medium,	24 × 38 "
Double royal,	26 × 40 "
Double super royal,	{ 28 × 42 "
	{ 29 × 43 "
Broad twelves,	23 × 47 "
Double imperial,	32 × 46 "

NOTES.—1. The sizes of paper given in the table are regular or trade sizes.

Names	Each sheet of paper is folded into
xxiv. Books.	Folio, 2 leaves.
	Quarto, 4 "
	Octavo or 8vo, 8 "
	Duodecimo or { 12 "
	12mo, }
	16mo, 16 "
	18mo, 18 "
	24mo, 24 "
	32mo, 32 "
	64mo, 64 "

2. The sizes of books are known by a combination of the name of the sheet of paper used and the book class.

e. g. A medium octavo, or an octavo medium; a royal quarto, etc.

3. The marks A, B, C; 1, 2, 3; 1*, 2*, etc., found at the bottom of pages of books, are termed *signatures*, and are for the direction of binders in folding the sheets.

4. The thickness of paper is described by the weight per ream,—as 40 lb. medium; 38 lb. medium.
5. A *printers' thousand*, sometimes called a *printers' token*, is 2 reams or 960 sheets.
6. Some paper manufacturers now sell paper in packages of 500 sheets instead of the ream.

xxv. MISCELLANEOUS DENOMINATIONS.

	Units	Uses
Linear.	1 Barleycorn	} For grading the sizes of boots and shoes.
	1 in.	
	1 Hand	} For measuring the height of horses.
	4 in.	
Capacity.	1 Cubit	} A unit of length mentioned in the Bible.
	21.666 in.	
	1 Pace	} For measuring short distances approximately.
	3 ft.	
Capacity.	1 Stricken Bushel	} For measuring grains, seeds, peas, beans, small fruits, etc. (U. S. standard bushel.)
	2150.42 cu. in.	
	1 Heaped Bushel	} For measuring corn in the ear, coarse fruits, vegetables, lime, coal, etc.
	1½ stricken bu.	
	1 Barrel	} The standard of exchange and transportation for petroleum and its products.
	42 gal.	
	100 qt. dry meas.	} For apples and potatoes.
	81.5 gal.	
	1 Hogshead	} For estimating the capacity of vats, cisterns, reservoirs, etc.
	63 gal.	
	1 Pipe	} Original package of Madeira wine;
	110 gal.	
	120 gal.—Barcelona, Vidonia, or Teneriffe wine;	
	180 gal.—Sherry wine;	
	186 gal.—Port wine;	
	140 gal.—Lisbon wine.	

	Units	Uses
Volume.	1 Perch 24.75 cu. ft.	} For estimating rough building-stone.
	1 Load 1 cu. yd.	
	1 Chaldron 36 bu.	} For wholesaling charcoal.
	1 Cord 128 cu. ft.	
	1 Ton 100 cu. ft. 40 cu. ft.	For wood, earth, undressed and refuse stone, manure, etc. } For estimating the carrying capacity of ships and other sea-going vessels. } For estimating amount of freight on vessels, and in making freight charges.
Weight.	1 Barrel 196 lb. 200 lb.—Of salt meats and fish; 280 lb.—Of salt. (New York standard.)	} For the packing, sale, and transportation } Of flour;
	1 Cental 100 lb.	
	1 Quintal 100 lb.	} For grain. Used by some wholesale dealers and shippers. } A measure for dry, salt fish.
	1 Keg 100 lb.	
		} A measure for nails.

xxvi. STANDARD UNITS.

	Unit	Value	Derived from
U. S. Measures of	Extension; { <i>Linear,</i> { <i>Superficial,</i> { <i>Solid.</i>	Yard, 36 in.	} Scale of Troughton.
	Capacity; { <i>Liquid,</i> { <i>Dry.</i>	Gallon, 231 cu. in. Bushel, 2,150.42 cu. in.	
	Weight; { <i>Troy,</i> { <i>Apothecaries',</i> { <i>Avotirdupois.</i>	Pound, 5,760 gr. Pound, 7,000 gr.	} A cubic inch of distilled water that weighs 252.458 gr.
	Money;	Dollar, 412½ gr.	
			Standard silver.

British Measures of			Unit	Value	Derived from
	Extension;	$\left\{ \begin{array}{l} \text{Linear,} \\ \text{Superficial,} \\ \text{Solid.} \end{array} \right\}$	Yard,	36 in.	The restored Stand-ard Yard.
	Capacity;	$\left\{ \begin{array}{l} \text{Liquid,} \\ \text{Dry.} \end{array} \right\}$	Imperial Gallon,	277.274 cu. in.	
			Imperial Bushel,	2,218.193 cu. in.	
	Weight;	$\left\{ \begin{array}{l} \text{Troy,} \\ \text{Apothecaries',} \\ \text{Avoirdupois.} \end{array} \right\}$	Pound,	5,760 gr.	A cubic inch of dis-tilled water that weighs 252.458 gr.
			Pound,	7,000 gr.	
Money;			$\left\{ \begin{array}{l} \text{Pound or} \\ \text{Sovereign,} \end{array} \right\}$	123 $\frac{1}{4}$ gr.	Gold 23 carats fine.

NOTES.—1. The *unvarying standard unit* from which all measures and weights of both the English and U. S. governments are derived, is the day of 24 h.; the linear standard yard being $\frac{1}{3600000}$ of the length of a pendulum that vibrates once a second in a vacuum, at the level of the sea, in the latitude of London.

2. The impossibility of absolute accuracy in the application of the foregoing standard makes its use impracticable; hence, each of the governments that has adopted it, has provided itself with a metal standard linear measure which is securely kept in government depositories.

3. The *Scale of Troughton* is a brass rod or scale 82 inches long, prepared for the U. S. Coast Survey by the celebrated English mechanician, Troughton. It is the standard with which U. S. standard units are compared.

4. The *Imperial standard yard of Great Britain* was destroyed at the burning of the Houses of Parliament in 1834. The *Restored standard yard* consists of a solid bar of bronze, 36 inches long and 1 inch square. Near each end of this bar is a hole about one half-inch deep, and the distance between the centres of these holes is 36 inches, or the *Imperial yard*.

5. The old English standard of dry measure was the *Winchester bushel* of 2,150.42 cu. in., so called from the standard bushel measure kept at Winchester. The *U. S. standard bushel* is of the same dimensions as the Winchester standard, viz., a cylinder $18\frac{1}{4}$ inches in diameter and 8 inches deep, inside measurements.

6. The *U. S. standard for gold coin* is 25.8 gr. of standard gold to the dollar.

§ 5. Standard Sets of Measures and Weights.

In 1842, the General Government furnished to each of the States a uniform set of measures and weights, consisting of

1. A yard;
2. A wine gallon and its subdivisions;
3. A half-bushel and its subdivisions;
4. A set of Troy weights;
5. A set of avoirdupois weights.

County authorities receive, from the State Government, sets of standard measures and weights, each set consisting of

1. A *brass yard measure* graduated on one side to feet and inches (the first foot graduated to tenths and also to eighths of an inch), and on the opposite side to cloth measure.
 2. A *set of liquid measures* (copper);—1 gal., $\frac{1}{2}$ gal., 1 qt., 1 pt., $\frac{1}{2}$ pt., 1 gi.
 3. A *set of dry measures* (copper);— $\frac{1}{2}$ bu., 1 pk., $\frac{1}{2}$ pk., 2 qt., 1 qt.
 4. A *large brass balance, with set of brass weights*, 50 lb., 20 lb., 10 lb., and 5 lb.
 5. A *small brass balance, with set of brass weights*, 4 lb., 2 lb., 1 lb., 8 oz., 4 oz., 2 oz., 1 oz., $\frac{1}{2}$ oz., $\frac{1}{4}$ oz.
-

132. METRIC SYSTEM OF MEASURES AND WEIGHTS.

Page 38; Art. 132.

§ 1. In the *metric system* the *meter* is the standard unit for all measures, whether of extension, weight, or money value; and the successive multiples and divisors of this unit are in the decimal scale.

NOTES.—1. The term *meter* means measure.

2. An endeavor to select from nature an unchangeable, scientific, standard unit for the metric system, led to the adoption of one ten-millionth part of the distance, on the earth's surface, between the equator and the north pole, as such unit.

In the attempt to conform to this plan, the French Government made an accurate survey of the part of the terrestrial meridian between Dunkirk and Barcelona; and from this survey was computed the length of the entire meridian, and the consequent length of the meter or standard unit of the French metric system.

It was afterward found that an error had been made in the computation; and the United States Government, upon authorizing the use of a metric system, attempted to correct the error. Later computations show the U. S. standard meter incorrect also. Hence,

The only correct scientific part of the metric systems now in use is their uniform decimal scale.

§ 2. The cube of one tenth of the meter is the unit of measures of volume. This unit is denominated *liter*.

§ 3. The weight of distilled water at its greatest density (39° Fahrenheit) which this cube will contain is called a kilogram, of which one thousandth part is the unit of weight. This unit is denominated *gram*.

§ 4. The *meter* is one ten-millionth of the distance between the equator and the pole.

§ 5. The *liter* is a cube whose edge is one tenth of a meter.

§ 6. The *gram* is the weight of a cube of distilled water whose edge is one hundredth of a meter.

§ 7. The multiples of these units are distinguished by placing before their names the Greek prefixes *Deka*, *Hekto*, *Kilo*; and the divisors, by prefixing the Latin numerals *deci*, *centi*, *milli*.

§ 8. Metric Nomenclature.

Multiple Prefixes.			Standard Units.			Divisor Prefixes.		
Spelling	Pronunciation	Symbol	Spelling	Pronunciation	Symbol	Spelling	Pronunciation	Symbol
Deka	dēka	D.	meter	mēter	m.	deci	dēsi	d.
Hekto	hēktō	H.	liter	lēter	l.	centi	sēnti	c.
Kilo	kilō	K.	gram	grām	g.	milli	mīli	m.
Myria	mīria	M.	ster	stayer	st.			

Deka signifies 10; hekto, 100; kilo, 1,000; myria, 10,000; deci, .1; centi, .01; and milli, .001. Hence,

§ 9. In writing metric numbers, the names of the places used correspond in meaning to the English names of the same places used in writing mixed decimal numbers,—as shown below:

Names of places used in writing a metric number:	Kilo	Hekto	Deka	{ ones or units	deci	centi	milli
	0	0	0	0.	0	0	0
Names of the same places used in writing a mixed decimal number:	thousands	hundreds	tens	{ ones or units	tenths	hundredths	thousandths
	0	0	0	0.	0	0	0

§ 10. All the multiples and divisors are in the scale of 10, and may be written thus:

	Km.	Hm.	Dm.	m.	dm.	cm.	mm.
Extension.—	7	6	5	4.	3	2	1 meters.
	Kl.	Hl.	Dl.	l.	dl.	cl.	ml.
Volume.—	7	6	5	4.	3	2	1 liters.
	Kg.	Hg.	Dg.	g.	dg.	cg.	mg.
Weight.—	7	6	5	4.	3	2	1 grams.

These numbers may be read, giving name to each unit. Thus, 7 Km. 6 Hm. 5 Dm. 4 m. 3 dm. 2 cm. 1 mm., etc.

In the same manner read

1. The numbers expressing volume.
2. The numbers expressing weight.

§ 11. Any one of these numbers may be expressed in units of any one of its denominations, and read as ones and decimals of that denomination; *e.g.*,

The number expressing extension may also be read in the denomination of the standard unit (meters) thus: 7,654 and 321 thousandths meters.

Again, this number may be expressed either as millimeters, centimeters, decimeters, meters, dekameters, hektometers, or kilometers, as here shown, and read accordingly.

{	7	6	5	4	3	2	1	mm.	
	7	6	5	4	3	2	cm.		.1
		7	6	5	4	3	dm.		.21
			7	6	5	4	m.		.321
				7	6	5	Dm.		.4321
					7	6	Hm.		.54321
						7	Km.		.654321

The Three Standard Units.

§ 12. Surfaces and volumes are simply the squares and cubes of the measure of length; the *ar* is only another name for the square dekameter or 100 square meters of land; the *ster* is only another name for the cubic meter of fire-wood; and the *tonneau* is only another name for the weight of a cubic meter of water. Hence,

Every possible dimension (length, area, volume), can be measured with the meter; every possible capacity, with the liter; and every possible weight, with the gram.

§ 13. The various metric units—standard and derived—should be used in measurements and computations or estimates as follows:

- i. *Meter*.—All short distances; as feet and yards.
- ii. *Kilometer*.—All long distances; as miles.
- iii. *Liter*.—Liquids, small fruits, etc., now measured by the quart and gallon.
- iv. *Dekaliter*.—Grains, fruits, vegetables, etc., now measured by the peck and bushel.
- v. *Hektoliter*.—Liquids and dry substances now measured by the barrel and hogshead.
- vi. *Gram*.—Medicines and small articles now weighed by the grain.
- vii. *Kilogram*.—Articles now weighed by the avoirdupois pound.
- viii. *Milligram*.—Very precious and very delicate articles.

§ 14. *In measuring and weighing,*

Measure all dimensions in *meters*; all capacities in *liters*; and all weights in *grams*, using decimals only.

§ 15. *In computations, use*

1. *Ar* and *hectar* for area of land;
2. *Square meter*, *square kilometer*, *square decimeter*, and *square centimeter* for other areas;
3. *Ster* for solidity of wood;
4. *Cubic meter*, *cubic decimeter*, and *cubic centimeter* for other volumes; and
5. *Tonneau* for the weight of heavy articles now estimated by the ton.

§ 16. *Approximate Equivalents.*

Meter—a little less than 1.1 yd.

Liter—about $\frac{1}{4}$ dry qt., or $1\frac{1}{4}$ liquid qt.

Gram—about $\frac{1}{16}$ oz. avoirdupois.

Kilogram (or **Kilo**)—about 2.2 lb. avoirdupois.

The scale below is a **decimeter**, or .1 of a meter long; and each division of the scale is a **centimeter**, or .01 of a meter long.



§ 17. *Table of Comparison of Standard Metric and English Units.*

Length.	{ Meter,	39.37 in.	Yard,	.914 $\frac{1}{2}$ meter.
	{ Kilometer,	{ .62137 mi., or 3,280 ft. 10 in. }		Mile,
Area.	{ Ar,	119.6 sq. yd.	Square yard,	.836 square meter.
	{ Hectar,	2.471 A.		Acre,
Solidity.	{ Ster or	1.308 cu. yd.	Cubic yard,	{ .764 $\frac{1}{2}$ cubic meter. (ster).
	{ Cubic meter,			
Capacity.—Liter,	{	1.0567 liquid qt.	Liquid quart,	.946 $\frac{1}{2}$ liter.
	{	.908 dry qt.		Dry quart,
Weight.	{ Gram,	15.432 gr. Troy.	Troy grain,	.064 $\frac{1}{2}$ gram.
	{ Kilo,	2.2046 lb. avoird.		Avoird. pound,
	{ Tonneau,	2,204.6 lb.	Ton { net, gross.	.907 tonneau. 1.016 tonnes.

This Table is correct to four decimal places.

The exact equivalents of metric units and denominate units, each in terms of the other, are given in the following

§ 18. Exact Equivalents.

	<i>Metric</i>	=	<i>Denominate</i>	<i>Denominate</i>	=	<i>Metric</i>
Linear.	1 cm.	=	.8937 in.	1 in.	=	.0254 m.
	1 dm.	=	.328 ft.	1 ft.	=	.3048 m.
	1 m.	=	1.0936 yd.	1 yd.	=	.9144 m.
	1 Dm.	=	1.9884 rd.	1 rd.	=	5.029 m.
	1 Hm.	=	19.884 rd.	1 ml.	=	1.6093 Km.
	1 Km.	=	.62137 mi.			
Solid.	1 cu. cm.	=	.061 cu. in.	1 cu. in.	=	16.387 cu. cm.
	1 cu. dm.	=	.0353 cu. in.	1 cu. ft.	=	28.317 cu. dm.
	1 cu. m.	=	1.308 cu. yd.	1 cu. yd.	=	.7645 cu. m.
	1 ster	=	.2759 cd.	1 cd.	=	3.624 sters.
Weight.	1 mg.	=	.01543 gr.	1 oz. Troy	=	31.104 g.
	1 g.	=	.03215 oz. Troy	1 oz. avoird.	=	28.35 g.
		=	.03527 oz. avoird.	1 lb. Troy	=	.3732 Kg.
	1 Kg.	=	2.2046 lb. avoird.	1 lb. avoird.	=	.4536 Kg.
	1 tonneau	=	1.1023 tons	1 T.	=	.9072 tonneau
Area.	1 sq. cm.	=	.155 sq. in.	1 sq. in.	=	6.452 sq. cm.
	1 sq. dm.	=	.1076 sq. ft.	1 sq. ft.	=	.0929 sq. m.
	1 sq. m.	=	1.196 sq. yd.	1 sq. yd.	=	.8361 sq. m.
	1 ar	=	8.954 sq. rd.	1 sq. rd.	=	25.293 sq. m.
	1 Ha.	=	2.471 A.	1 A.	=	40.47 ars
	1 sq. Km.	=	.3861 sq. mi.	1 sq. mi.	=	259 Ha.
Capacity.	1 l.	=	1.0567 qt. Liq.	1 liq. qt.	=	.9463 l.
		=	.908 qt. Dry	1 dry qt.	=	1.101 l.
	1 Dl.	=	2.6417 gal. Liq.	1 gal.	=	.8785 Dl.
	1 Dl.	=	1.135 pk.	1 pk.	=	.881 Dl.
	1 Hl.	=	2.8375 bu.	1 bu.	=	.8524 Hl.

§ 19. Metric Dimensions and Metric Weights of American Coins.

1. **Dimensions.**—The 5-cent nickel measures .03 meter (= .2 centimeter) in diameter, and .002 meter (= .2 millimeter) in thickness.

2. **Weights.**—The 5-cent nickel weighs 5 grams; Silver coin weighs 4 cents to the gram; \$1 weighs 25 grams, and \$10 of nickel coin weigh 1,000 grams, or 1 kilo. \$40 weigh 1,000 grams or 1 kilo.

The sum of the diameters of 50 nickels,
also the sum of the thickness of 500 nickels, } is 1 meter.

§ 20. Since the values of the successive units of any number expressing metric measure or weight are in the scale of 10, the writing and reading of any such number is simply an exercise in decimal notation and numeration. Hence, properly speaking,

In the metric system there are no tables, no varying scales, no complicated relations of the different orders of units.

PROBLEMS.

In interchanging metric numbers and compound numbers, the table of approximate equivalents is sufficiently accurate when the numbers are small and of ordinary value. For larger numbers and greater values, use the table of exact equivalents.

Reduce

- | | |
|----------------------------------|--|
| 1. 27 ^{Dm} . to rods. | 6. 46 lb. to kilograms. |
| 2. 12 ^{Kg} . to pounds. | 7. 13.5 bu. to hectoliters. |
| 3. 9 ^m . to yards. | 8. 3.895 qt. (liquid measure) to liters. |
| 4. 53 ^{Dl} . to pecks. | 9. 175 $\frac{1}{4}$ A. to hectara. |
| 5. 20 ^{Km} . to miles. | 10. 2 ft. 7 in. to decimeters. |

11. The railroad distance from New York to Philadelphia is 92 miles. What is the distance in kilometers?

12. An invoice of 5 pieces of French laces showed the numbers of meters as follows:—97.875, 84.82, 103.7, 97.85, and 73.125. How many yards of lace were in the invoice?

Change to the corresponding metric unit

- | | |
|---------------------------|----------------------------------|
| 13. 137 lb. 9 oz. avoird. | 17. 92 mi. 116 rd. 12 ft. 5 in. |
| 14. 6 lb 7 oz. 5 dr. | 18. 65 cd. 117 cu. ft. |
| 15. 132 bu. 1 pk. 5 qt. | 19. 84 A. 54 sq. rd. 216 sq. ft. |
| 16. 48 rd. 13 ft. 7 in. | 20. 9 hhd. 1 bar. 27 gal. 3 qt. |

21. How many 5-cent nickels placed edge to edge would reach a distance of 3 miles?

22. How much will 40 A. 90 sq. rd. of land cost @ \$137.50 per hectar?

23. What is the value of 119 $\frac{1}{4}$ yd. of broadcloth @ \$5.13 $\frac{1}{4}$ per meter?

24. How many suits of clothes, each requiring 7 $\frac{1}{4}$ yd. of cloth, can be cut from a piece containing 86.172^m?

25. Find the sum of 26.1783^m, 23.9654^{Dl}, and 4.86^{Dl}.

26. What is the sum of 34.87^{q. Km}, 491.517^{Ha}, .5265^a, and 846.75^{q. m}?

TESTS OF DIVISIBILITY BY 7, 11, AND 13.

Page 49; Art. 161.

1. Tests by 7. I. *Special test*—for small numbers.

7 is an exact divisor of any number whose right-hand digit is either one half or one ninth of the number expressed by the other digits; or, if the number expressed by the two right-hand digits is either one third or five times the number expressed by the remaining digits.

In 168, 8 is $\frac{1}{2}$ of 16; in 735, 35 is 5 times 7;

in 728, 28 is $\frac{1}{3}$ of 72; in 6,923, 23 is $\frac{1}{5}$ of 69.

Hence, 7 is an exact divisor of 168, of 728, of 735, and of 6,923.

II. *General test*.

7 is an exact divisor of any number, if the sum of the products of the ones, tens, and hundreds of the odd periods, multiplied respectively by 1, 3, and 2, equals the sum of the products of the ones, tens, and hundreds of the even periods multiplied respectively by the same multipliers; or, if the difference between such sums is a multiple of 7.

EXAMPLES.—Show by the General Test that 7 is an exact divisor

1. Of 423,325 | 2. Of 564,998 | 3. Of 51,687,405

1			PROCESSES.		2		
	Even Periods	Odd Periods		Even Periods	Odd Periods		
<i>ones,</i>	$3 \times 1 = 3$	$5 \times 1 = 5$	<i>ones,</i>	$4 \times 1 = 4$	$8 \times 1 = 8$		
<i>tens,</i>	$2 \times 3 = 6$	$2 \times 3 = 6$	<i>tens,</i>	$6 \times 3 = 18$	$9 \times 3 = 27$		
<i>hundreds,</i>	$4 \times 2 = 8$	$3 \times 2 = 6$	<i>hundreds,</i>	$5 \times 2 = 10$	$9 \times 2 = 18$		
	<u>Sums,</u>	<u>17</u>		<u>Sums,</u>	<u>32</u>	<u>53</u>	
				$53 - 32 = 21; 21 \div 7 = 3$			

The sums of the products are equal; hence, 7 is an exact divisor of the number.

The difference between the sums of the products is 21, a multiple of 7; hence, 7 is an exact divisor of the number.

3				
	Even Periods	Odd Periods		
ones,	$7 \times 1 = 7$	$5 \times 1 = 5$		
tens,	$8 \times 3 = 24$	$0 \times 3 = 0$		$43 - 29 = 14$
hundreds,	$6 \times 2 = 12$	$4 \times 2 = 8$		
ones,		$1 \times 1 = 1$		$14 \div 7 = 2$
tens,		$5 \times 3 = 15$		
hundreds,				
	<u>Sums,</u>	<u>43</u>		<u>29</u>

2. Tests by 11.

11 is an exact divisor of any number, if the sum of the digits in the odd places equals the sum of the digits in the even places; or, if the difference between such sums is a multiple of 11.

Ex. 1. In 14,377 the sums of alternate digits are equal ($\overbrace{14377}$). Hence, 11 is an exact divisor of the number.

Ex. 2. In 53,809,173 the difference between the sums of the alternate digits is 22,—a multiple of 11 ($\overbrace{53809173}$). Hence, 11 is an exact divisor of the number.

3. Tests by 13.

13 is an exact divisor of a number, if the sum of the products of the ones, tens, and hundreds of the odd periods, multiplied respectively by 1, 10, and 9, equals the sum of the products of the ones, tens, and hundreds of the even periods, multiplied respectively by the same multipliers; or, if the difference between such sums is a multiple of 13.

NOTE 1.—13 is an exact divisor of a number of but one period, if the sum of the products of the ones, tens, and hundreds of this period multiplied by 1, 10, and 9, as above, is a multiple of 13.

EXAMPLES.—Show by the test that 13 is an exact divisor

1. Of 334,243 | 2. Of 930,345

PROCESSSES.

1			2		
	Even Periods	Odd Periods		Even Periods	Odd Periods
ones,	$4 \times 1 = 4$	$3 \times 1 = 3$	ones,	$0 \times 1 = 0$	$5 \times 1 = 5$
tens,	$3 \times 10 = 30$	$4 \times 10 = 40$	tens,	$3 \times 10 = 30$	$4 \times 10 = 40$
hundreds,	$3 \times 9 = 27$	$2 \times 9 = 18$	hundreds,	$9 \times 9 = 81$	$3 \times 9 = 27$
	<u>Sums,</u>	<u>61</u>		<u>Sums,</u>	<u>72</u>
				$111 - 72 = 39$;	$39 \div 13 = 3$

The sums of the products are equal; hence, 13 is an exact divisor of the number.

The difference between the sums of the products is 39, a multiple of 13; hence, 13 is an exact divisor of the number.

NOTE 2.—The learner should have sufficient practice to become thoroughly familiar with these tests.

PRIME FACTORS.

Page 50; Art. 162.

To find prime factors greater than 11, in numbers less than 10,000, by means of the following table—

I. Cancel from the given number all prime factors from 2 to 11 inclusive.

II. Of the number remaining after these factors have been cancelled, cut off the two right-hand figures, and find in the table the division marked by the left-hand figure or figures in bold-face type.

III. In the column of factors in this division, and at the right of the figures cut off from the number, find a factor sought.

IV. Divide by this factor; and proceed with the quotient as before, until the resulting number can not be found in the table.

Hundreds		Thousands															
Numbers	Factors	Numbers	Factors	Numbers	Factors	Numbers	Factors	Numbers	Factors	Numbers	Factors	Numbers	Factors	Numbers	Factors	Numbers	Factors
1 69 13	93 13	41 17	49 17	63 13	63 17	47 41	77 17	97 43	49 23	69 13							
2 21 13	99 17	47 29	51 13	20 21 43	69 23	59 31	97 19	34 01 19	57 13	87 61							
47 13	8 17 19	61 13	79 23	38 19	24 07 29	71 17	31 03 29	08 41	63 53	97 17							
89 17	41 29	71 31	81 41	41 13	13 19	73 47	07 13	19 13	81 19	41 17 23							
99 13	51 23	73 19	91 19	47 23	19 41	28 09 53	27 53	27 23	91 17	21 13							
3 23 17	71 13	13 13 13	17 03 13	59 29	49 31	13 29	31 31	31 47	99 29	41 41							
61 19	93 19	33 31	11 29	71 19	61 23	31 19	33 13	89 19	38 09 13	63 23							
77 13	99 29	39 13	17 17	77 31	79 37	39 17	39 43	73 23	11 37	71 43							
91 17	9 01 17	43 17	39 37	21 17 29	88 13	67 47	49 47	81 59	27 43	81 37							
4 03 13	23 13	49 19	51 17	19 13	89 19	69 19	51 23	97 13	41 23	83 47							
37 19	43 23	57 23	63 41	47 19	91 47	73 13	61 29	35 08 31	59 17	87 53							
81 13	49 13	63 29	69 29	59 17	25 01 41	81 43	73 19	23 13	69 53	89 59							
93 17	61 31	69 37	81 13	71 13	07 23	99 13	93 31	51 53	87 13	99 13							
5 27 17	89 23	87 19	18 07 13	73 41	09 13	29 11 41	97 23	69 43	93 17	42 23 41							
29 23	10 03 17	91 13	17 23	83 37	33 17	21 23	32 11 13	87 17	39 01 47	87 19							
33 13	07 19	14 03 23	19 17	97 13	37 43	23 37	33 53	89 37	37 31	47 31							
51 19	27 13	11 17	29 31	22 01 31	61 13	29 29	39 41	99 59	53 59	67 17							
59 13	37 17	17 13	43 19	09 47	67 17	41 17	47 17	36 01 13	59 37	43 03 13							
89 19	73 29	57 31	49 43	27 17	73 31	51 13	63 13	11 23	61 17	07 59							
6 11 13	79 13	69 13	53 17	81 23	81 29	77 13	77 29	29 19	73 29	09 31							
29 17	81 23	15 01 19	91 31	49 13	87 13	83 19	81 17	49 41	77 41	13 19							
67 23	11 21 19	13 17	19 09 23	57 37	99 23	87 29	87 19	53 13	79 23	21 29							
89 13	39 17	17 37	19 19	63 31	26 03 19	93 41	93 37	67 19	91 13	31 61							
97 17	47 31	37 29	21 17	79 43	23 43	30 07 31	33 17 31	79 13	40 09 19	43 43							
7 03 19	57 13	41 23	27 41	91 29	27 37	13 23	37 47	83 29	31 29	51 19							
13 23	59 19	77 19	37 13	23 23 23	41 19	29 13	41 13	37 13 47	33 37	69 17							
81 17	89 29	91 37	43 29	27 13	69 17	43 17	49 17	21 61	43 13	79 29							
67 13	12 07 17	16 33 23	57 19	29 17	27 01 37	53 43	79 31	37 37	61 31	81 13							
79 19	19 23	43 31	61 37	53 13	43 13	71 37	83 17	48 19	63 17	87 41							

Thousands															
Numbers	Factors	Numbers	Factors	Numbers	Factors	Numbers	Factors	Numbers	Factors	Numbers	Factors	Numbers	Factors	Numbers	Factors
93 23	4901 13	61 43	89 53	11 17	61 23	19 73	23 71	85 07	47	61 13	71 17				
99 53	13 17	73 13	93 13	27 61	67 37	81 17	27 23	09 67	71 47	77 61					
44 27 19	27 13	91 17	60 01 17	33 47	81 73	43 19	33 29	31 19	73 43	89 43					
29 43	79 13	97 23	19 13	39 13	87 19	71 67	47 13	49 83	77 29	93 53					
39 23	81 17	55 13 37	23 19	41 31	93 41	97 71	51 83	51 17	83 31	99 29					
53 61	97 19	39 29	81 87	57 79	97 47	76 13 23	77 41	57 43	89 61	96 07 13					
69 41	50 17 29	43 23	49 23	83 29	99 31	19 19	83 59	67 13	91 01 19	17 59					
71 17	29 47	49 31	59 73	93 19	71 11 13	27 29	81 19 23	79 23	13 13	37 23					
89 67	41 71	61 67	71 13	66 13 17	23 17	81 13	81 47	87 81	81 23	41 31					
45 11 13	53 31	67 19	77 59	17 13	41 87	83 17	87 79	93 13	39 13	59 13					
31 23	57 13	87 37	61 03 17	23 37	53 23	57 13	43 17	86 11 79	43 41	71 19					
37 13	63 61	97 29	07 31	31 19	57 17	61 47	49 29	21 87	67 89	73 17					
41 19	69 37	56 03 13	09 41	41 29	63 13	63 79	53 31	83 89	69 53	83 23					
53 29	83 13	09 71	19 29	47 17	69 67	97 43	59 41	89 53	79 67	97 01 89					
59 47	51 11 19	11 31	37 17	49 61	71 71	77 09 13	77 13	51 41	93 29	03 31					
73 17	23 47	17 41	57 47	67 59	81 43	29 59	89 19	53 17	97 17	07 17					
77 23	29 23	27 17	61 61	83 41	99 23	39 71	82 01 59	71 13	92 11 61	27 71					
79 19	41 53	29 13	69 31	97 37	72 01 19	47 61	03 13	83 19	17 13	31 37					
89 13	43 37	33 43	79 37	67 07 19	23 31	51 23	07 29	87 11 31	23 23	61 43					
46 01 43	49 19	71 53	87 23	81 53	41 13	69 17	13 43	17 23	53 19	63 13					
07 17	61 13	81 13	91 41	39 23	61 53	71 19	27 19	49 13	59 47	73 29					
19 31	77 31	99 41	62 27 13	49 17	67 13	81 31	49 73	59 19	63 59	97 97					
33 41	83 71	57 07 13	33 23	51 43	77 19	83 43	51 37	73 31	69 13	99 41					
61 59	91 29	13 29	39 17	57 29	79 29	87 13	57 23	77 67	71 73	98 09 17					
67 13	52 07 41	23 59	41 79	67 67	89 37	78 04 29	79 17	91 59	87 37	27 31					
81 31	13 13	29 17	53 13	73 13	91 23	07 37	99 43	97 19	99 17	41 13					
87 43	19 17	59 13	83 61	99 13	73 08 67	11 73	83 03 19	88 01 13	33 01 71	47 43					
93 13	21 23	67 73	89 19	68 17 17	13 71	13 13	21 53	09 23	07 41	53 59					
99 37	39 13	71 29	63 13 59	21 19	19 13	31 41	33 13	43 37	13 67	69 71					
47 09 17	49 29	73 23	19 71	47 41	27 17	37 17	39 31	51 53	22 19	81 41					
17 53	51 59	77 53	31 13	51 13	39 41	49 47	41 19	57 17	47 13	93 13					
27 29	63 19	58 09 37	41 17	59 19	61 17	59 29	47 17	73 19	53 47	99 19					
47 47	67 23	33 19	71 23	77 13	63 37	71 17	57 61	79 13	67 17	99 13 23					
57 67	87 17	37 13	83 13	87 71	67 53	91 13	53 13	81 83	79 83	17 47					
69 19	93 67	91 43	64 01 37	89 83	73 73	97 53	81 17	91 17	89 41	37 19					
71 13	53 11 47	93 71	03 19	93 61	79 47	79 13 41	33 83	89 03 29	64 07 23	43 61					
77 17	17 13	99 17	07 43	69 01 67	87 83	21 89	99 37	09 59	09 97	53 37					
48 11 17	21 17	59 09 19	09 13	13 31	91 19	39 17	84 01 31	17 37	51 13	59 23					
19 61	29 73	11 23	31 59	29 13	97 13	43 13	11 41	27 79	69 17	71 13					
41 47	39 19	17 61	37 41	31 29	74 09 31	57 73	13 47	47 23	81 19	79 17					
43 29	53 53	21 31	39 47	43 53	21 41	61 19	17 19	57 13	87 53	83 67					
47 37	59 23	33 17	43 17	53 17	23 13	67 31	41 23	59 17	95 03 13	91 97					
49 13	63 31	41 13	63 23	73 19	29 17	69 13	53 79	77 47	09 37	97 13					
53 23	71 41	47 19	67 29	89 29	39 43	79 79	71 43	83 13	17 31						
59 43	77 19	59 59	87 13	70 03 47	53 29	81 23	73 37	89 89	23 89						
67 31	89 17	63 67	93 43	09 43	63 17	91 61	79 61	93 17	29 13						
83 19	54 29 61	69 47	97 73	31 79	71 31	99 19	83 17	90 17 71	53 41						
91 67	47 13	77 43	99 67	33 13	93 59	80 03 53	89 13	19 29	57 19						
97 59	59 53	83 31	65 09 23	37 31	75 01 13	21 13	97 29	47 83	63 73						

**GREATEST COMMON DIVISOR AND LEAST COMMON
MULTIPLE OF FRACTIONS.**

Page 63; Art. 179.

1. The greatest common divisor or the least common multiple of fractions may be found by reducing dissimilar to similar fractions, and then finding the greatest common divisor or least common multiple (as may be required) of the numerators of the similar fractions for a numerator, under which is written their common denominator.

The following is a shorter method :

2. FOR GREATEST COMMON DIVISOR.

Important Facts.

- I. *An exact divisor gives an integral quotient (143).*
- II. *An exact divisor of a fraction must be a fraction.*

PRINCIPLE I.—*The numerator of an exact divisor of a fraction is an exact divisor of the given numerator, and the denominator is a multiple of the given denominator.*

ILLUSTRATION.— $\frac{1}{2}$ is an exact divisor of $\frac{3}{4}$, because when the divisor is inverted in the process of division ($\frac{3}{4} \times \frac{2}{1} = 6$), the numerator of the divisor is cancelled by a factor of the numerator of the dividend, the denominator of the dividend is cancelled by a factor of its multiple—the denominator of the divisor,—and the quotient is an integer.

PRINCIPLE II.—*The numerator of a common divisor of two or more fractions is a common divisor of their numerators, and the denominator is a common multiple of their denominators.*

Since the greater the numerator and the smaller the denominator, the greater the value of a fraction (79),—

PRINCIPLE III.—*The numerator of the greatest common divisor of two or more fractions is the greatest common divisor of their numerators, and the denominator is the least common multiple of their denominators. Hence, To find the greatest common divisor of two or more fractions,—*

RULE.—*Find the greatest common divisor of the numerators for a numerator, and the least common multiple of the denominators for a denominator.*

3. FOR LEAST COMMON MULTIPLE.

A multiple contains any of its divisors some exact or integral number of times (148).

Important Facts.

- I. *The multiple of a fraction may be either an integer or a fraction.*
- II. *A common multiple of two or more fractions may be either an integer or a fraction.*

PRINCIPLE I.—*If the multiple is a fraction, the numerator is a multiple of the given numerator, and the denominator is a divisor of the given denominator.*

ILLUSTRATION.— $\frac{3}{4}$ is a multiple of $\frac{1}{4}$ because when the divisor is inverted in the process of division, the denominator of the divisor contains the denominator of the dividend, the numerator of the dividend contains the numerator of the divisor, and the quotient is an integer.

PRINCIPLE II.—*If the multiple is a fraction, the numerator is a common multiple of the given numerators, and the denominator is a common divisor of the given denominators.*

Since the smaller the numerator and the greater the denominator, the less the value of the fraction (79),—

PRINCIPLE III.—*The numerator of the least common multiple of two or more fractions is the least common multiple of their numerators, and the denominator is the greatest common divisor of their denominators.* Hence,

To find the least common multiple of two or more fractions,—

RULE.—*Find the least common multiple of the numerators for a numerator, and the greatest common divisor of the denominators for a denominator.*

4. The rules for greatest common divisor and least common multiple of fractions may be given in one

GENERAL RULE.—*Perform the required operation with the numerators for a numerator, and the converse operation with the denominators for a denominator.*

PROBLEMS.

Find the least common multiple

1. Of $\frac{2}{3}$, $\frac{4}{5}$, $\frac{1}{6}$, $\frac{3}{7}$, and $\frac{5}{8}$.
2. Of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$.
3. Of $\frac{3}{4}$, $\frac{1}{2}$, and $\frac{2}{3}$.
4. Of $\frac{2}{3}$, $\frac{3}{4}$, $\frac{1}{5}$, $\frac{2}{6}$, and $\frac{1}{7}$.
5. Of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$.

Find the greatest common divisor

6. Of the numbers in problem 1.
7. Of the numbers in problem 2.
8. Of the numbers in problem 3.
9. Of the numbers in problem 4.
10. Of the numbers in problem 5.

REDUCTION OF FRACTIONS.

Page 72; Art. 193.

1. In *Art. 191* it is shown that fractions may be reduced by either multiplication or division of their terms.

They may also be reduced by subtracting unequal parts from, or adding unequal parts to the terms, provided the ratio of the parts subtracted or added is equal to the ratio of the terms. Hence,

2. To reduce a fraction to lower terms by subtraction, or to higher terms by addition,—

RULE.—I. Find the ratio of the numerator of the fraction to the denominator, either in lowest integral terms, or by assuming one term of the ratio to be 1.

II. $\left\{ \begin{array}{l} 1. \text{ To lower terms.—Subtract from} \\ 2. \text{ To higher terms.—Add to} \end{array} \right\} \begin{array}{l} \text{the terms of the fraction the} \\ \text{corresponding terms of the} \\ \text{ratio thus found.} \end{array}$

Ex. 1. Reduce $\frac{3}{5}$ to lower terms.

PROCESS.

EXPLANATION.—I first reduce the ratio expressed by the fraction to its lowest integral terms, i. e. 5 : 6. I then subtract the term of the ratio corresponding to the numerator from the numerator, and the term of the ratio corresponding to the denominator from the denominator of the given fraction, obtaining $\frac{2}{4}$ equal in value to $\frac{3}{5}$. In like manner I subtract the terms of the ratio from $\frac{2}{4}$; and so on, until the terms of the fraction are the terms of the ratio.

$$25 : 30 :: 5 : 6$$

$$\frac{3}{5} = \frac{25-5}{30-6} = \frac{20}{24}$$

$$\frac{20}{24} = \frac{20-5}{24-6} = \frac{15}{18}$$

$$\frac{15}{18} = \frac{15-5}{18-6} = \frac{10}{12}$$

$$\frac{10}{12} = \frac{10-5}{12-6} = \frac{5}{6}$$

The terms of the several fractions $\frac{20}{24}$, $\frac{15}{18}$, $\frac{10}{12}$, $\frac{5}{6}$, are the only lower terms of the given fraction that can be expressed by integers.

Ex. 2. Find the three fractional expressions, next in order, less than $\frac{9}{25}$, the denominators to be integers.

PROCESS.

$$9 : 25 :: \frac{9}{25} : 1$$

$$\frac{9}{25} = \frac{9 - \frac{9}{25}}{25 - 1} = \frac{8\frac{16}{25}}{24}$$

$$\frac{8\frac{16}{25}}{24} = \frac{8\frac{16}{25} - \frac{9}{25}}{24 - 1} = \frac{8\frac{7}{25}}{23}$$

$$\frac{8\frac{7}{25}}{23} = \frac{8\frac{7}{25} - \frac{9}{25}}{23 - 1} = \frac{7\frac{22}{25}}{22}$$

EXPLANATION.—I first reduce the ratio expressed by the fraction to a ratio in which the term corresponding to the denominator is 1 (i. e., $\frac{9}{25} : 1$).

I then subtract the terms of the ratio, and also of each result, successively, from the corresponding terms of the fraction, until I obtain the three fractions required.

PROBLEMS.

Find all the values expressed by lower integral terms of

1. $\frac{1}{2}$ 2. $\frac{1}{3}$ 3. $\frac{1}{4}$ 4. $\frac{1}{5}$ 5. $\frac{1}{6}$ 6. $\frac{1}{7}$

Find the three fractional expressions, with integral denominators, next in order,

- | | |
|--|--|
| 7. In terms less than $\frac{1}{8}$. | 10. In terms greater than $\frac{1}{9}$. |
| 8. In terms less than $\frac{1}{9}$. | 11. In terms greater than $\frac{1}{10}$. |
| 9. In terms less than $\frac{1}{10}$. | 12. In terms greater than $\frac{1}{11}$. |

REPETENDS.

Page 78; Art. 201.

1. It has already been shown in this article that, in reducing fractions to decimals, the division of the numerator with decimal ciphers annexed, is at times exact or terminable, and at others not exact or interminable. This gives rise to the classification of decimals as *finite*, and *infinite* or *circulating*.

2. A *finite decimal* is a decimal that terminates at some certain decimal place; as .25, .872, etc.

3. An *infinite* or *circulating decimal* is a decimal that never terminates.

An infinite decimal is expressed wholly or in part by a figure or a set of figures constantly repeated; as .3333 +, .545454 +, .386386386 +, etc.

4. A *repetend* is the repeated figure or set of figures used to express a circulating decimal.

A repetend is indicated by a dot placed over a repeated figure or over the first and the last figure of a repeated set; as $\dot{3}$ in .333 +, $\dot{54}$ in .545454 +, and $\dot{386}$ in .386386 +, etc.

5. A *pure circulating decimal* is a decimal expressed wholly by a repetend.

6. A *mixed circulating decimal* is a decimal expressed only in part by a repetend; as .3672672 or .367 $\dot{2}$.

NOTE.—The decimal places preceding the repetend are called *finite places*.

EXERCISES.

Write, and indicate the repetend in each,

- | | |
|---|--|
| 1. A pure circulating decimal.
2. A mixed circulating decimal.
3. A mixed decimal number, the decimal part being a mixed circulating decimal. | 4. A circulating decimal with a repetend of four figures.
5. Reduce the fraction $\frac{1}{4}$ to a decimal of a sufficient number of places to prove the repetend. |
|---|--|

7. PROPERTY I.—*Any fraction in its lowest terms, whose denominator contains no other prime factors than 2 or 5, can be reduced to a finite decimal.*

DEMONSTRATION.—Since to obtain a finite decimal, the division must be exact or terminable, all the prime factors of the denominator must be found in the numerator with decimal ciphers annexed (165, V). The prime factors 2 and 5 are found in the composite factor 10 indicated by each decimal cipher annexed to the numerator. Hence, the property.

PROPERTY II.—*Any fraction in its lowest terms, whose denominator contains other prime factors than 2 or 5, will produce a circulating decimal.*

DEMONSTRATION.—The division can not be exact or terminable when the denominator contains factors not found in the numerator with decimal ciphers annexed (165, V). No prime factor of the denominator is found in the numerator alone (192); and no prime factor other than 2 or 5 is found in the composite factor 10 introduced by each decimal cipher. Hence, the property.

PROPERTY III.—*Any fraction in its lowest terms, reduced to a decimal, produces as many finite places in the decimal as equal the greatest number of factors 2 or 5 in the denominator.*

DEMONSTRATION.—Each factor 10 ($= 2 \times 5$) of the numerator will allow the cancellation either of only one factor 2 or 5, or of both 2 and 5, of the denominator. Therefore, there must be as many decimal ciphers used as equal the greatest number of factors either 2 or 5, in the denominator. Hence, the property.

PROPERTY IV.—*Any fraction in its lowest terms, reduced to a decimal, produces a number of repetend places less than the number expressed by the product of all the prime factors other than 2 or 5, in the denominator.*

Important Fact.—*In dividing a number in the decimal scale by an integer, the greatest number of different remainders that can possibly occur must be less than the number expressed by the divisor.*

ILLUSTRATIONS.—With 5 as a divisor, the several possible remainders are 1, 2, 3, and 4; with 19 as a divisor, only eighteen different remainders are possible; with 247 as a divisor, only two hundred forty-six different remainders are possible; etc.

DEMONSTRATION OF PROPERTY IV.—In reducing a fraction to a decimal, the partial dividends are composed of remainders with ciphers annexed. Therefore, the same partial dividend,—and a consequent repetition of decimal figures,—must recur before there are partial dividends, equal in number to the divisor. Also, upon introducing decimal ciphers into the numerator, the factors 2 and 5 of the denominator may be cancelled (*191*, I), and the divisor remaining will be the product of the prime factors other than 2 or 5 of the denominator. Hence, the property.

8. From the properties and demonstrations above given, prove and illustrate by examples the following

Important Facts.

1. *Any fraction in its lowest terms, whose denominator contains either or both of the factors 2 or 5 and also other prime factors, will produce a mixed circulating decimal.*

2. *The finite places of the decimal will equal the greatest number of factors 2 or 5 in the denominator.*

3. *The number of repetend places will not exceed the number expressed by the product of the prime factors other than 2 or 5 of the denominator.*

PROBLEMS.

By inspection, tell

1st. Which of the following eighteen fractions will produce finite decimals, and how many places there will be in each;

2d. Which will produce pure circulating decimals, and the limit of number of repetend places in each; and

3d. Which will produce mixed circulating decimals, the number of finite places, and the limit of number of repetend places in each.

Ex. 1. $\frac{3}{16}$ will produce a finite decimal of four places.

PROCESS.

$$\frac{3}{16} = \frac{3}{2 \times 2 \times 2 \times 2}. \quad \text{Properties I, III.}$$

Ex. 2. $\frac{18}{105} = \frac{6 \times 3}{5 \times 3 \times 7}$, and will produce a mixed circulating decimal (Fact 1) having one finite place (Property III), and the greatest possible number of repetend places is 21 ($= 3 \times 7$), Property IV.

1. $\frac{1}{16}$	4. $\frac{3}{16}$	7. $\frac{1}{16}$	10. $\frac{1}{16}$	13. $\frac{1}{16}$	16. $\frac{1}{16}$
2. $\frac{1}{16}$	5. $\frac{1}{16}$	8. $\frac{1}{16}$	11. $\frac{1}{16}$	14. $\frac{1}{16}$	17. $\frac{1}{16}$
3. $\frac{1}{16}$	6. $\frac{1}{16}$	9. $\frac{1}{16}$	12. $\frac{1}{16}$	15. $\frac{1}{16}$	18. $\frac{1}{16}$

9. I. The denominator of a finite decimal expressed in fractional form is 1 with as many ciphers annexed as there are places in the decimal.

The same is true of the finite part of a mixed circulating decimal.

II. To determine the denominator of a repetend in the fractional form, Art. 193, Supplement, page 391, may be applied, as follows:

PROPOSITION I.—Prove that $\dot{3} = .3\frac{1}{3} = \frac{3\frac{1}{3}}{10} = \frac{10}{9}$, and that $\dot{5}\dot{4} = \frac{54}{99}$.

DEMONSTRATION.—The ratio of $8\frac{1}{3}$ to 10 equals the ratio of $\frac{1}{3}$ to 1; i. e.,

$$8\frac{1}{3} : 10 = \frac{1}{3} : 1. \quad \text{Hence, } \frac{8\frac{1}{3}}{10} = \frac{8\frac{1}{3} - \frac{1}{3}}{10 - 1} = \frac{8}{9}.$$

$$\text{Again, } \dot{5}\dot{4} = .54\frac{54}{99} = \frac{54\frac{54}{99}}{100}. \quad 54\frac{54}{99} : 100 = \frac{54}{99} : 1. \quad \text{Hence, } \frac{54\frac{54}{99}}{100} = \frac{54\frac{54}{99} - \frac{54}{99}}{100 - 1} = \frac{54}{99}.$$

PROPOSITION II.—Prove that the terminal fraction of a repetend expressed in the common decimal form (as $\dot{6} = .6\frac{2}{3}$) always bears the same ratio to 1 as the entire repetend bears to its denominator when expressed in the fractional form ($\dot{6}\frac{2}{3} = \frac{6\frac{2}{3}}{10}$); i. e., $\frac{2}{3} : 1 = 6\frac{2}{3} : 10$.

III. Any repetend is but another form for a decimal expressed by the figures of the repetend and a terminal fraction,—or for a fraction having for its numerator the mixed fractional number expressed by the figures of the repetend and the terminal fraction, and for its denominator 1 with as many ciphers annexed as there are places in the repetend,—or for any reduction of such fraction; and

The fraction reduced to lower terms by subtracting the terminal fraction from the numerator and 1 from the denominator, has for its numerator the figures of the repetend, and for its denominator as many 9's as there are figures in the repetend. Hence,

To reduce a repetend to a fraction,—

RULE.—Write the repetend for the numerator, and as many 9's as there are repetend places for the denominator.

10. A mixed circulating decimal may be expressed in the common decimal form with a terminal fraction,

$$\text{as } .4\dot{3} = .4\frac{3}{9} = \frac{4}{10} + \frac{3}{90} = \frac{36+3}{90} \text{ (or } \frac{40-4+3}{90}, \text{ see Art. 259)} = \frac{39}{90};$$

$$\text{also, } .24\dot{3}\dot{6} = \frac{24}{100} + \frac{36}{9900} = \frac{2376+36}{9900} \text{ (or } \frac{2400-24+36}{9900}, \text{ Art. 259)} = \frac{2412}{9900}$$

The numerator of each of these results may be obtained by subtracting the finite part from the entire circulating decimal (see page 391, 6);

and the denominator, by writing as many 9's as there are repetend places and as many ciphers following as there are finite places. Hence,

To reduce a mixed circulating decimal to a fraction,—

RULE.—I. For the numerator,—*Subtract the finite part from the entire circulating decimal; and*

II. For the denominator,—*Write as many 9's as there are repetend places with as many ciphers annexed as there are finite places.*

Ex. 1. Reduce $.45\dot{6}$ to a fraction.

PROCESS.

$$.45\dot{6} = \frac{456}{111} = \frac{152}{37}$$

Ex. 2. Reduce $.3746\dot{8}$ to a fraction.

PROCESS.

$$.3746\dot{8} = \frac{37468}{11111} = \frac{37468}{37431}$$

Reduce to fractions

PROBLEMS.

1. $.3\dot{3}$	5. $.812\dot{7}$	9. $.23076\dot{9}$	13. $.241\dot{2}$	17. $8.21\dot{6}$
2. $\dot{6}$	6. $.14285\dot{7}$	10. $\dot{9}$	14. $.81\dot{6}$	18. $27.3\dot{3}$
3. $.8\dot{4}$	7. $.1428571\dot{4}$	11. $.999\dot{9}$	15. $.8161\dot{6}$	19. $18.1\dot{8}$
4. $.12\dot{3}$	8. $.455\dot{4}$	12. $.32\dot{7}$	16. $.475\dot{2}$	20. $72.164\dot{5}$

11. Circulating decimals may be changed in form by expanding them. In any such expansion, observe

1. That the order of succession of the repetend figures must remain unchanged.

2. That the number of repetend places must remain unchanged, or be changed to a multiple of that number of places.

e. g., $.3\dot{6} = .36\dot{3} = .363\dot{6} = .3636\dot{3} = .363636\dot{3}$, and so on.

In the first and second of the expansions above, both the order of succession and the number of places in the repetend remain unchanged. In the third and fourth, the order of succession remains unchanged, but the number of repetend places is changed to a multiple of 2, the original number of repetend places.

PROPOSITION.—Prove, by reducing to fractions in their lowest terms, that the expansions above given produce no change in the value of the decimal.

- 12. Similar repetends** begin at the same decimal place;
Conterminous repetends end at the same decimal place;
Like repetends are both similar and conterminous; hence,

PRINCIPLE.—*Only like repetends can be added or subtracted.*

13. To make repetends similar,—

RULE.—*Expand the circulating decimals till the number of finite places in each equals the greatest number of finite places in any of them.*

14. To make similar repetends conterminous,—

RULE.—*Expand each repetend to as many places as are indicated by the least common multiple of the numbers of places in the several repetends.*

Ex. Make the circulating dec-

imals $.8\dot{6}$, $.875\dot{4}$, $.36\dot{3}$, $.427\dot{5}\dot{8}$

similar and conterminous.

	PROCESS.	
	Similar	Similar and conterminous
$.8\dot{6}$	$= .866\dot{6}$	$= .866\dot{6}6666\dot{6}$
$.875\dot{4}$	$= .875\dot{4}\dot{5}$	$= .875\dot{4}5454\dot{5}$
$.36\dot{3}$	$= .363\dot{3}6\dot{3}$	$= .363\dot{3}6363\dot{3}$
$.427\dot{5}\dot{8}$	$= .427\dot{5}\dot{8}$	$= .427\dot{5}858\dot{5}\dot{8}$

EXPLANATION.—I first examine the given decimals to ascertain what is the greatest number of finite places in any one of them, and find it is three in the fourth

number. I therefore expand each repetend to the fourth decimal place, over which I place the repetend dot, and where necessary continue the expansion until I have the original number of repetend places in each. The repetends are now similar.

I find that the numbers of repetend places in the several repetends are expressed by 1, 2, 3, and 2. I therefore find the least common multiple of 1, 2, 3, 2—which is 6,—and expand each repetend until it contains six repetend places. The repetends are now similar and conterminous.

15. To add or subtract circulating decimals,—

RULE.—I. *If necessary, make the repetends similar and conterminous.*

II. *Add or subtract as in integers, always carrying from the left-hand repetend column when adding or subtracting the right-hand column.*

16. To multiply or divide repetends,—

RULE.—I. *Reduce the repetends to fractions.*

II. *Multiply or divide as in fractions.*

III. *Reduce the fractional result to a decimal.*

PROBLEMS.

<i>a.</i> .863̄	<i>n.</i> 15.912̄	Add all the given numbers 1. From <i>f</i> to <i>j</i> inclusive. 2. “ <i>u</i> to <i>y</i> inclusive. 3. “ <i>a</i> to <i>e</i> inclusive. 4. “ <i>g</i> to <i>k</i> inclusive.
<i>b.</i> .754̄	<i>o.</i> 74.237̄	
<i>c.</i> .929̄	<i>p.</i> 21.468̄	
<i>d.</i> .765̄	<i>q.</i> 54.31̄	
<i>e.</i> .82313̄	<i>r.</i> 2.750̄	Subtract { 5. <i>n</i> from <i>o</i> . 6. <i>w</i> from <i>v</i> . 7. <i>f</i> from <i>r</i> . 8. <i>q</i> from <i>v</i> . 9. <i>t</i> from <i>e</i> .
<i>f.</i> .7543̄	<i>s.</i> .0018̄	
<i>g.</i> .4836̄	<i>t.</i> .81234̄	
<i>h.</i> .8216̄	<i>u.</i> 27.1235̄	
<i>i.</i> .73̄	<i>v.</i> 89.6543̄	Multiply { 10. <i>d</i> by <i>i</i> . 11. <i>p</i> by <i>h</i> .
<i>j.</i> .2287̄	<i>w.</i> .8184̄	
<i>k.</i> .64513̄	<i>x.</i> 2.5733̄	Divide { 12. <i>p</i> by <i>f</i> . 13. <i>x</i> by <i>n</i> .
<i>l.</i> .8114̄	<i>y.</i> 91.25̄	
<i>m.</i> 8.723̄	<i>z.</i> 4.36̄	

14. Add $\frac{1}{4}$ and $\frac{1}{4}$; then reduce the fractions to decimals, add these results, and reduce the sum to a fraction, making the results agree.

15. Add $\frac{1}{4}$ and $\frac{1}{4}$ and reduce the result to a decimal; then reduce the fractions to decimals separately, and add the decimals, making the results agree.

SHORT METHODS OF REDUCTION.

Page 84; Art. 209.

Pupils should give reasons for all Short Methods not analyzed.

I. To reduce hundred-weight and pounds to the decimal of a ton.

Ex. Reduce 3 T. 5 cwt. 24 lb. to tons and decimals.

PROCESS.

$$\begin{aligned} 5 \text{ cwt. } 24 \text{ lb.} &= 5.24 \text{ cwt.} \\ 5.24 \times 5 &= .262 \text{ T.} \\ 3 \text{ T.} + .262 \text{ T.} &= 3.262 \text{ T.} \end{aligned}$$

Annex the pounds as hundredths to the number of hundred-weight, and multiply the number thus formed by 5, pointing off two additional decimal places.

PROBLEMS.

Reduce to decimals of a ton,		Reduce to tons and decimals,	
1. 8 cwt. 56 lb.	3. 18 cwt. 48 lb.	5. 7 T. 12 cwt.	7. 2 T. 14 cwt. 9 lb.
2. 7 cwt.	4. 5 cwt. 17 lb.	6. 14 T. 5 cwt. 12 lb.	8. 1 T. 17 cwt. 5 lb.

II. To reduce shillings and pence to the decimal of a pound.**Ex.** Reduce £18 5s. 9d. to pounds and decimals.

Multiply the number of shillings by 5 and write the product as hundredths of a pound.

Multiply the number of pence by $4\frac{1}{3}$ and write the product as thousandths of a pound.

The sum of the two products will be the decimal required.

PROCESS.

$$\begin{array}{rcl}
 5 \times 5 & = 25 & = \text{£} .25 \\
 9 \times 4\frac{1}{3} & = 0375 & = \text{£} .0375 \\
 \hline
 & & \text{£} 18.2875
 \end{array}$$

PROBLEMS.

Reduce to decimals of a £,

Reduce to £'s and decimals,

1. 17s.

3. 13s. 5d.

5. £2 3s. 9d.

7. £204 10s. 6d.

2. 9s. 8d.

4. 11s. 6d.

6. £17 11s. 10d.

8. £193 1s. 1d.

SHORT METHODS OF MULTIPLICATION.**Page 131; Art. 264.****I. To multiply by 29, 39, and so on to 99.****Ex.** Multiply 747×59 . $59 = 60 - 1$, and $60 = 6 \times 10$.

6 times 10 times a number minus once the number is 59 times the number.

PROCESS.

$$\begin{array}{r}
 7,470 \\
 6 \\
 \hline
 44,820 \\
 747 \\
 \hline
 44,073
 \end{array}$$

PROBLEMS.

Using this method, find the product

1. Of 973×89 3. Of $3,470 \times 69$ 5. Of 48.72×4.9 2. Of 426×19 4. Of $4,802 \times 59$ 6. Of $3.164 \times .29$ **II. To find the product of two factors, when $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., of one of them is a convenient multiplier.****Ex.** Multiply 42 by 21.

Introducing the factor 3 into the multiplicand, and rejecting it from the multiplier does not affect the quotient.

PROCESS.

$$\begin{array}{rcl}
 42 \times 3 & = & 126 \\
 21 \div 3 & = & 7 \\
 \hline
 & & 882
 \end{array}$$

III. To find the product of two factors each expressed by two figures,—the tens alike and the sum of the ones, 10.**Ex.** Multiply 83 by 87.**PROCESS.**

$$80 \times 90 + 21 = 7,221$$

Multiply the tens of one factor by the next greater number of tens, and to this product add the product of the ones of both factors.

PROBLEMS.

Using this method, find the product

- | | | | |
|----------------------|----------------------|----------------------|----------------------|
| 1. Of 76×74 | 3. Of 29×21 | 5. Of 65×65 | 7. Of 49×41 |
| 2. Of 52×58 | 4. Of 97×93 | 6. Of 13×17 | 8. Of 82×88 |

IV. To find the product of two factors, one of them more and the other less by a convenient amount, than an order of units (*i.e.*, than 10, 100, 1000, etc.).

PROCESS.

Ex. Multiply 106 by 97.

Cancel or reject the left-hand figure of the greater factor, to the remaining figures add the less factor, and to the sum annex as many ciphers as there are places in the less factor.

From the number thus obtained subtract the product of the excess of the greater factor over the unit and the complement of the smaller factor.

$$106 + 97 = 10,300$$

$$6 = \text{excess}; 3 = \text{complement.}$$

$$10,300 - 6 \times 3 = 10,282$$

PROBLEMS.

Using this method, find the product

- | | | |
|-----------------------|--------------------------|-----------------------------|
| 1. Of 103×91 | 3. Of $1,004 \times 995$ | 5. Of $10,006 \times 9,993$ |
| 2. Of 96×106 | 4. Of $1,003 \times 992$ | 6. Of 107×91 |

V. To find the product of two factors that have the same number of places, and are either both more or both less than—but near to—an order of units (*i.e.*, to 10, 100, 1000, etc.).

Ex. 1. Multiply 106 by 104.

Cancel or reject the left-hand figure of one of the factors, and to the other factor add the number expressed by the remaining figures. To the sum thus obtained annex as many ciphers less one as there are places in each of the factors, and to the number thus formed

add the product of the excesses of the factors over a unit of the nearest order.

PROCESS.

$$106 + 104 = 110$$

$$6 = \text{excess of } 106 \text{ over } 100$$

$$4 = \text{excess of } 104 \text{ over } 100$$

$$11,000 + 24 = 11,024$$

Ex. 2. Multiply 997 by 992.

Add the two factors, from their sum cancel or reject the left-hand figure, and to the remaining figures annex as many ciphers as there are places in each factor.

To the number thus formed, add the product of the complements of the two factors.

PROCESS.

$$997 + 992 = 1,989$$

$$989,000 + 24 = 989,024$$

PROBLEMS.

Using this method, find the product

1. Of 103×106 | 3. Of $1,007 \times 1,004$ | 5. Of 92×98 | 7. Of 18×16
 2. Of 109×105 | 4. Of $10,009 \times 10,002$ | 6. Of 993×995 | 8. Of 97×93

VI. To find the product of two factors between which there is a mean number of tens.

Multiply 84×76 .

Square the tens of the greater factor, and from the result thus obtained subtract the square of the ones of the same factor.

PROCESS.

$$\begin{array}{r} 80^2 = 6,400 \\ 4^2 = \quad 16 \\ \hline 6,384 \end{array}$$

ANALYSIS.— $76 \times 84 = 80 \times 84 - 4 \times 84$.

Adding 4 to the factor 76 adds 4 times the other factor (84) to the product.

$$80 \times 84 = 80 \times 80 + 4 \times 80.$$

Subtracting 4 from the factor 84 subtracts 4 times the other factor (80) from the product.

$$\text{Hence, } 76 \times 84 = 80 \times 80 - 4 \times 84 + 4 \times 80; \text{ or,}$$

$$76 \times 84 = 80 \times 80 - 4 \times (84 - 80); \text{ or,}$$

$$76 \times 84 = 80 \times 80 - 4 \times 4 = 80^2 - 4^2$$

PROBLEMS.

Write a multiplicand to which this method will apply for each, and multiply

1. By 92 | 2. By 46 | 3. By 37 | 4. By 246 | 5. By 127 | 6. By 117

VII. To find the product of two mixed numbers that end with the same fraction.

Ex. Multiply $27\frac{1}{3}$ by $19\frac{1}{3}$.

Find the sum of the product of the integral parts, of the product of the sum of the integral parts and the common fraction, and the square of the fraction.

PROCESS.

$$\begin{array}{r} 19 \times 27 = 513 \\ 19 + 27 \times \frac{1}{3} = 15\frac{1}{3} \\ \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} \\ \hline 528\frac{1}{3} \end{array}$$

PROBLEMS.

Using this method, find the product

1. Of $8\frac{1}{2} \times 28\frac{1}{2}$ | 3. Of $11\frac{2}{3} \times 10\frac{2}{3}$ | 5. Of $18\frac{1}{3} \times 9\frac{1}{3}$ | 7. Of $14\frac{1}{2} \times 16\frac{1}{2}$
 2. Of $12\frac{1}{3} \times 3\frac{1}{3}$ | 4. Of $9\frac{2}{3} \times 12\frac{2}{3}$ | 6. Of $12\frac{2}{3} \times 7\frac{2}{3}$ | 8. Of $13\frac{1}{3} \times 11\frac{1}{3}$

VIII. To find the product of two mixed numbers whose fractional parts have a common denominator, and whose integral parts are the same.

Ex. Multiply $9\frac{3}{4}$ by $9\frac{3}{4}$.

Add the fraction of one factor to the other factor, multiply the sum by the remaining integral part, and to this product add the product of the fractional parts.

PROCESS.

$$\begin{aligned} 9\frac{3}{4} + \frac{3}{4} &= 9\frac{6}{4} \\ 9 \times 9\frac{6}{4} &= 87\frac{3}{2} \\ \frac{3}{4} \times \frac{3}{4} &= \frac{9}{16} \\ &\underline{87\frac{15}{8}} \end{aligned}$$

PROBLEMS.

Using this method, find the product

1. Of $6\frac{1}{2} \times 6\frac{1}{2}$ | 3. Of $27\frac{1}{2} \times 27\frac{1}{2}$ | 5. Of $10\frac{1}{2} \times 10\frac{1}{2}$ | 7. Of $11\frac{1}{11} \times 11\frac{1}{11}$
2. Of $12\frac{1}{3} \times 12\frac{1}{3}$ | 4. Of $16\frac{2}{11} \times 16\frac{2}{11}$ | 6. Of $8\frac{1}{2} \times 8\frac{1}{2}$ | 8. Of $5\frac{1}{4} \times 5\frac{1}{4}$

IX. To find the product of two factors the difference of whose integral parts is 1, and the sum of whose fractional parts is also 1.

Ex. Multiply $8\frac{2}{3}$ by $7\frac{1}{3}$.

Add 1 to the greater integer, multiply the sum by the smaller integer, and to the product add the difference between 1 and the square of the fraction of the greater number.

PROCESS.

$$\begin{aligned} 8 + 1 (= \frac{2}{3} + \frac{1}{3}) &= 9 \\ 7 \times 9 &= 63 \\ 1 - (\frac{2}{3} \times \frac{2}{3}) &= \frac{5}{9} \\ &\underline{63\frac{5}{9}} \end{aligned}$$

PROBLEMS.

Using this method, find the product

1. Of $9\frac{1}{2} \times 8\frac{1}{2}$ | 2. Of $8\frac{1}{2} \times 7\frac{1}{2}$ | 3. Of $5\frac{2}{3} \times 6\frac{1}{3}$ | 4. Of $13\frac{2}{3} \times 12\frac{1}{3}$ | 5. Of $26\frac{1}{2} \times 25\frac{1}{2}$

X. To find the integer nearest to the product of two mixed fractional numbers.

Ex. Find the nearest approximate integral value of $16\frac{1}{2} \times 8\frac{1}{2}$.

To the product of the integers, add the nearest integer to the product of the whole number in each factor by the fraction in the other factor.

PROCESS.

$$\begin{aligned} 16 \times 8 &= 128 \\ \frac{1}{2} \text{ of } 16 &= 8 \text{ (nearest integer).} \\ \frac{1}{2} \text{ of } 8 &= 4 \text{ (nearest integer).} \\ &\underline{134} \end{aligned}$$

PROBLEMS.

Find the nearest approximate integral value

1. Of $16\frac{1}{2} \times 7\frac{1}{2}$ | 3. Of $12\frac{2}{11} \times 4\frac{1}{2}$ | 5. Of $18\frac{3}{4} \times 28\frac{1}{4}$
2. Of $17\frac{1}{3} \times 11\frac{1}{3}$ | 4. Of $19\frac{1}{2} \times 20\frac{1}{2}$ | 6. Of $50\frac{1}{2} \times 6\frac{1}{2}$

XI. Cross multiplication,—or omitting partial products in written work.

ILLUSTRATIVE PROCESS.

Ex. 1. Multiply 35 by 24.

Multiply ones by ones (4×5) and write the ones (0) of this product for the ones of the required result. Add (carry) the tens (2) to the sum of the products of ones and tens ($4 \times 3 + 2 \times 5$) and write the ones (4) of this result (24) for the tens of the required result. Add (carry) the tens (2) to the product of the tens and tens (2×3) and write the result for the hundreds of the required result.

	35		
	24		
Prod. of ones and ones,	20	ones	—from 4×5
prod. of ones and tens,	22	tens	—from $4 \times 3 + 2 \times 5$
prod. of tens and tens,	6	hundreds	—from 2×3
	<u>840</u>		

PROCESS.

$$\begin{array}{r} 35 \\ 24 \\ \hline 840 \end{array}$$

COMPUTATION.

4 times 5, 20—write 0;
2 plus 4 times 3, plus 2 times 5, 24—write 4;
2 plus 2 times 3, 8—write 8.

Ex. 2. Multiply 345 by 256.

ILLUSTRATIVE PROCESS.

	345		
	256		
Prod. of ones and ones,	30	ones	—from 6×5
prod. of ones and tens,	49	tens	—from $6 \times 4 + 5 \times 5$
prod. of {ones and hundreds, } {tens and tens, }	48	hundreds	—from $6 \times 3 + 5 \times 4 + 2 \times 5$
prod. of tens and hundreds,	23	thousands	—from $5 \times 3 + 2 \times 4$
prod. of hundreds and hundreds,	6	ten-thousands	—from 2×3
	<u>88,320</u>		

PROCESS.

$$\begin{array}{r} 345 \\ 256 \\ \hline 88,320 \end{array}$$

COMPUTATION.

6 times 5, 30—write 0;
3 plus 6 times 4, plus 5 times 5, 59—write 2;
5 plus 6 times 3, plus 5 times 4, plus 2 times 5, 53—write 8;
5 plus 5 times 3, plus 2 times 4, 28—write 8;
2 plus 2 times 3, 8—write 8.

PROBLEMS.

Omitting partial products in the written work, multiply

- | | | | |
|-------------|-------------|---------------|---------------|
| 1. 26 by 32 | 3. 48 by 36 | 5. 532 by 324 | 7. 329 by 487 |
| 2. 75 by 23 | 4. 96 by 43 | 6. 618 by 275 | 8. 826 by 623 |

XII. To square any number that is expressed by 9's.

Ex. Square 999.

PROCESS.

Write as many 9's less one as there are 9's in the given number, an 8, as many ciphers as 9's written, and a 1.

$$999^2 = 998,001$$

XIII. To square any number that ends with 5.

Ex. Square 35.

PROCESS.

Omit the 5, multiply the number expressed by the remaining figures by the next greater integral number, and to this product annex 25 ($=5^2$).

$$\begin{aligned} 3 \times 4 &= 12 \\ 12 \text{ annex } 25 &= 1,225 \end{aligned}$$

XIV. To square any number expressed by two places.

Ex. 1. Square 28.

Ex. 2. Square 84.

PROCESSES.

Find the difference between the number and the nearest number of tens. If the number be less than the nearest number of tens, subtract the difference from the number; but if the number be greater, add the difference. Multiply the sum or the difference thus obtained by the number of tens, and add the square of the difference to the product. (See Analysis, Case VI.)

$$\begin{aligned} 1. \quad & \begin{cases} 30 - 28 = 2 \\ 30 \times 26 + 4 (=2^2) = 784 \end{cases} \\ 2. \quad & \begin{cases} 84 - 80 = 4 \\ 80 \times 88 + 16 (=4^2) = 7,056 \end{cases} \end{aligned}$$

XV. To square a number that ends with $\frac{1}{2}$.

Ex. Square $11\frac{1}{2}$.

PROCESS.

Multiply the integral part by the next higher integer and to the product annex $\frac{1}{4}$ ($=(\frac{1}{2})^2$).

$$11\frac{1}{2}^2 = 11 \times 12 + \frac{1}{4} (= (\frac{1}{2})^2) = 132\frac{1}{4}$$

PROBLEMS.

Using the methods above given, find

1. 99^2	3. 85^2	5. 64^2	7. 93^2	9. $9\frac{1}{4}^2$	11. 45^2
2. $9,999^2$	4. 55^2	6. 82^2	8. $13\frac{1}{4}^2$	10. 37^2	12. $20\frac{1}{4}^2$

DIVISION.

Page 132, Art. 265.

The proper definition for division has long been a matter of dispute among arithmeticians. The three definitions representing the different views that have been held may be formulated substantially as follows:

1. *Division* is the *process of separating a number into equal parts*.
2. *Division* is the *process of measuring one of two like numbers by the other*, to find the times or the part of 1 time the former number contains the latter.

3. The two preceding definitions combined, as given in this work (265).

According to the first definition, *the divisor is always abstract*, and the dividend and quotient are like numbers.

According to the second definition, the divisor and dividend are like numbers, and *the quotient is always abstract*.

According to the third definition, *either the divisor or the quotient is abstract*, as the varying condition of problems may demand.

The analysis of the solution of problems according to the first and second definitions is shown in the two problems following.

PROBLEM 1. If 42 apples be divided equally among 6 boys, how many apples will each boy receive?

Analysis (1st definition).—Since 42 apples are to be divided equally among 6 boys, each boy will receive one of the 6 equal parts into which 42 apples can be divided, or $\frac{1}{6}$ of 42 apples,—which is 7 apples.

Analysis (2d definition).—If 1 apple be given to each boy, the 6 boys will receive 6 apples; and, of the 42 apples, each will receive as many times 1 apple as the times 6 apples are contained in 42 apples, which are 7 times. Therefore, each boy will receive 7 apples.

PROBLEM 2. At \$6 each, how many sheep can be bought for \$42?

Analysis (1st definition).—At \$1 each, 42 sheep can be bought for \$42; and, since \$1 is $\frac{1}{6}$ of \$6, at \$6 each, $\frac{1}{6}$ of 42 sheep, or 7 sheep, can be bought for \$42.

Analysis (2d definition).—Since 1 sheep can be bought for \$6, as many times 1 sheep can be bought for \$42 as the times \$6 are contained in \$42, which is 7 times. Therefore, at \$6 each, 7 sheep can be bought for \$42.

EXERCISES.

Which of the foregoing definitions (1, 2) most naturally complies with the conditions of the problem

1. When the number of equal parts and the whole are given, to find one equal part?
2. When the whole and one equal part are given, to find the number of equal parts?

TRUE REMAINDER.

Page 133, Art. 267.

1. In dividing by the factors of a number, there may be as many remainders as there are factors used as divisors.

2. The *true remainder* is the remainder after all integral places in the quotient have been obtained by the common method of division.

	PROCESS.	Partial remainders.
Ex. Divide 64,322 by the factors of 308 ($= 4 \times 7 \times 11$) and find the true remainder.	$308 \left\{ \begin{array}{l} 4 \overline{) 64,322} \\ 7 \overline{) 16,080} \\ 11 \overline{) 2,297} \end{array} \right.$	$\begin{array}{r} 2 \\ 1 \times 4 = 4 \\ 9 \times 7 \times 4 = 252 \\ \hline \text{True remainder, } 258 \end{array}$
An examination of the process in a reverse order shows that	208	

$$\begin{aligned} 2,297 &= 11 \text{ times } 208 \text{ plus } 9, \text{—or } 11 \times 208 + 9 \\ 16,080 &= 7 \text{ times } 2,297 \text{ plus } 1, \text{—or } 7 \times (11 \times 208 + 9) + 1 \\ 64,322 &= 4 \text{ times } 16,080 \text{ plus } 2, \text{—or } 4 \times [7 \times (11 \times 208 + 9) + 1] + 2 \end{aligned}$$

Performing indicated operations and removing signs of aggregation, successively,

$$\begin{aligned} 4 \times [7 \times (11 \times 208 + 9) + 1] + 2 &= \\ (a) \ 28 \times (11 \times 208 + 9) + 4 + 2 &= \\ (b) \ 308 \times 208 + 252 + 4 + 2 &= \\ (c) \ 308 \times 208 + 258, \text{ the true remainder.} & \end{aligned}$$

3. An examination of the foregoing equations shows,—

- I. That the remainder from the first division remains unchanged.
- II. That the remainders from all succeeding divisions are multiplied respectively by all divisors of preceding divisions.
- III. That the remainder from the first division plus the products of the other remainders by preceding divisors is the true remainder.

GENERAL METHOD OF PROOF BY SUMMATION OF DIGITS.

Pages 101, 111, 124, 141.

The *summation of the digits of a number to a final digit* is performed as follows:—

- I. Add digits of the number until a sum greater than 9 is obtained.
- II. Add the digits of this sum and a digit or digits of the given number until the sum exceeds 9.
- III. Add the digits of this sum; and so continue until all the digits in the number are added and a final sum is obtained that is expressed by a single digit.

The last digit obtained is the *final digit*.

Ex. By summation of its digits find the final digit of 84,375,624.

FULL PROCESS.

$$\begin{aligned}
 8 + 4 &= 12, (1 + 2 =) 3 \\
 3 + 3 + 7 &= 13, (1 + 3 =) 4 \\
 4 + 5 + 6 &= 15, (1 + 5 =) 6 \\
 6 + 2 + 4 &= 12, (1 + 2 =) 3 \\
 &\quad 3, \text{ final digit.}
 \end{aligned}$$

COMMON PROCESS.

$$\begin{aligned}
 &8, 12; \\
 &3, 6, 13; \\
 &4, 9, 15; \\
 &6, 8, 12; \\
 &3, \text{ final digit.}
 \end{aligned}$$

NOTES.—1. The addition may begin with any digit, and proceed from left to right, from right to left, without any systematic order.

2. All the digits may be added and the digits of this sum added, until a final digit is obtained. Thus, 8, 12, 15, 22, 27, 33, 35, 39; 12; 3, final digit.

3. The summation may continue to any partial sum above 9; this partial sum may be dropped, the sum of its digits used in its stead, and the summation continued. Thus, beginning at the left, 8, 12, 15, 22; (2 + 2 =) 4, 9, 15, 17, 21; (2 + 1 =) 3, final digit. Or, beginning at the right,—4, 8, 12, 17, 24; (2 + 4 =) 6, 9, 13, 21; (2 + 1 =) 3, final digit.

PROBLEMS.

By summation of digits find the final digit in

1. 78	3. 5,734	5. 72,781	7. 64,538	9. 537,682
2. 643	4. 24,861	6. 21,873	8. 97,557	10. 445,743

11. In the first four of the above taken together.

12. In the last six of the above taken together.

On the summation of digits is based *French's General Method of Proof* of the Four Fundamental Processes in Decimal Numbers.*

ADDITION.

Page 101; Art. 229.

EXPLANATION.—Summing the digits in the parts, I obtain the digit 5; and summing the digits in the sum, I also obtain the digit 5. The sum total is therefore supposed to be correct.

Apply this method of proof to the problems in Art. 223.

EXAMPLE.

$$\begin{array}{r} 7,834 \\ 6,593 \\ 12,347 \\ 9,855 \\ 4,326 \\ \hline 40,955 \end{array} \left. \vphantom{\begin{array}{r} 7,834 \\ 6,593 \\ 12,347 \\ 9,855 \\ 4,326 \end{array}} \right\} 5$$

SUBTRACTION.

Page 111; Art. 238.

First Method.

EXPLANATION.—Summing the digits in the subtrahend and remainder, I obtain the final digit 7; and summing the digits in the minuend, I also obtain the final digit 7. The remainder is therefore supposed to be correct.

EXAMPLE.

$$\begin{array}{r} 86,758 \\ 22,578 \\ \hline 64,180 \end{array} \left. \vphantom{\begin{array}{r} 86,758 \\ 22,578 \end{array}} \right\} 7$$

Second Method.

EXPLANATION.—Summing the digits in minuend, subtrahend, and remainder, I obtain 7, 6, and 1. Adding the final digits (6 and 1) of subtrahend and remainder I obtain 7, which is also the final digit of the minuend. Or, subtracting the final digit (6) of the subtrahend from the final digit (7) of the minuend, I obtain 1, which is also the final digit of the remainder. The remainder is therefore supposed to be correct.

EXAMPLE.

$$\begin{array}{r} 86,758 \\ 22,578 \\ \hline 64,180 \end{array} \left. \vphantom{\begin{array}{r} 86,758 \\ 22,578 \end{array}} \right\} 7$$

Third Method.

EXPLANATION.—Summing the digits in minuend, subtrahend, and remainder separately, I obtain the final digits 3, 5, and 7. Since the final digit of the subtrahend is more than the final digit of the minuend, I add 9 to the final digit 3, and from the sum I subtract the final digit 5, and obtain 7, which is the same as the final digit of the remainder. The remainder is therefore supposed to be correct.

EXAMPLE.

$$\begin{array}{r} 213,573 \\ 64,238 \\ \hline 149,335 \end{array} \left. \vphantom{\begin{array}{r} 213,573 \\ 64,238 \end{array}} \right\} 7$$

Apply these methods of proof to the problems in Art. 232.

* Named from its discoverer, John H. French, LL.D., the author of "French's Binomial Theorem." (See "Robinson's Elementary Algebra," pages 174-179.)

MULTIPLICATION.

Page 124; Art. 254.

EXPLANATION.—Summing the digits in the factors, I obtain the final digits 6 and 8; and summing the digits in the product (48) of these digits, I obtain the final digit 3. Summing the digits in the product of the given factors, I also obtain the final digit 3. The product is therefore supposed to be correct.

Note.—In proving continued multiplication, the final digit obtained by summing the product of the final digits in the several factors equals the digit obtained by summing the digits of the final product.

EXAMPLE 1.

$$\begin{array}{r}
 68,487 \quad 6 \\
 \underline{9,674 \quad 8} \\
 273948 \quad 48 \quad 12 \quad 3 \\
 479409 \\
 410922
 \end{array}$$

$$\begin{array}{r}
 616383 \\
 \underline{662,543,238} \quad 39 \quad 12 \quad 3
 \end{array}$$

EXPLANATION.—The final digit obtained by summing the product (86,614,986) of the factors 258, 3,461, and 97 is 8; the final digits in the factors, in their order, are 6, 5, and 7; and the final digit in the product (310) of the final digits of the

EXAMPLE 2.

several factors is also 3. The product is therefore supposed to be correct.

Apply this method of proof to the problems in Art. 246.

DIVISION.

Page 141; Art. 275.

CASE I. The quotient a decimal number and no remainder.

EXPLANATION.—Summing the digits in divisor and quotient, I obtain the final digits 4 and 7, and summing the digits in the product of 4 and 7 (28), I obtain the final digit 1. Summing the digits of the dividend, I also obtain the final digit 1. The quotient is therefore supposed to be correct.

EXAMPLE.

$$\begin{array}{r}
 873,872,722 \div 92,317 = 9,466 \\
 46 \quad 10 \quad 1 \quad 22 \quad 4 \quad 25 \quad 7 \\
 4 \times 7 = 28 \quad 10 \quad 1
 \end{array}$$

CASE II. The quotient a mixed number.**EXAMPLE.**

$$1,940,605 \div 2,538 = 764 \text{ and } 1,573 \text{ rem. (i.e., } 764\frac{1573}{2538})$$

$$25 \quad 7 \quad 18 \quad 9 \quad 17 \quad 8 \quad 16 \quad 7$$

$$9 \times 8 + 7 = 79 \quad 16 \quad 7$$

EXPLANATION.—Summing the digits in the dividend, I obtain the final digit 7; and summing the digits in the divisor—the integral part of the quotient—and the remainder, I obtain the final digits 9, 8, and 7. Summing the digits of the sum of the products of the final digits (9 and 8) of the divisor and the integral part of the quotient plus the final digit (7) of the remainder, I also obtain the final digit 7. The quotient is therefore supposed to be correct.

Apply this method of proof to problems in Art. 269.

The application of summation of digits to proving the Four Fundamental Processes in Decimal Numbers may be formally stated as

PRINCIPLES OF FRENCH'S GENERAL METHOD OF PROOF.

I. The final digit obtained by summing the digits in all the parts added, is the same as the final digit obtained by summing the digits in the amount.

II. The final digit obtained by summing the digits in both subtrahend and remainder, is the same as the final digit obtained by summing the digits in the minuend.

III. The final digit obtained by summing the digits in the product of the final digits in the factors, is the same as the final digit obtained by summing the digits in the product of the factors.

IV. The final digit obtained by summing the digits in the product of the final digits of the divisor and quotient plus the final digit obtained by summing the digits in the remainder (when there is a remainder), is the same as the final digit obtained by summing the digits in the dividend.

THE DIVISOR ANY NUMBER OF 9's.

Page 142; Art. 276.

Ex. 1. Divide 843 by 99.

PROCESS.

$$8 \mid 43 \div 99 = 8 \text{ and } 43 + 8 = 51 \text{ rem.}$$

EXPLANATION.—I cut off from the right of the dividend as many figures as there are 9's in the divisor, and write the remaining figure (8) of the dividend for a partial quotient.

$$8\frac{43}{99} = 8\frac{43}{99}$$

I then add the quotient 8 to the number expressed by the figures cut off from the right of the dividend, and obtain 51, the remainder.

This remainder is less than the divisor; I therefore write it over the divisor at the right of the quotient figure 8, and I have $8\frac{51}{99} = 8\frac{17}{33}$, the quotient of the complete division.

Ex. 2. Divide 486,723 by 999.

PROCESS.

$$486 \mid 723 \div 999 = 486 \text{ and } 486 + 723 = 1,209 \text{ rem.}$$

$$1 \mid 209 \div 999 = \underline{1} \text{ and } 209 + 1 = 210 \text{ rem.}$$

$$487\frac{210}{999} = 487\frac{70}{333}$$

EXPLANATION.—I cut off from the right of the dividend as many figures as there are 9's in the divisor, and write the remaining figures, 486, for a partial quotient.

I then add this quotient to the number expressed by the figures cut off from the right of the dividend, and obtain the remainder 1,209.

This remainder is greater than the divisor; I therefore divide it by the divisor, and obtain for a second partial quotient 1, and for the remainder 210.

I next add the partial quotients 486 and 1, and obtain 487, at the right of which I write the remainder 210 over the divisor, and I have $487\frac{210}{999} = 487\frac{70}{333}$, the quotient of the complete division.

PROBLEMS.

Divide	{	1. 4,385 by 99	4. 87,435 by .99	7. 373.64 ÷ 99
		2. 6,278,349 by 999	5. 921,873 by 9,999	8. 4,875,632 ÷ 999
		3. 864,275 by 9,999	6. 4,128,737 by 99.9	9. 43,276 ÷ 9.999

POWERS AND ROOTS.

SHORT METHODS.

Page 208; Art. 353.

1. To find an approximate root of a surd fraction.

Ex. 1. What is the approximate square root of $\frac{8}{15}$?

PROCESS.

$$\frac{8}{15} = \frac{8 \times 15}{15 \times 15} = \frac{120}{15^2}; \quad \sqrt{\frac{120}{15^2}} = \frac{\sqrt{120}}{\sqrt{15^2}} = \frac{11-}{15} = .733-$$

EXPLANATION.—Multiplying both terms of the fraction, $\frac{8}{15}$, by the denominator, 15, I obtain $\frac{120}{225}$, or $\frac{120}{15^2}$. Extracting the square root of the denominator (15^2), I have 15 for the denominator of the root; and, since 120, the numerator, is very near and a little less than the square of 11, I write 11— for the numerator of the root.

Reducing $\frac{11-}{15}$ to a decimal, I obtain .733—, which is correct to two decimal places.

Ex. 2. What is the approximate cube root of $\frac{5}{7}$?

PROCESS.

$$\frac{5}{7} = \frac{5 \times 7 \times 7}{7 \times 7 \times 7} = \frac{245}{7^3}; \quad \sqrt[3]{\frac{245}{7^3}} = \frac{\sqrt[3]{245}}{\sqrt[3]{7^3}} = \frac{6+}{7} = .857+$$

Ex. 3. What is the approximate square root of $\frac{1}{2}$?

PROCESS.

$\frac{1}{2} = \frac{1 \times 2}{2 \times 2} = \frac{1}{2}$, from which no near approximation can be obtained.

$\frac{2 \times 2}{1 \times 9} = \frac{4}{9}$; $\frac{\sqrt{4}}{\sqrt{1 \times 9}} = \frac{2+}{3} = \frac{2}{3} +$, a remote approximation.

$\frac{2 \times 25}{1 \times 25} = \frac{50}{25}$; $\frac{\sqrt{50}}{\sqrt{1 \times 25}} = \frac{7+}{5} = \frac{7}{5} + = .7 +$, a near approximation.

a. All factors introduced into the terms of the fraction after the denominator is so introduced, must be perfect powers; and such factors should be introduced as will make the numerator a near approximation to a perfect power.

b. By ingenuity in introducing factors, near approximations can readily be found.

PROBLEMS.

1. Find an approximate square root of $\frac{1}{2}$.
2. Find an approximate cube root of $\frac{1}{11}$.
3. What is an approximate square root of $\frac{1}{2}$?
4. What is an approximate square root of $\frac{1}{2}$?

2. Contracted decimal division (page 144, Case VI) applied to the extraction of the approximate square and cube roots of surds.

Ex. 1. Find the square root of 23, correct to six decimal places.

PROCESS.

$$\begin{array}{r} 23 \overline{) 4.795831+} \\ 16 \end{array}$$

EXPLANATION.—Extracting the root by the ordinary process until more than one half the number of places in the required root are found, I have for a remainder, 7,975.

Finding the next trial divisor in the usual manner, I contract it two places by cutting off the ciphers; and, using 7,975 for a dividend, divide as in contracted division of decimals; and I so continue until I have the required number of places in the root.

$$\begin{array}{r} 80 \overline{) 700} \\ 87 \quad 609 \\ \hline 940 \quad 9100 \\ 949 \quad 8541 \\ \hline 9580 \quad 55900 \\ 9585 \quad 47925 \\ \hline 9599 \quad 7975 \\ 7672 \\ \hline 303 \\ 288 \\ \hline 15 \\ 10 \end{array}$$

Ex. 2. Find the cube root of 37,184, correct to four decimal places.

PROCESS.

$$\begin{array}{r} 37,184 \overline{) 33.3773+} \\ 27 \end{array}$$

$$\begin{array}{r} 3 \times 30^2 = \quad \quad \quad 2700 \quad 10184 \\ 2700 + 3 \times 30 \times 3 + 3^2 = 2979 \quad 8937 \\ \hline 3 \times 330^2 = \quad \quad \quad 326700 \quad 1247000 \\ 3 \times 330 \times 3 + 3^2 \quad \quad \quad 2979 \\ \hline \quad \quad \quad 329679 \quad 989037 \\ \hline 3 \times 3330^2 = \quad \quad \quad 33266700 \quad 257963 \\ 3 \times 3330 \times 7 + 7^2 = \quad \quad \quad 69079 \\ \hline \quad \quad \quad 333366 \quad 233356 \\ \hline \quad \quad \quad 33407 \quad 24607 \\ \quad \quad \quad \quad 6 \\ \hline \quad \quad \quad 33413 \quad 23389 \\ \hline \quad \quad \quad 3341 \quad 1218 \\ \quad \quad \quad \quad 1002 \end{array}$$

EXPLANATION.—Extracting the root by the ordinary process until more than one third the number of places in the required root are found, I have for a remainder 257,963.

I find the next trial divisor in the usual manner, contract it three places, and omit to annex a period of decimal ciphers to the remainder.

I find the next figure of the root, 7, and also the correction, 69,979, in the usual manner, but contract it three places and add it to the trial divisor, obtaining 83,836 for the approximate complete divisor.

Multiplying and subtracting, I obtain 24,607 for a new dividend.

Contracting the sum of the last complete divisor and the last correction one place, I find a new trial divisor.

I add the correction contracted two places to the trial divisor, and obtain 8,341 for a complete divisor, with which I obtain 7 for the next figure of the root.

Multiplying and subtracting, I obtain 1,218 for a new dividend; and, since the corrections (having disappeared by contraction) can have no effect upon the succeeding divisors, I complete the work by contracted division.

The ordinary processes of extraction of roots make the computations cumbrous when extended to many decimal places; hence the great practical utility of these contracted processes.

PROBLEMS.

Find, correct to 5 decimal places each, the following indicated roots:

- | | | | | | |
|---------------------|------------------------------------|---------------------|---------------------------------------|--------------------|-------------------------------|
| 1. $\sqrt{3,146}$ | 3. $\sqrt{\frac{1}{3}\frac{1}{4}}$ | 5. $\sqrt{.34891}$ | 7. $\sqrt[3]{\frac{1}{4}\frac{1}{4}}$ | 9. $\sqrt{5}$ | 11. $\sqrt[3]{.2534}$ |
| 2. $\sqrt{972,813}$ | 4. $\sqrt{431\frac{1}{4}}$ | 6. $\sqrt[3]{86.5}$ | 8. $\sqrt[3]{3,214}$ | 10. $\sqrt[3]{11}$ | 12. $\sqrt[3]{18\frac{1}{3}}$ |

ROOTS OF HIGHER DEGREES.

Page 208; Art. 353.

Since $7^6 = 7 \times 7 \times 7 \times 7 \times 7 \times 7 = (7 \times 7 \times 7) \times (7 \times 7 \times 7) = (7 \times 7 \times 7)^2 = (7^3)^2$: or $7 \times 7 \times 7 \times 7 \times 7 \times 7 = (7 \times 7) \times (7 \times 7) \times (7 \times 7) = (7 \times 7)^3 = (7^2)^3$, hence the following deductions,—

1. *Any power of a number can be found by separating the exponent of the power into factors, and raising the root by continued involution to the powers indicated by the several factors;—and*

Conversely,—

2. *Any root of a number can be found by separating the index of the root into factors and extracting by successive evolution the roots indicated by the several factors. And, in general,*

3. *Any root whose index contains no other prime factors than 2 and 3 can be extracted by the methods already shown for extraction of square and cube roots.*

b. With reference to the powers of the tens and the ones in each of the several terms:—*That the highest power of the tens is in the first term, and that the exponent of the power decreases regularly by 1, each succeeding term; and that the first power of the ones is found in the second term, and that the exponent of the power increases regularly by 1, each succeeding term, until in the last term it corresponds to the exponent of the tens in the first term.*

c. With reference to the numeral factor in each term except the first and the last:—*That it is the quotient of the continued product of the exponents of the powers of the tens in all the preceding terms DIVIDED by the continued product of the integers in order from 1 to the number of the preceding term inclusive.*

6. The foregoing analysis enables us to deduce any power of a number considered as tens and ones.

Ex. Formulate the 7th power of a number considered as tens and ones.

In the 7th power of a number considered as tens and ones there are 8 terms.

The first term is.....tens⁷;
 the second term.....7 × tens⁶ × ones;
 the third term.....($\frac{7 \times 6}{1 \times 2} = 21$).....21 × tens⁵ × ones²;
 the fourth term.....($\frac{7 \times 6 \times 5}{1 \times 2 \times 3} = 35$).....35 × tens⁴ × ones³;
 the fifth term.....($\frac{7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4} = 35$).....35 × tens³ × ones⁴;
 the sixth term.....($\frac{7 \times 6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4 \times 5} = 21$).....21 × tens² × ones⁵;
 the seventh term.....($\frac{7 \times 6 \times 5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4 \times 5 \times 6} = 7$).....7 × tens × ones⁶;
 the eighth term.....ones⁷, or
 tens⁷ + 7 × tens⁶ × ones + 21 × tens⁵ × ones² + 35 × tens⁴ × ones³ + 35 × tens³ × ones⁴ +
 21 × tens² × ones⁵ + 7 × tens × ones⁶ + ones⁷.

EXTRACTION OF THE FIFTH ROOT.

$$23^5 = (20+3)^5 = 20^5 + 5 \times (20^4 \times 3) + 10 \times (20^3 \times 3^2) + 10 \times (20^2 \times 3^3) + 5 \times (20 \times 3^4) + 3^5 = 3,200,000 + 2,400,000 + 720,000 + 108,000 + 8,100 + 243 = 6,436,343.$$

Pointing off the power, 6,4.36,343, into five-figure periods, and comparing the parts with the power, it will be seen

(1) That the fifth power (32) of the tens (2) of the root is wholly in the first period of the power (64).

(2) That the product of five times the fourth power of the tens (2) and the ones (3) is wholly in that part of the power expressed by the first period of the power minus the fifth power of the tens of the root ($64 - 32 =$) 32 with the *first* figure (3) of the second period of the power annexed (323); and $323 - (5 \times 2^4 \times 3) = 83$.

(3) That the product of ten times the cube of the tens (2) and the square of the ones (3) is wholly in that part of the power expressed by ($323 - 240 =$) 83 with the *second* figure (6) of the second period of the power annexed (836); and $836 - (10 \times 2^3 \times 3^2) = 116$.

(4) That the product of ten times the square of the tens (2) and the cube of the ones (3) is wholly in that part of the power expressed by ($836 - 720 =$) 116 with the *third* figure (3) of the second period of the power annexed (1,163); and $1,163 - (10 \times 2^2 \times 3^3) = 83$.

(5) That the product of five times the tens (2) and the fourth power of the ones (3) is wholly in that part of the power expressed by ($1,163 - 1,080 =$) 83 with the *fourth* figure of the second period of the power annexed (834); and $834 - (5 \times 2 \times 3^4) = 24$.

(6) That the fifth power of the ones (3) is wholly in that part of the power expressed by ($834 - 810 =$) 24 with the last figure of the power annexed (243); and $243 - 3^5 = 0$.

IMPORTANT FACTS.

The product of five times the fourth power of the tens and the ones of the fifth root of a decimal number is wholly in the number formed by subtracting the fifth power of the tens of the root from the first period of the power, and to the remainder annexing the first figure of the second period of the power.

The product of ten times the cube of the tens and the square of the ones is wholly in the number formed by subtracting the product of five times the fourth power of the tens and the ones from the preceding minuend, and to the remainder annexing the second figure of the second period of the power.

The product of ten times the second power of the tens and the cube of the ones is wholly in the number formed by subtracting the product of ten

times the cube of the tens and square of the ones from the preceding minuend, and to the remainder annexing the third figure of the second period of the power.

The product of five times the tens and the fourth power of the ones is wholly in the number formed by subtracting the product of ten times the square of the tens and the cube of the ones from the preceding minuend, and to the remainder annexing the fourth figure of the second period of the power.

The fifth power of the ones is wholly in the number formed by subtracting the product of five times the tens and the fourth power of the ones from the preceding minuend, and to the remainder annexing the last figure of the power.

Ex. Extract the fifth root of 6,436,343.

PROCESS.		6436343 23
$t^5 = 2^5 =$		32
$5 \times t^4 = 5 \times 2^4 =$	80	) 323
$5 \times t^4 \times 0 = 5 \times 2^4 \times 3 =$		240
		836
$10 \times t^3 \times 0^2 = 10 \times 2^3 \times 3^2 =$		720
		1163
$10 \times t^2 \times 0^3 = 10 \times 2^2 \times 3^3 =$		1080
		834
$5 \times t \times 0^4 = 5 \times 2 \times 3^4 =$		810
		243
$t^5 = 3^5 =$		243

STEPS IN THE PROCESS OF EXTRACTING ROOTS OF ANY DEGREE.

- I. Point off into periods.
- II. Find first figure of required root.
- III. Subtract the power of the number expressed by the first figure of the

root from the first period, and to the remainder annex the first figure of the second period.

IV. *Find the trial divisor*,—by raising the root already found to a power one less than the index of the required root, and multiply the result by the index of the root.

V. *Find the next figure, or ones of the root*,—by dividing the dividend by the trial divisor.

VI. *Subtract the corrections in consecutive order*:

The first correction,—the trial divisor times the ones of the root. To the remainder annex the *second* figure of the second period for a minuend.

The second correction,—the tens raised to a power one less than in the first correction, *times* the square of the ones, *times* the quotient of the product of the index of the root and the exponent of the power of the tens in the first correction divided by 2. To the remainder annex the *third* figure of the second period for a minuend.

The third correction,—the tens raised to a power one less than in the second correction, *times* the cube of the ones, *times* the quotient of the continued product of the index of the root and the exponents of the powers of the tens in the preceding corrections divided by 2×3 . To the remainder annex the *fourth* figure of the second period for a minuend.

The fourth correction,—the tens raised to a power one less than in the third correction, *times* the fourth power of the ones, *times* the quotient of the continued product of the index of the root and the exponents of the powers of the tens in the preceding corrections divided by $2 \times 3 \times 4$.

Continue in like manner until the number of corrections is equal to the index of the root. If a remainder arises by subtracting the last correction, the number expressed by the two figures of the root found becomes the *tens* of the root; and

VII. *Find the trial divisor, the next figure of the root*, and proceed as before with the corrections arising; and so continue to any required degree of accuracy, or until the exact root is obtained.

Ex. What is the fifth root of 716,703,146,875 ?

SHORT PROCESS.

$$\begin{array}{r}
 t^5 = 2^5 = 32 \\
 5 \times t^4 = 5 \times 2^4 = 80 \overline{) 396:7031} \\
 5 \times t^4 \times o = 5 \times 2^4 \times 3 = 240 \\
 10 \times t^3 \times o^2 = 10 \times 2^3 \times 3^2 = 720 \\
 10 \times t^2 \times o^3 = 10 \times 2^2 \times 3^3 = 1080 \\
 5 \times t \times o^4 = 5 \times 2 \times 3^4 = 810 \\
 o^5 = 3^5 = 243 \\
 5 \times t^4 = 5 \times 23^4 = 1399205 \overline{) 7306884:6875} \\
 5 \times t^4 \times o = 5 \times 23^4 \times 5 = 6996025 \\
 10 \times t^3 \times o^2 = 10 \times 23^3 \times 5^2 = 3041750 \\
 10 \times t^2 \times o^3 = 10 \times 23^2 \times 5^3 = 661250 \\
 5 \times t \times o^4 = 5 \times 23 \times 5^4 = 71875 \\
 o^5 = 5^5 = 3125
 \end{array}$$

The last example is an extension of the short methods of extracting the square and cube roots to the extraction of roots of the fifth degree. The same methods also apply to the extraction of roots of any degree.

PROBLEMS.

Find the following indicated roots:

9. $\sqrt[5]{1,350,125,107}$	12. $\sqrt[7]{24,160,660,561,265,139}$
10. $\sqrt[5]{66,338,290,976}$	13. $\sqrt[5]{7,148,929,565,430,432}$
11. $\sqrt[4]{298,248,146,641}$	14. $\sqrt[11]{2,384,185,791,015,625}$

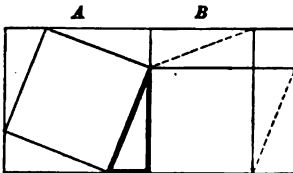


Diagram showing:—*The square of the hypotenuse of a right-angled triangle equals the sum of the squares on the other two sides.*

EXPLANATION.—*A* and *B* are equal squares. In *A* is inscribed a square, and from its point of contact with the side of *B* a line is drawn across *B*, parallel with its base. A line is drawn at right-angles to this line to resolve *B* into two squares and two equal rectangles. The sum of these rectangles equals the sum of the triangles of *A*. Subtracting these equals from the equal squares the remainders are equal. Hence, will be seen the truth of the proposition.

MEDIAL PROPORTION.*Page 167; Art. 304.*

1. *Medial proportion* is the relation which different quantities or qualities bear to their *average* or *mean* (304).

2. *Alligation* is the process of determining medial proportions.

a. *Simples* are the different quantities or qualities combined to produce a given average or mean.

b. A *compound* is the whole, formed by a combination of simples.

There are two general processes of alligation, known as Alligation Medial and Alligation Alternate.

3. *Alligation medial* is Average (305).

4. *Alligation alternate* is the process of finding the proportions of two or more simples that will produce a given average or mean.

a. The numbers expressing the proportions in alligation alternate are *medial ratios*, and are subject to the principles governing partitive ratios (Art. 391).

b. Alligation medial and alligation alternate are applied to converse problems, but are *not* converse operations.

c. Alligation medial, or average, is largely applied to business affairs. Alligation alternate is more a mathematical curiosity; still there is a limited class of problems that have no other arithmetical solution.

5. Since an average or mean always implies simples, some less and some greater than the average, we have

PRINCIPLE I.—*The required average must be greater than some of the simples and less than others.*

6. Since in average there is neither gain nor loss (the gain or gains in one instance balancing the loss or losses in another), we have

PRINCIPLE II.—*To produce a compound of a given average, a gain on any simple or simples must be balanced by a corresponding loss on some other simple or simples.*

Ex. 1. In what ratios must paints worth \$1.20 and \$1.80 per gallon be mixed to make the mixture worth \$1.55 per gallon?

EXPLANATION.—I write the average price at the left of a vertical line and the simples or prices to be compared with the average in a column (headed *simples*) at the right.

In a second column (headed *gains and losses*), I write the gain or loss on a gallon of each kind of paint when valued at the average price—i. e., 85¢, the gain or advance on a gallon worth \$1.20 when made worth \$1.55, and 25¢, the loss on a gallon worth \$1.80, under the same conditions.

Since the ratio must be such that the gains and losses balance each other, I write in a third column (headed *medial ratios*) that part of a gallon of each of the two simples on which the gain or loss respectively will be 1¢,—i. e., $\frac{1}{3}$ gal. @ \$1.20 and $\frac{1}{7}$ gal. @ \$1.80.

To reduce this ratio to its lowest integral terms, I multiply both terms by the least common multiple of the denominators, and obtain 5 gal. @ \$1.20 and 7 gal. @ \$1.80.

7. The preceding process may be varied as shown in the second process.

SECOND PROCESS.

EXPLANATION.—I place the average price, simples, and gains and losses respectively, as in the preceding process.

Since the medial ratios are directly derived from the gain and the loss, multiplying or dividing both of them by the same number will be, in effect, multiplying or dividing the terms of the medial ratio by that number. Again,

Since the gain and the loss are obtained by a comparison of the average, \$1.55, and the simples, \$1.20 and \$1.80, multiplying or dividing both the average and the simples by the same number will be, in effect, multiplying or dividing the gain and the loss by that number. Hence,

THIRD PROCESS.

Average price	Simples	Ratio of average and simples	Ratio of gains and losses	Medial ratios	Ratio in lowest integers
\$1.20		2 4 ¢	7 ¢	$\frac{1}{7}$ gal.	5 gal.
\$1.55		3 1 ¢			
\$1.80		3 6 ¢	5 ¢	$\frac{1}{5}$ gal.	7 gal.

8. IMPORTANT FACTS.

I. In the process of alligation, the numbers in each column except the

columns of simples and of gains and losses are the terms of a complete medial ratio.

II. The numbers from which the medial ratios are derived, are

- (1) The set in the column of gains and losses; and
- (2) The set composed of the average and the column of simples.

PRINCIPLE III.—*Multiplying or dividing by the same number all the terms of any medial ratio, or of any set of numbers from which a medial ratio is derived, does not change the relation of the parts to each other or to the compound.*

Ex. 2. In what parts must silver $\frac{2}{3}$ pure, $\frac{1}{3}$ pure, and $\frac{1}{4}$ pure, be combined to form a compound $\frac{7}{9}$ pure?

STEPS IN PROCESS;—BY COLUMNS.—(1) Average quality. (2) Simples. (3) Proportions of average quality and simples in lowest integers.

PROCESS.

(1)	(2)	(3)		(4)	(5)	(6)	(7)	(8)	(9)
$\frac{7}{9}$	$\frac{2}{3}$		60	10	$\frac{1}{6}$	$\frac{1}{6}$	1	1	2
	$\frac{1}{3}$	.. 70		...	...	...	...	...	...
	$\frac{1}{4}$		72	2		$\frac{1}{2}$		5	5
	$\frac{1}{4}$		75	5	$\frac{1}{4}$		2		2

(4) Gain and losses. (5) Ratio of gain and second loss.

(6) Ratio of gain and first

loss. (7) Lowest integral terms of (5). (8) Lowest integral terms of (6). (9) Sum of (7) and (8) (ratios of like simples united).

Result,—2 parts $\frac{2}{3}$ pure, 5 parts $\frac{1}{3}$ pure, and 2 parts $\frac{1}{4}$ pure.

Ex. 3. In what ratio must teas worth 40¢, 55¢, 63¢, and 70¢ per pound be mixed to form a compound worth 60¢ per pound?

STEPS IN PROCESS.—BY COLUMNS.—(1) Average price. (2) Simples. (3) Gains and losses. (4) Ratio from first gain combined with second loss. (5) Ratio from second gain combined with first loss. (6) Lowest integral terms of (4). (7) Lowest integral terms of (5). (8) Sum of (6) and (7). (There are no two ratios of the same simple to be united.)

FIRST PROCESS.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	40¢	20¢	$\frac{1}{6}$ lb.		1 lb.		1 lb.
	55¢	5¢		$\frac{1}{2}$ lb.		3 lb.	3 lb.
60¢							
	63¢	3¢		$\frac{1}{2}$ lb.		5 lb.	5 lb.
	70¢	10¢	$\frac{1}{6}$ lb.		2 lb.		2 lb.

Result,—1 lb. @ 40¢, 3 lb. @ 55¢, 5 lb. @ 63¢, and 2 lb. @ 70¢.

By COLUMNS.—(1) Average price. (2) Simples. (3) Gains and losses. (4) Ratio from first gain combined with first loss. (5) Ratio from second gain combined with second loss. (6) Lowest integral terms of (4). (7) Lowest integral terms of (5). (8) Sum of (6) and (7).

SECOND PROCESS.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	40¢	20¢	$\frac{1}{2}$ lb.		3 lb.		3 lb.
	55¢	5¢		$\frac{1}{2}$ lb.		2 lb.	2 lb.
60¢							
	63¢	3¢	$\frac{1}{2}$ lb.		20 lb.		20 lb.
	70¢	10¢		$\frac{1}{2}$ lb.		1 lb.	1 lb.

Result,—3 lb. @ 40¢, 2 lb. @ 55¢, 20 lb. @ 63¢, and 1 lb. @ 70¢.

9. From an inspection of the foregoing operations may be deduced

PRINCIPLE IV.—*An original medial ratio in alligation alternate consists of two terms only.*

PRINCIPLE V.—*The sum of two or more medial ratios is itself a medial ratio.*

PRINCIPLE VI.—*A medial ratio derived from other medial ratios by addition consists of a number of terms not less than 2 nor greater than the number of simples.*

10. Resuming the consideration of the first process, Ex. 3, at the columns containing the ratios in their lowest integral terms (6) and (7), viz.:

(6)	(7)	(8)	(9)	(10)		(11)	(12)	(13)		(14)	(15)	(16)		(17)	(18)	(19)
1 lb.		3 lb.		3 lb.	, or	3 lb.	6 lb.	9 lb.	, or	12 lb.	2 lb.	3 lb.	, or	5 lb.	6 lb.	2 lb.
	3 lb.		6 lb.	6 lb.			12 lb.	12 lb.		18 lb.	3 lb.	3 lb.		6 lb.	9 lb.	3 lb.
	5 lb.		10 lb.	10 lb.			20 lb.	20 lb.		30 lb.	5 lb.	5 lb.		10 lb.	15 lb.	5 lb.
2 lb.		6 lb.		6 lb.		6 lb.	12 lb.	18 lb.		24 lb.	4 lb.	6 lb.		10 lb.	12 lb.	4 lb.

(8) Terms of (6) multiplied by 3, Prin. III. (9) Terms of (7) multiplied by 2, Prin. III. (10) Sum of (8) and (9), Prin. V. (11) Ratio in (8). (12) Terms of (10) multiplied by 2, Prin. III. (13) Sum of (11) and (12), Prin. V. (14) Sum of (10) and (13), Prin. V. (15) Terms of (14) divided by 6, Prin. III. (16) Sum of (6) and (15), Prin. V. (17) Sum of (15) and (16), Prin. V. (18) Sum of (6), (7), and (17), Prin. V. (19) Terms of (18) divided by 3, Prin. III.

The results obtained in (10), (13), (15), (16), (17), (19) are all expressed in lowest integers, are all correct, and are therefore of equal credit with the result first obtained in Ex. 3.

EXPLANATION.—I proceed by the general process until I have obtained the general ratio in lowest integers—2 bu. @ 90¢, 2 bu. @ 95¢, 1 bu. @ \$1.10, and 1 bu. @ \$1.20. The sum of the terms of the general ratio is 6 bu.; and since the sum of the simples is limited to 360 bu., I multiply the terms of the general ratio by 60 ($= 360 \text{ bu.} \div 6 \text{ bu.}$), and obtain 120 bu. @ 90¢, 120 bu. @ 95¢, 60 bu. @ \$1.10, and 60 bu. @ \$1.20.

Another correct result to the preceding example may be obtained by combining the first gain with the first loss, and the second gain with the second loss. Using all the variations under both methods of combination, 450 correct results may be obtained.

PROBLEMS.

1. Required the lowest integral terms of the proportions in which sugars worth 6¢, 7¢, and 9¢ per pound, can be combined to form a compound worth $7\frac{1}{2}$ ¢ per pound.

2. What is the least number of gallons of vinegar @ 20¢ and 15¢ that must be mixed with $31\frac{1}{2}$ gallons @ 22¢, to form a compound worth 18¢ per gallon?

3. A merchant has four grades of pepper worth respectively 24¢, 30¢, 33¢, and 42¢ per pound. What is the least number of pounds of each that can be taken, to form a mixture worth 32¢ per pound?

4. A boy paid 20¢ for 20 lead-pencils at the rates of 4 pencils for 1¢, 2 pencils for 1¢, and 1 pencil for 4¢. How many pencils at each price did he purchase?

A cubic foot of gold weighs $1,203\frac{1}{2}$ pounds avoirdupois, and a cubic foot of silver $654\frac{3}{4}$ pounds. A bar of gold alloyed with silver is 6 inches long and 1 inch square, and weighs $3\frac{1}{4}$ pounds.

5. Required the weight of each metal in the bar.

6. Required the bulk of each metal.

7. Required the value of each metal.

8. A farmer purchased 52 sheep for \$156, paying for different qualities \$2.50, \$2.80, \$3.15, and \$3.45 per head. Required 9 proportions or results that will answer the conditions of the problem.

9. How many gallons of water must be mixed with vinegar worth 30¢ per gallon, to make 80 gallons, worth 24¢ per gallon?

ADDITIONAL FORMULÆ.

ARITHMETICAL PROGRESSION.

Page 236; Art. 402.

$$a = \frac{2 \times S - n \times (n - 1) \times d}{2 \times n}$$

$$a = \frac{2 \times S}{n} - l$$

$$l = \frac{2 \times S + n \times (n - 1) \times d}{2 \times n}$$

$$l = \frac{2 \times S}{n} - a$$

$$l = \frac{\sqrt{8 \times d \times S + (2 \times a - d)^2} - d}{2}$$

$$n = \frac{2 \times S}{a + l}$$

$$n = \frac{d - 2 \times a + \sqrt{(d - 2 \times a)^2 + 8 \times d \times S}}{2 \times d}$$

$$n = \frac{d + 2 \times l - \sqrt{(d + 2 \times l)^2 - 8 \times d \times S}}{2 \times d}$$

$$d = \frac{2 \times (S - a \times n)}{n \times (n - 1)}$$

$$d = \frac{2 \times (n \times l - S)}{n \times (n - 1)}$$

$$d = \frac{(l + a) \times (l - a)}{2 \times S - l - a}$$

$$S = \frac{n \times [2 \times a + (n - 1) \times d]}{2}$$

$$S = \frac{n \times [2 \times l - (n - 1) \times d]}{2}$$

$$S = \frac{(l + a) \times (l - a + d)}{2 \times d}$$

GEOMETRICAL PROGRESSION.

Page 240; Art. 412.

$$a = \frac{r - 1}{r^n - 1} \times S$$

$$a = S - (S - l) \times r$$

$$l = \frac{(r - 1) \times S + a}{r}$$

$$l = S - \frac{S - a}{r}$$

$$l = \frac{r^n - r^{n-1}}{r^n - 1} \times S$$

$$r = \frac{S - a}{S - l}$$

$$r^n = \frac{(r - 1) \times S + a}{a}$$

$$S = \frac{l - a}{r^n - 1} + l$$

$$S = \frac{l \times (r^n - 1)}{(r - 1) \times r^{n-1}}$$

INSURANCE.

Page 258; Art. 443.

Define *insurance company* (Art. 443).

1. Insurance companies are classified according to principles of organization, as stock, mutual, mixed, co-operative, or assessment.

a. A *stock* insurance company is one that has a capital stock owned by stockholders who alone share the profits and losses of the business.

- b.* A *mutual* insurance company is one of which every person insured therein becomes a member and shares in the profits and losses of the business.
- c.* A *mixed* insurance company is one that has a capital stock owned by stockholders whose profits are limited to a certain per cent. on their stock. The surplus profits, if any, are divided among the policy-holders.
- d.* A *co-operative* or *assessment* insurance company is one having a general fund raised by membership fees and dues, from which fund the general expenses are paid, and another fund raised by assessments upon the members, from which losses are paid.

2. The *risk* is the person or property upon which the insurance is taken.

Risks are divided into various classes, according to liability to loss; and the cost of insurance depends upon the class in which the risks are placed.

a. The chief conditions affecting *fire insurance risks and rates* are

- (1) The use to which the property is put.
e.g. A match factory is more liable to burn than an ice-house.
- (2) The certainty of title of the insured.
e.g. If a man's title were uncertain he might prefer the amount of insurance to the property.
- (3) The distance from other inflammable property.
e.g. If two buildings touch each other, there is greater probability that both will burn than that one would burn if it stood alone.
- (4) The material and construction of a building.
e.g. A wooden building is more liable to burn than a stone building; tin roofs are safer than shingle roofs; stove-pipes that pass through floors or ceilings are dangerous.
- (5) The facilities for extinguishing fires.
e.g. Nearness to water, street hydrants, good fire departments—all tend to lessen the chances of conflagration.

b. The chief conditions affecting *marine risks and rates* are

- (1) The kind of trade in which the vessel is employed.
e.g. There is more danger in a whaling trip to the Arctic Ocean than in a trip across the Atlantic.
- (2) The nature of the cargo.
e.g. A cargo of salt is safer than a cargo of petroleum.
- (3) The certainty of title of the insured. See *a*, (2), *e.g.*
- (4) The seaworthiness or stanchness of the vessel.

c. The chief conditions affecting *life and accident risks and rates* are

- (1) The age of the insured.

- (2) The occupation of the insured.
- (3) The place of residence of the insured.

3. Insurance policies are of several kinds, the principal of which are as follows :

a. IN FIRE INSURANCE,

- (1) The *ordinary* policy,—the conditions of which are, that the property insured shall remain in a certain place, and that all losses shall be paid in full unless they exceed the valuation given in the policy.
- (2) The *floating* policy,—which permits the location of the property to be changed in the ordinary course of trade and manufacture.
- (3) The *average clause* policy,—which in case of partial loss indemnifies the insured for only such a part of the loss as the policy valuation is part of the real value of the property insured.

b. IN MARINE INSURANCE,

- (1) The *valued* or *closed* policy,—in which the property insured is distinctly specified in the ship's *Manifest* or *Bill of Lading*. This policy does not indemnify against loss to property taken on board the vessel at a later date.
- (2) The *open* policy,—which indemnifies against loss to property shipped at different times and ports, by indorsing or entering the same on the policy, the conditions and rates being already fixed in the policy. This form of policy allows goods shipped in foreign countries to be insured in home companies.

c. IN LIFE INSURANCE,

- (1) The *ordinary life* policy,—which requires payments of premium during life, and which is payable only upon the death of the insured.
- (2) The *limited payment* life policy,—which requires all premiums to be paid in a specified term, as 5 years, 10 years, 15 years.
- (3) The *endowment* policy,—which is payable at the end of a specified term, or sooner upon the death of the insured, the premiums being payable during the continuance of the policy.
- (4) The *limited payment* endowment policy,—which provides for the payments of all premiums in 5 years, 10 years, 15 years, 20 years.
- (5) The *term* life policy,—which indemnifies only in case of death during a specified term.
- (6) The *annuity* policy,—which secures by a single payment of premium, the payment to the insured of a certain sum annually during his life-time.
- (7) The *joint life* policy,—which secures the payment of a certain sum on the death of either one of two persons, to the other.

4. The adjustment of losses in marine insurance follows the plan of the *average clause* fire-insurance policy.

5. Insurance companies usually refuse to insure unoccupied buildings because of their liability to be burned by incendiaries.

a. In no case can the insured collect more than the value of the property destroyed, even if the valuation in the policy is greater and the risk is a total loss.

b. For terms of insurance less than one year, the rates of fire insurance are proportionally much greater than regular or year rates. The rates for such short terms are known as *short rates*.

6. The *assets* of an insurance company include all property of the company,—as cash, real-estate, securities, balances due, etc.

The *liabilities* are all of its debts, outstanding obligations, reserve for re-insurance, premiums paid in advance, etc.

LIFE AND ACCIDENT INSURANCE.

7. *Life insurance* is a guaranty to pay a specified sum of money to the heirs or beneficiary of a person when he dies, or to himself, if living at the end of a specified term.

a. The *reserve* of a life insurance policy is such a sum as, put at interest at the rate established by law, and increased by the net premiums still to be paid, will amount to the sum guaranteed by the policy, when it will become due as shown by average life tables.

b. The *surrender value* of a life-insurance policy is the amount of cash which the company will pay the holder at any time upon the surrender of the policy. It is the legal reserve less an estimated per cent for expenses.

c. The *expectation of life* is the number of years that a person of a given age will probably live. It is the average life of persons of that age as determined by carefully kept records of mortality.

d. The premium may be payable annually, semi-annually, or quarterly—as prescribed by the terms of the policy.

8. *Accident insurance* is a guaranty to the insured of a weekly indemnity in case of disability from accident, or of the full valuation in case of death from accident.

a. The payment of the weekly indemnity is usually limited, by the terms of the policy, to the term of six months, whether the disability has ceased or not; and in any case the weekly indemnity ceases when the disability ceases.

b. Death resulting from accident must occur within 90 days of the accident to secure the payment of the full valuation.

9. TABLE OF RATES.

Annual Premium for an Insurance of \$1,000.

LIFE POLICIES.					ENDOWMENT POLICIES.				
Age	Annual Premium	10 Annual Payments	15 Annual Payments	20 Annual Payments	10-Year Endowment	15-Year Endowment	20-Year Endowment	25-Year Endowment	30-Year Endowment
25	\$20 50	\$ 43 50	\$33 10	\$28 10	\$105 90	\$67 40	\$48 70	\$38 00	\$31 40
26	21 00	44 80	33 80	28 60	106 00	67 50	48 90	38 20	31 60
27	21 50	45 20	34 40	29 20	106 10	67 60	49 00	38 40	31 80
28	22 10	46 10	35 10	29 80	106 30	67 80	49 20	38 60	32 00
29	22 70	47 00	35 90	30 50	106 40	68 00	49 40	38 80	32 30
30	23 80	48 00	36 60	31 10	106 60	68 20	49 60	39 10	32 60
31	24 00	49 10	37 40	31 80	106 80	68 30	49 80	39 30	32 90
32	24 70	50 10	38 30	32 60	107 00	68 60	50 10	39 60	33 20
33	25 50	51 20	39 10	33 30	107 20	68 80	50 30	39 90	33 60
34	26 30	52 40	40 00	34 10	107 40	69 00	50 60	40 30	34 00
35	27 10	53 60	41 00	35 00	107 60	69 30	50 90	40 60	34 50
36	28 00	54 80	42 00	35 80	107 80	69 60	51 30	41 10	35 00
37	29 00	56 20	43 00	36 80	108 10	69 90	51 70	41 50	35 60
38	30 00	57 50	44 10	37 70	108 40	70 20	52 10	42 00	36 20
39	31 10	59 00	45 30	38 80	108 70	70 60	52 50	42 60	36 80
40	32 20	60 40	46 50	39 80	109 10	71 00	53 00	43 20	37 60
41	33 40	62 00	47 70	41 00	109 40	71 50	53 60	43 90	38 40
42	34 70	63 60	49 00	42 20	109 80	72 00	54 20	44 60	39 30
43	36 10	65 30	50 40	43 50	110 30	72 50	54 80	45 40	40 30
44	37 50	67 10	51 90	44 80	110 80	73 10	55 60	46 30	41 30
45	39 10	69 00	53 40	46 20	111 30	73 80	56 40	47 30	42 50
46	40 70	70 90	55 10	47 80	112 00	74 60	57 30	48 40	43 80
47	42 50	73 90	56 80	49 40	112 60	75 40	58 30	49 60	45 20
48	44 40	75 10	58 60	51 10	113 40	76 30	59 40	51 00	46 80
49	46 40	77 30	60 50	52 90	114 20	77 30	60 70	52 40	48 50
50	48 50	79 60	62 50	54 80	115 10	78 40	62 00	54 00	50 30
51	50 80	82 10	64 60	56 90	116 10	79 70	63 50	55 30	
52	53 30	84 60	66 90	59 10	117 20	81 00	65 20	57 70	
53	55 90	87 30	69 20	61 40	118 40	82 50	67 00	59 90	
54	58 70	90 10	71 80	63 90	119 80	84 10	68 90	62 20	
55	61 60	93 00	74 40	66 60	121 20	85 90	71 10	64 70	
56	64 80	96 10	77 30	69 50	122 80	87 90	73 50		
57	68 20	99 30	80 30	72 60	124 60	90 10	76 10		
58	71 80	102 70	83 50	75 80	126 50	92 50	78 90		
59	75 70	106 30	86 90	79 40	128 70	95 10	82 10		
60	79 90	110 10	90 60	83 20	131 00	98 00	85 50		

- a. The tables here given are those of the Equitable Life Assurance Society of the United States.
- b. The Tables of Rates of different life insurance companies vary according to the plan of insurance followed by the company ; but the real cost of insurance is very nearly the same in all, for the benefits secured.

Ex. 1. I desire to take an ordinary life policy for \$2,500. If my age is 33 years, what annual premium must I pay ?

PROCESS.

\$25.50, premium on \$1,000 at 33 years. (See Table.)

$\$25.50 \times 2,500 = \$63.75$, the annual premium required.

Ex. 2. Find the total amount of the payments, without interest, on a life policy of \$12,000 secured by 15 annual payments, and issued to a person 42 years old.

PROCESS.

\$49, premium on \$1,000 at 42 yr., 15 payments.

$\$49 \times 12,000 = \$588$, annual premium on \$12,000.

$\$588 \times 15 = \$8,820$, the total sum of payments required.

Ex. 3. Find the total amount of the payments to secure a 20-year endowment of \$10,000, age 38.

PROCESS.

\$52.10, premium on \$1,000. (See Table.)

$\$52.10 \times 10,000 \times 20 = \$10,420$, total payments required.

PROBLEMS.

Find the annual payment required at the age of 40 for a policy of \$4,250

1. By the ordinary life plan.
2. By the 20-payment life plan.
3. By the 20-year endowment plan.
4. By the 30-year endowment plan.
5. A clergyman at the age of 35 had his life insured by the ordinary plan for \$5,000, and died after making 17 payments. How much more did his family receive than he had paid to the company?
6. If I take an endowment policy of \$3,000 at the age of 28, payable to myself in 20 years, and the dividends increase the policy \$912.25, how much more will I receive from the company than I have paid?

7. Two persons aged 35 years and 42 years each take a life policy premiums payable until death. What will be the age of each when the payments of premium exceed the face of the policy?

8. A gentleman 36 years of age has his life insured for \$4,000 premium during life, after the sixth payment the premium is reduced one-half by dividends. Find the amount of the first 20 payments.

GENERAL AVERAGE.

10. *General average* is the adjustment of losses of property at sea—voluntarily made for the purpose of saving the vessel and the remaining cargo.

a. That a loss may be subject to general average, three conditions are necessary,—

1st. A *necessity* for sacrificing a part to save the remainder;

2d. The actual *sacrifice* of a part to save the remainder; and

3d. The *remainder must be saved* as a result of the sacrifice.

b. *Jettison* is the portion of a cargo thrown overboard in order to lighten the ship and save the remainder of the cargo.

c. *Salvage* is the amount paid for assistance rendered by another vessel in time of danger.

11. The *contributory interests* in general average are

(1) The value of the vessel when refitted.

(2) The freight less $\frac{1}{4}$ for seamen's wages.

In New York, Virginia, California, and some other States $\frac{1}{4}$ of the freight is deducted for seamen's wages.

(3) The full value of the cargo at its port of destination.

12. The *losses* and *expenses* subject to general average are

(1) Jettison, and also the damage to other parts of the cargo from the removal of a part, either by water or by breakage.

(2) Repairs to the ship less $\frac{1}{4}$ on account of superior worth of new work.

(3) Expenses of detention, including salvage, wages, and provisions of crew, expense of unloading and reloading, adjuster's fees, etc.

13. In computations in general average,

1. *Total contributory interest* = base. | 2. *Loss + expense* = the percentage.

PROBLEMS.

1. The ship *Silas Richards* on a voyage from New York to Charleston became stranded on the coast of North Carolina, when it was found necessary to throw overboard 506 barrels of flour belonging to Goodrich & Co., worth in Charleston \$6.87 per barrel. The expense of getting the ship afloat was \$197; of supplying new rigging, \$240; and the necessary expenses of detention were \$240. The ship, at the time of average, was valued at \$10,232; the freight at \$4,800. Goodrich & Co. had on board 1,000 barrels of flour; Burke & Malone goods worth \$4,000; Miller Bros. goods worth \$5,236; and Jones, Howe & Co. goods worth \$9,000. Find how much Goodrich & Co. realize on their shipment of flour; how much each interest contributes towards the loss; and the per cent on the contributory interests, the adjuster's fees being \$75.

Statement of Charges.

SHIP AND OWNERS.

\$

\$

CONTRIBUTORY INTERESTS AND APPORTIONMENTS IN GENERAL AVERAGE.

				Losses.	
Ship,	\$			Jettison,	\$
Freight, less $\frac{1}{2}$,				Repairs, less $\frac{1}{2}$,	
				Cost of detention,	
Cargo:				Total,	\$
Goodrich & Co.,	\$		$\div$	\$	=
Burke & Malone,	\$	$\times$	=	, payable by ship.	
Miller Bros.,		$\times$	=	"	freight.
Jones, Howe & Co.,		$\times$	=	"	Goodrich & Co.
Total,	\$	$\times$	=	"	Burke & Malone.
		$\times$	=	"	Miller Bros.
		$\times$	=	"	Jones, Howe & Co.

SETTLEMENT.

	Dr.		Cr.		To pay.	To receive.
Ship,	\$				\$	
Freight,	\$					
Goodrich & Co.,						\$
Burke & Malone,						
Miller Bros.,						
Jones, Howe & Co.,						
	\$		\$		\$	\$
\$	$\times$ (	) + \$	= \$		, Goodrich & Co. realize.	
			2 E			

NOTE.—1. The Statement of Charges contains a column headed "Ship and Owners" for any items payable by ship-owner and not subject to general average.

2. The footings of columns in Settlement headed "Dr." and "Cr." should agree; also those headed "To pay" and "To receive."

3. The bark *La Belle* from Boston to Rio Janeiro being in distress at sea, cut away her mainmast, threw overboard cargo belonging to Williams & Co. valued at \$3,650 in Rio Janeiro, and cargo belonging to Briggs, Sewell & Co. valued at \$4,190, and put into Havana for repairs. The repairs cost \$1,764; the expense of detention was \$935.80; and the adjuster's fees were \$95. The vessel was valued at \$26,700; and the freight at \$3,925. The cargo consisted of goods of Williams & Co. valued at \$9,700; goods of Briggs, Sewell & Co. valued at \$13,250; goods of Gates & Brown valued at \$5,000; and goods of M. L. Hair valued at \$11,300. Apportion the losses by General Average.

STOCKS.

Page 258; Case V.

1. Define a *bond*.

- a.* Bonds bearing interest are issued by nations, states, cities, and other corporations as securities upon which to borrow money.
- b.* *Coupon* bonds are those to which interest certificates or coupons are attached. The coupons are cut off as the interest named in each becomes due, and are retained by the official paying the interest as his receipts for the money paid.
- c.* *Registered* bonds are those which are registered or recorded in the name of the owner in the books or records of the government or corporation issuing them. The interest on registered bonds is payable at a designated place, or by draft payable to the order of their owner.
- d.* Coupon bonds are bought and sold by mere transfer of the bonds from one party to another; but registered bonds are bought and sold by assignment, which must be entered on the records of the party issuing the bonds.

2. A *stock broker* is a broker who deals in the purchase and sale of stocks.

3. A *stock exchange* is an association of stock brokers who meet and transact business by certain recognized forms, regulations, and usages.

Stock quotations are reports of prices bid and paid for stocks listed in the stock exchange.

4. In order to read intelligently the published reports of stock transactions, a knowledge of the technical terms used among brokers is essential.

Bear, a broker who operates to effect a decline in the price of stocks.

Block of stocks, a large number of shares bought or sold in a lump.

Bull, a broker who operates to effect a rise in the price of stocks.

Buyer 30 da. or **B. 30**, buyer's option to continue 30 days.

B. 20 & int. }
B. 20 flat } buyer's option to continue 20 days, and include interest.

B. 20 w. n., buyer's option to continue 20 days, and take effect without notice.

b. c., between calls.

Buyer's option,¹ buyer to name the time for delivery. (See Option.)

Buying flat, all accrued interest to go to the purchaser.

Bucket shop, an office kept by a broker not a member of the stock exchange.

Call,¹ a privilege of buying of the signer or seller. (See Stock privilege.)

Collateral, shares of stock or bonds given in pledge as security for money loaned.

Cons. or **cn.**, consolidated.

Convertible } applied to bonds that are privileged to be exchanged for the capital
Conv. } stock of the company issuing them.

Corner, the control of some particular stock by the bulls.

Cover, the buying of stocks to meet contracts for delivery when the seller is short of the stock to be delivered.

Curb-stone broker, a broker who transacts business on the sidewalk, and is not a member of any regular board of brokers.

Dead duck, a bankrupt broker.

Deposit, a sum of money placed in the hands of a broker to cover losses he may sustain in behalf of his principal.

Double privilege, a spread or a straddle.

Dividend on, including a pending dividend.

Earnest money. (See Deposit.)

Ex-div., not including a dividend recently made or soon to be made.

Ex-coup., not including the coupons for the next payment of interest.

Flyer, a purchase of stocks at random ; a small side operation.

G't'd, guaranteed.

Hypothecate, to deposit or put up collaterals.

Investment stocks, stocks that pay dividends, and are in demand for income.

Lame duck, a broker who has failed to meet his engagements.

Lamb, a green outsider who undertakes to speculate in stocks.

l. g., land grants.

Long of stocks,² holding a large quantity of some particular stocks for a rise.

Margin. (See Deposit.)

Opening day, the day the books of a corporation are first opened after a dividend has been declared.

On the street, outside of the exchange.

Option,² a contract for the purchase or sale of a certain number of shares of a given stock, at a specified price and at any day within a limited time.

Outsider, is a broker not a member of the stock exchange.

Overloaded, under contract to purchase more stock than financially able to pay for.

Pool, stock and money contributed by a clique to carry through a prearranged speculation.

Put,¹ a privilege of selling to the signer or buyer. (See Stock privilege.)

Pref., preferred; applied to stocks entitled to priority of dividend.

Reg., or **R. bds.,** registered bonds.

Regular way, to be delivered or paid for the next business day.

Seller 30 da., or **S. 30,** seller's option to continue 30 days.

S. 20 & int. } seller's option 20 da., and include interest.
S. 20 flat }

S. 20 w. n., seller's option 20 da., without notice.

Seller's option,¹ seller to name the time for delivery. (See Option.)

Selling long. (See Long of stocks.)

Selling short. (See Short of stocks.)

Short of stocks,³ not having sufficient stocks to meet contracts. .

Settling day, the last day allowed in a contract for the delivery of stock.

S. f., sinking funds.

Speculative stocks, stocks that pay no dividends; not valuable for income.

Spread,¹ a privilege to either buy of or sell to the maker, at the option of the holder. (See Stock privilege.)

Stag. (See Outsider.)

Stock jobber, a broker who deals in stocks for speculative purposes.

Stock privilege,⁴ a contract securing to the holder the right to buy of or to sell to the signer or maker, or to either buy or sell as he may elect, a certain number of shares of a stock, at a given price, at his option within a time named, but does not bind the *holder* to complete the contract unless he so elects. The holder forfeits the money paid for the contract, if it is not his interest to use it.

Straddle,¹ a privilege to buy of or to sell to the maker at the same fixed price for either buying or selling. (See Stock privilege.)

Syndicate, a combination of bankers who undertake to place a loan.

Wash, a mock sale for the purpose of keeping the stock quoted.

Watering stock, increasing the number of shares of stock without a corresponding increase in the value of the property which it represents.

1st m. bds., 2d m. bds., first-mortgage bonds, etc.

5 up, 10 up, the per cent on the market value required, as a deposit, of a broker whose financial standing is questioned.

¹ On options for more than 3 days one day's notice must be given by the holder to enforce the option. In dealing in options and stock privileges the broker against whom a loss accrues, instead of delivering or taking the stock, settles by paying the *difference*—i. e., the difference between the market value of the stock on the day of settlement and the sum fixed by the contract.

² Parties who are "long of stocks" are the "bulls."

³ Parties who are "short of stocks" are the "bears."

⁴ Transactions in stock privileges—i. e., puts, calls, spreads, and straddles—are considered irregular by the Stock Exchange.

5. Read the following extracts of Stock Quotations from the New York stock market, July 28, 1887.

<i>Government Securities.</i>		<i>Bid.</i>	<i>Ask.</i>	<i>R. R. Bonds.</i>		<i>Bid.</i>	<i>Ask.</i>
U. S. 4½'s reg. 1891		109½	110	Chicago & Alton 1st m. 7's		116	—
U. S. 4½'s coup. 1891		109½	110	N. Y. Elevated 1st 7's		119	—
U. S. 4's reg. 1907		128½	128½	N. Y. C. & H. R. 1st 7's		186½	
U. S. 4's coup. 1907		129½	129½	St. Paul & Duluth 1st 5's		112½	113
U. S. 3's reg. option		100	—	Chic. & Ind. Ry. & Coal 1st 8's		100	100½
U. S. 6's currency 1899		134½	—	L. S. & Mich. S. consol. 2d 7's		126½	—
<i>State Securities.</i>				<i>Stocks.</i>			
N. Y. 6's coup.		102½	—	Western Union Tel.		78½	74
N. C. consolidated 6's		120	125	St. Paul & Duluth s. 30		76	76
N. C. do. do. ex-coup.		117	—	Ches. & Ohio 1st pref.		10½	13
Tenn. 6's new bonds		68	—	Ches. & Ohio 2d pref.		7	10
Missouri 6's reg.		102	—	Illinois Central R. R.		122	123
Ga. 7's gold bonds		108	109	Sutro Tunnel b. 3		20	—
Ark. 7's, Ark. Central R. R.		10	14	Phoenix Bank b. c.		113	—

LIST OF THE PRINCIPAL OUTSTANDING U. S. BONDS, 1889.

Names of Bonds.	When Redeemable.	Rates of Interest.	Interest Payable.
Four-and-a-halves of 1891	After Sept. 1, 1891.	4½%	Quarterly.
Fours of 1907	After July 1, 1907.	4%	Quarterly.
Fours, refunding certificates	At option of Gov't.	4%	Quarterly.
U. S. Pacific RR. currency sixes	{ From 1895 to 1899, inclusive }	6%	Semi-annually.

CUSTOMS OR DUTIES.

Page 262; Art. 458.

1. What kinds of taxes are levied by the General Government?

2. *Customs or duties* are taxes levied on imported articles.

a. Ad valorem duties are taxes computed at a certain per cent on the net value of the imports in the country where purchased.

b. Specific duties are taxes levied on the net weight, measure, or number of articles without regard to value.

c. In most cases, duties are *ad valorem*; on some articles they are *specific*; and on some others they are both *ad valorem* and *specific*.

d. Draft is an allowance deducted for waste, damage, and impurities;

Leakage is an allowance for loss on liquors imported in casks;

Breakage is an allowance for loss on liquors imported in bottles; and

Tare is an allowance in *specific* duties for the weight of the covering or package containing the goods.

NOTE.—Only actual leakage and breakage are allowed.

e. Net weight, amount, or value is what remains after all allowances have been deducted.

f. *Specific* duties are not computed by percentage.

3. A *tariff* is a schedule of the legal rates of duties on imports.

4. A *custom-house* is a government office for the transaction of business relating to customs or duties.

5. A *port of entry* is a seaport town in which a custom-house is located.

a. All legal introduction of imported goods into a country must be through regular ports of entry.

b. Smuggling is bringing goods into a country without paying duties required on their importation. It is a crime, and is subject to severe penalty.

6. A *bonded warehouse* is a building for the storage of property on which the duties or taxes have not been paid, but have been secured by a bond in double the amount of the tax.

a. Property left in bond is charged 10% extra duty for each full year that it remains in bond, and if left more than three years is considered as abandoned to the government.

b. Imported goods left in bond may be withdrawn and exported to a foreign country without paying the duty in bond.

7. An *invoice* or *manifest* is an itemized list of a cargo, including the prices at the place of purchase and other charges, to be exhibited at a custom-house.

8. A *clearance*, or *clearance papers*, is a certificate given by the collector of a port of entry that the vessel has complied with all the requirements of law in entering or leaving the port, in landing her cargo, etc.

9. *Internal revenue* is the taxes levied on domestic products and on certain kinds of business.

- a. Internal revenue taxes on articles are paid by affixing government revenue stamps of the required value to the package containing the article taxed.
- b. Internal revenue taxes on business are paid by the purchase of government licenses of the required value.
- c. Internal revenue is not computed by percentage.

10. In computations in customs or duties ad valorem,

1. *Net cost of imports* = base. | 2. *Ad valorem duty* = the percentage.

PROBLEMS.

Find the missing element :

<i>Net value</i>	<i>%</i>	<i>Duty</i>	<i>Net value</i>	<i>%</i>	<i>Duty</i>
1. \$4,866.25	8½	\$ ———	3. \$ ———	16½	\$975.50
2. \$15,425	—	\$1,388.25	4. \$ ———	22½	\$1,436.85

5. Find the duty on 75 chests of tea, each containing 75 lb., and invoiced at 50¢ per lb., at 25% ad valorem.

6. What is the duty at 6¢ per lb. on 95 cases of indigo, each case weighing 136 lb., allowing 12% tare?

7. I imported 102 hogsheads of molasses containing 72 gal. each. What duty must be paid at 5¢ per gal., leakage being allowed at 2½%?

8. What is the duty at 1½¢ per pound on 650 barrels of sugar, each weighing 270 pounds, allowing 5% tare?

9. Find the duty at 3¼% on 179 boxes of raisins, 32 pounds each, at 4½¢ per lb., tare 7½%.

10. What is the duty at 18% on 12,400 pounds of Australian wool invoiced at 28¢ per pound, draft allowed at 2%?

11. Find the duty on 45 pieces of three-ply carpet, 68 yd. in a piece, invoiced at 46¢ per yd., duty 16¢ per yard and 30% ad valorem.

12. If the duty at 16½% ad valorem on an importation of cottons is \$411.75, what is the total invoice of the goods?

13. The duty on 115 chests of tea, each chest containing 65 lb., invoiced at 48¢ per lb., was \$897. What was the per cent of duty?

14. A merchant imported 15 cases of silks, each case containing 10 pieces and each piece 65 yd., and paid \$1,937.50 duty at 12½¢ per yard and 24% ad valorem. What was the invoice price per yard?

INTEREST LAWS.

Page 272; Art. 465.

STATES AND TERRITORIES.	LEGAL RATE. <i>Lowest to Highest.</i>		PENALTY FOR USURY.
Alabama	8%	8%	Forfeiture of all interest.
Arizona	10%	any %	None.
Arkansas	6%	10%	Forfeiture of principal and interest.
California	10%	any %	None.
Colorado	10%	any %	None.
Connecticut	6%	6%	None.
Dakota	7%	12%	Forfeiture of contract.
Delaware	6%	6%	Forfeiture of contract.
Dist. of Columbia	6%	10%	Forfeiture of all interest.
Florida	8%	any %	None.
Georgia	7%	8%	Forfeiture of all interest.
Idaho	10%	18%	Forfeiture of three times the excess of interest over 18%.
Illinois	6%	8%	Forfeiture of all interest.
Indiana	6%	8%	Forfeiture of excess of interest over 6%.
Iowa	6%	10%	Forfeiture of all interest and costs.
Kansas	7%	12%	Forfeiture of excess of interest over 12%.
Kentucky	6%	10%	Forfeiture of excess of interest over 10%.
Louisiana	5%	8%	Forfeiture of all interest.
Maine	6%	any %	None.
Maryland	6%	6%	Forfeiture of excess of interest.
Massachusetts	6%	any %	None.
Michigan	7%	10%	Forfeiture of excess of interest over 10%.
Minnesota	7%	10%	Forfeiture of contract.
Mississippi	6%	10%	Forfeiture of excess of interest over 10%.
Missouri	8%	10%	Forfeiture of all interest.
Montana	10%	any %	None.
Nebraska	7%	10%	Forfeiture of all interest and costs.
Nevada	10%	any %	None.

STATES AND TERRITORIES.	LEGAL RATE.		PENALTY FOR USURY.
	<i>Lowest to Highest.</i>		
New Hampshire	6%	6%	Forfeiture of three times the excess of interest.
New Jersey	6%	6%	Forfeiture of all interest and costs.
New Mexico	6%	12%	None.
New York	6%	6%	Forfeiture of contract; \$1,000 fine; 6 mo. imprisonment.
North Carolina	6%	8%	Forfeiture of double the amount of interest.
Ohio	6%	8%	Forfeiture of excess of interest over 8%.
Oregon	10%	12%	Forfeiture of interest, principal, and costs.
Pennsylvania	6%	6%	Forfeiture of excess of interest.
Rhode Island	6%	any %	None.
South Carolina	7%	10%	Forfeiture of all interest.
Tennessee	6%	6%	Forfeiture of excess of interest.
Texas	8%	12%	Forfeiture of all interest.
Utah	10%	any %	None.
Vermont	6%	*7%	Forfeiture of excess of interest.
Virginia	6%	8%	Forfeiture of excess of interest over 8%.
Washington Terr.	10%	any %	None.
West Virginia	6%	6%	Forfeiture of excess of interest.
Wisconsin	7%	10%	Forfeiture of all the interest.
Wyoming	12%	any %	None.

* Allowed on Railroad Bonds only.

Any rate not exceeding the higher rate given in the table is legal if stipulated in writing; but if no rate is mentioned the lower rate given in the table is the legal rate.

DAYS OF GRACE—MATURITY.

Page 284; Art. 494.

Grace is not allowed on sight drafts in Arizona, California, Colorado, Connecticut, Delaware, District of Columbia, Florida, Georgia, Idaho, Illinois, Kansas, Louisiana, Maryland, Missouri, Nevada, New Mexico, New York, Ohio, Pennsylvania, Tennessee, Utah, Vermont, Virginia, and West Virginia.

Page 284; Art. 495.

It often happens that the day of maturity of commercial paper by the terms of the paper is Sunday or a legal holiday. In such cases the law of most of the States makes the paper due and payable on the *business day next preceding* the Sunday or the legal holiday; but in California, Louisiana, Nebraska, New Jersey, New Mexico, New York, and Virginia, it is made due and payable on the next *succeeding* business day. In Maine and Missouri the paper is due and payable on the nearest business day either preceding or succeeding.

COMPOUND INTEREST TABLE.

1. Showing the amount of \$1 at compound interest for any number of interest periods, from 1 to 40 inclusive.

Years.	1 per cent.	1½ per cent.	2 per cent.	2½ per cent.	3 per cent.	Years.
1	1.0100 000	1.0150 000	1.0200 0000	1.0250 0000	1.0300 0000	1
2	1.0201 000	1.0302 250	1.0404 0000	1.0506 2500	1.0609 0000	2
3	1.0303 010	1.0456 784	1.0612 0800	1.0768 9082	1.0927 2700	3
4	1.0406 040	1.0613 630	1.0824 3216	1.1038 1289	1.1255 0881	4
5	1.0510 101	1.0772 840	1.1040 8080	1.1314 0821	1.1592 7407	5
6	1.0615 202	1.0934 433	1.1261 6242	1.1596 9342	1.1940 5230	6
7	1.0721 854	1.1098 450	1.1486 8567	1.1886 8575	1.2298 7387	7
8	1.0828 567	1.1264 926	1.1716 5938	1.2184 0290	1.2667 7008	8
9	1.0936 853	1.1433 900	1.1950 9257	1.2488 6297	1.3047 7318	9
10	1.1046 231	1.1605 408	1.2189 9442	1.2800 8454	1.3439 1638	10
11	1.1156 693	1.1779 489	1.2433 7431	1.3120 8660	1.3842 3387	11
12	1.1268 250	1.1956 182	1.2682 4179	1.3448 8832	1.4257 6089	12
13	1.1380 933	1.2135 524	1.2936 0663	1.3785 1104	1.4685 3371	13
14	1.1494 742	1.2317 557	1.3194 7876	1.4129 7882	1.5125 8972	14
15	1.1609 690	1.2502 321	1.3458 6834	1.4482 9817	1.5579 6742	15
16	1.1725 786	1.2689 855	1.3727 8570	1.4845 0562	1.6047 0644	16
17	1.1843 044	1.2880 203	1.4002 4142	1.5216 1826	1.6528 4763	17
18	1.1961 475	1.3073 406	1.4282 4625	1.5596 5872	1.7024 3306	18
19	1.2081 090	1.3269 507	1.4568 1117	1.5986 5019	1.7535 0605	19
20	1.2201 900	1.3468 550	1.4859 4740	1.6386 1644	1.8061 1123	20
21	1.2323 919	1.3670 578	1.5156 6634	1.6795 8185	1.8602 9457	21
22	1.2447 159	1.3875 637	1.5459 7967	1.7215 7140	1.9161 0341	22
23	1.2571 630	1.4083 772	1.5768 9923	1.7646 1068	1.9735 8651	23
24	1.2697 346	1.4295 028	1.6084 3725	1.8087 2595	2.0327 9411	24
25	1.2824 320	1.4509 454	1.6406 0599	1.8539 4410	2.0937 7793	25
26	1.2952 563	1.4727 095	1.6734 1811	1.9002 9270	2.1565 9127	26
27	1.3082 089	1.4948 002	1.7068 8648	1.9478 0002	2.2212 8901	27
28	1.3212 910	1.5172 223	1.7410 2421	1.9964 9502	2.2879 2768	28
29	1.3345 039	1.5399 805	1.7758 4469	2.0464 0739	2.3565 6551	29
30	1.3478 490	1.5630 802	1.8113 6158	2.0975 6758	2.4272 6247	30
31	1.3613 274	1.5865 264	1.8475 8382	2.1500 0677	2.5000 8035	31
32	1.3749 407	1.6103 243	1.8845 4059	2.2037 5694	2.5750 8276	32
33	1.3886 901	1.6344 792	1.9222 3140	2.2588 5066	2.6523 3524	33
34	1.4025 770	1.6589 964	1.9606 7803	2.3153 2213	2.7319 0530	34
35	1.4166 028	1.6838 813	1.9998 8955	2.3732 0519	2.8138 6245	35
36	1.4307 688	1.7091 395	2.0398 8734	2.4325 3532	2.8982 7833	36
37	1.4450 765	1.7347 766	2.0806 8509	2.4933 4870	2.9852 2663	37
38	1.4595 272	1.7607 983	2.1222 9879	2.5556 8242	3.0747 6348	38
39	1.4741 225	1.7872 103	2.1647 4477	2.6195 7448	3.1670 2698	39
40	1.4888 637	1.8140 184	2.2080 3966	2.6850 6334	3.2620 3779	40

COMPOUND INTEREST TABLE.

2. Showing the amount of \$1 at compound interest for any number of interest periods, from 1 to 40 inclusive.

Years.	3½ per cent.	4 per cent.	4½ per cent.	5 per cent.	6 per cent.	Years.
1	1.0350 0000	1.0400 0000	1.0450 0000	1.0500 000	1.0600 000	1
2	1.0712 2500	1.0816 0000	1.0920 2500	1.1025 000	1.1236 000	2
3	1.1087 1787	1.1248 6400	1.1411 6612	1.1576 250	1.1910 160	3
4	1.1475 2300	1.1698 5856	1.1925 1860	1.2155 063	1.2624 770	4
5	1.1876 8631	1.2166 6290	1.2461 8194	1.2762 816	1.3382 256	5
6	1.2292 5533	1.2653 1902	1.3022 6012	1.3400 950	1.4185 191	6
7	1.2722 7926	1.3159 3178	1.3608 6183	1.4071 004	1.5036 303	7
8	1.3168 0904	1.3685 6905	1.4221 0061	1.4774 554	1.5938 481	8
9	1.3628 9785	1.4238 1181	1.4860 9514	1.5518 282	1.6894 790	9
10	1.4105 9876	1.4802 4428	1.5529 6942	1.6288 946	1.7908 477	10
11	1.4599 6972	1.5394 5406	1.6228 5305	1.7103 394	1.8982 966	11
12	1.5110 6866	1.6010 3222	1.6958 8143	1.7958 563	2.0121 965	12
13	1.5639 5806	1.6650 7351	1.7721 9610	1.8856 491	2.1329 233	13
14	1.6186 9452	1.7316 7645	1.8519 4492	1.9799 316	2.2609 040	14
15	1.6753 4833	1.8009 4351	1.9352 8244	2.0789 282	2.3965 582	15
16	1.7339 8601	1.8729 8125	2.0223 7015	2.1828 746	2.5408 517	16
17	1.7946 7555	1.9479 0050	2.1133 7681	2.2920 188	2.6927 728	17
18	1.8574 8920	2.0258 1652	2.2084 7877	2.4066 192	2.8543 892	18
19	1.9225 0182	2.1068 4918	2.3078 6031	2.5269 502	3.0255 995	19
20	1.9897 8886	2.1911 2314	2.4117 1402	2.6532 977	3.2071 855	20
21	2.0594 3147	2.2787 6807	2.5202 4116	2.7859 626	3.3995 636	21
22	2.1315 1158	2.3699 1879	2.6336 5301	2.9252 607	3.6035 374	22
23	2.2061 1443	2.4647 1555	2.7521 6635	3.0715 238	3.8197 497	23
24	2.2833 2849	2.5633 0417	2.8760 1883	3.2250 999	4.0489 346	24
25	2.3632 4498	2.6658 3633	3.0054 3446	3.3863 549	4.2918 707	25
26	2.4459 5856	2.7724 6979	3.1406 7901	3.5556 727	4.5493 830	26
27	2.5315 6711	2.8833 6858	3.2820 0956	3.7334 563	4.8223 459	27
28	2.6201 7196	2.9987 0332	3.4296 9999	3.9201 291	5.1116 867	28
29	2.7118 7798	3.1186 5145	3.5840 3649	4.1161 356	5.4188 879	29
30	2.8067 9370	3.2433 9751	3.7453 1813	4.3219 424	5.7434 912	30
31	2.9050 3148	3.3731 3341	3.9138 5745	4.5380 395	6.0881 006	31
32	3.0067 0759	3.5080 5875	4.0899 3104	4.7649 415	6.4533 867	32
33	3.1119 4235	3.6483 3110	4.2740 3018	5.0031 885	6.8405 899	33
34	3.2208 6033	3.7943 1634	4.4663 6154	5.2533 490	7.2510 253	34
35	3.3335 9045	3.9460 8699	4.6673 4781	5.5160 154	7.6860 868	35
36	3.4502 6611	4.1039 3253	4.8773 7846	5.7918 161	8.1472 520	36
37	3.5710 2543	4.2680 8986	5.0968 6049	6.0814 069	8.6360 871	37
38	3.6960 1182	4.4388 1345	5.3262 1921	6.3854 773	9.1542 524	38
39	3.8253 7171	4.6163 6599	5.5668 9908	6.7047 512	9.7035 075	39
40	3.9592 5972	4.8010 2063	5.8163 6454	7.0399 887	10.2857 179	40

NOTES.

\$100 *

BROOKLYN, N. Y., July 30, 1887.

to pay {	{	On demand, ¹	}	{	I, ³	}	promise				
		One day after date			we, ⁴						
		Nine months after date			{ we jointly and severally } ⁵						
		On or before May 1, 1888. ²			{ we or either of us }						
{	}	John Jones, or order,	}	one hundred dollars							
		to the order of John Jones,									
		John Jones, or bearer, ⁷									
		John Jones, ⁸									
{	}	{ with interest, } ⁹	{	at the Long Island Bank, ¹¹	}	{	Value received, ¹³				
		{ with use, }						}	}		
		 ¹⁰								 ¹²	 ¹⁴

Signature.....

- ¹ Demand note.
² Time note.
³ Individual note.
⁴ Joint note.
⁵ Joint and several note.
⁶ Negotiable note transferable by indorsement.
⁷ Negotiable note transferable without indorsement.

- ⁸ Non-negotiable note.
⁹ Interest-bearing note.
¹⁰ Non-interest-bearing note.
¹¹ Place of payment specified.
¹² No place of payment specified.
¹³ Consideration admitted by maker.
¹⁴ Consideration must be proven by the holder.

BANKABLE NOTE.

\$50.

New York, Jan. 2, 1889.

One year after date, I promise to pay James Goodwin, or order, Fifty Dollars, value received, with interest, at Park National Bank, New York.

Thomas Hunter.

JUDGMENT NOTE.

For value received, I promise to pay to the order of Henry Clay, of Ithaca, N. Y., Three Hundred Dollars, with interest, on the first day of March next; and I hereby nominate, constitute, and appoint any attorney-at-law of this State my true and lawful attorney, irrevocable, for me and in my name to appear in any court of record of this State, at any time after the above promissory note becomes due, and to waive all process and service thereof, and to confess judgment in favor of the holder hereof for the sum that may be due and owing hereon, with interest and costs, waiving all errors, etc., with stay of execution until the first day of March next.

Witness my hand and seal at Syracuse, N. Y., this fifteenth day of June, in the year one thousand eight hundred and eighty-eight.

Signed, sealed, and delivered in the presence of
 Richard White, James Reynolds.

James Bruce. [L. S.]

PROPERTY NOTE.

\$800.

Utica, N. Y., Jan. 20, 1888.

For value received, one year after date I promise to pay Charles A. Mott, or order, Eight Hundred Dollars in merchantable spruce logs, at his mill, at the market value on the maturity of this note.

Edwin Utter.

CHATTEL NOTE.

\$80.

Syracuse, N. Y., June 25, 1889.

Six months after date I promise to pay Bradley & Company, or order, Eighty Dollars, value received in one combined Mower and Reaper.

The express condition of the delivery of said Mower and Reaper to the said William Whitney is such that the title, ownership, or possession does not pass from the said Bradley & Company until this note is paid in full; and they may take possession of said Mower and Reaper at any time in case said note is not paid.

William Whitney.

BANK DRAFT.

\$5000.

Bank of the Metropolis.

No. 1846.

Chicago, Ill., July 5, 1890.

Pay to the order of Walter Dunham Five Thousand Dollars.

To Elliot Nat. Bank, Boston, Mass.

G. A. Copeland, Cashier.

CHECK.

Rochester, N. Y., Sept. 9, 1889.

No. 28.

First National Bank.

Pay to Charles Hunt, or order, One Hundred Dollars.

\$100.

Hamilton Brown.

ORDER FOR MONEY.

Mr. Edwin Foster,—

Binghamton, N. Y., Apr. 2, 1890.

Please pay Alexander Worth Fifty Dollars on my account.

Spencer Hoyt.

DUE-BILL FOR MONEY.

\$200.

London, Ont., July 13, 1889.

Due Thomas White, on demand, Two Hundred Dollars.

George M' Clellan.

RECEIPT FOR MONEY ON ACCOUNT.

\$50.

Baltimore, June 7, 1888.

Received of John Flagler Fifty Dollars to apply on account.

Nelson Dwight.

PROTEST.

\$1,500.

Buffalo, N. Y., Aug. 7, 1886.

Please take notice that a Promissory Note for Fifteen Hundred Dollars, made by Robert Ray, May 4, 1886, and indorsed by you, having been duly presented and payment thereof demanded, which was refused, is therefore protested for non-payment, to you, Neal Dow.

Look out.

I. N. Henderson, Notary Public.

BILL OF A SET OF EXCHANGE.

£500 10s.

Chicago, Jan. 15, 1886.

At twenty days sight of this First of Exchange (Second and Third of the same tenor and date unpaid), pay to the order of Peter Cooper, Five Hundred Pounds Ten Shillings Sterling, value received, and charge the same to account of

James Walden, Jr.

To James Kent & Co., Bankers, London.

CIRCULAR LETTER OF CREDIT.

No. $\frac{4}{A}$ 4796.

New York, Jan. 20, 1885.

Gentlemen,—We request that you will have the goodness to furnish Richard Lee, the bearer, whose signature is at foot, with any funds he may require to the extent of £1000 (say One Thousand Pounds Sterling), against his drafts upon Messrs. Brown, Shipley & Co., London; each draft must bear the number (No. $\frac{4}{A}$ 4796) of this letter, and we engage that the same shall meet due honor.

Whatever sums Mr. Lee may take up you will please indorse on the back of this Circular Letter, which is to continue in force till Jan. 20, 1886, from the present date, Jan. 20, 1885.

We are respectfully, gentlemen,

Your obedient, humble servants,

Brown Brothers & Co.

The signature of Richard Lee.

To Messieurs the Bankers,

Mentioned on the third page of this Letter of Credit.

CERTIFICATE OF DEPOSIT.

New York, Dec. 10, 1888.

The Chemical National Bank.

George Williams has deposited in this bank Five Hundred Dollars, to the credit of himself, payable on the return of this Certificate, properly indorsed.

Julius Fielding, Cashier.

PARTIAL PAYMENTS.

Page 288; Art. 506.

Special State Laws.

CONNECTICUT.

I. *When a year's interest or more has accrued at the time of a payment, or when any payment is less than the interest due, and also in case of the last payment, the interest is computed by the U. S. Court Rule.*

II. *When less than a year's interest has accrued at the time of any payment, except the last, the amount of the payment from its date to the end of the full year is deducted from the amount of the principal for the full year; the remainder is the new principal for the next interest term.*

VERMONT.

When notes, bills, or other obligations draw annual interest, and any part or the whole of this interest remains unpaid,—

I. *Payments draw interest to the end of the yearly interest terms in which they are made.*

II. *The amount of any payment or payments made in any interest term is applied*

1st. *To cancel interest due on unpaid yearly interests;*

2d. *To cancel unpaid yearly interests;*

3d. *To cancel the principal.*

The last balance must be computed to the date of settlement.

NEW HAMPSHIRE.

Under the law in this State, payments are applied

1st. *To the payment of any simple interest that may have accrued upon the annual interest;*

2d. *To the payment of the annual interest;*

3d. *To the payment of the principal.*

Payments not exceeding interest due at the end of the year, and made expressly on account of interest accruing but not yet due, do not draw interest. At the end of the year they must be applied to the payment of the interest then accrued.

In all other respects the law is the same as in Vermont.

ESTIMATE OF VALUES OF FOREIGN COINS.

FROM HOMANS'S BANKER'S ALMANAC, 1881.

Country.	Monetary unit.	Standard.	Value in U. S. money.	Standard coin.
Argentine Republic.	Peso	Gold and silver.	\$0.96, 5	$\frac{1}{10}$, $\frac{1}{20}$, $\frac{1}{4}$, $\frac{1}{8}$, and 1 Argentine and Argentine.
Austria	Florin	Silver	.39, 8	
Belgium	Franc	Gold and silver.	.19, 3	5, 10, and 20 francs.
Bolivia	Boliviano	Silver	.80, 6	Boliviano.
Brazil	Milreis of 1,000 reis	Gold	.54, 6	
Brit. Pos. in N. A.	Dollar	Gold	1.00	
Chili	Peso	Gold and silver.	.91, 2	Condor, doubloon, and escudo.
Cuba	Peso	Gold and silver.	.93, 2	$\frac{1}{10}$, $\frac{1}{20}$, $\frac{1}{4}$, $\frac{1}{8}$, and 1 doubloon.
Denmark	Crown	Gold	.26, 8	10 and 20 crowns.
Ecuador	Peso	Silver	.80, 6	Peso.
Egypt	Piaster	Gold	.04, 9	5, 10, 25, 50, and 100 piasters.
France	Franc	Gold and silver.	.19, 3	5, 10, and 20 francs.
German Empire	Mark	Gold	.23, 8	5, 10, and 20 marks.
Great Britain	Pound sterling	Gold	4.86, 6	$\frac{1}{2}$ sov. and sov.
Greece	Drachma	Gold and silver.	.19, 3	5, 10, 20, 50, and 100 drachmas.
Hayti	Gourde	Gold and silver.	.96, 5	1, 2, 5, and 10 gourdes.
India	Rupce, of 16 annas	Silver	.38, 3	
Italy	Lira	Gold and silver.	.19, 3	5, 10, 20, 50, and 100 lire.
Japan	Yen	Silver	.86, 9	1, 2, 5, 10, and 20 yen, gold and silver yen.
Liberia	Dollar	Gold	1.00	
Mexico	Dollar	Silver	.87, 5	Peso or Dollar, 5, 10, 25, and 50 centavo.
Netherlands	Florin	Gold and silver.	.40, 2	
Norway	Crown	Gold	.26, 8	10 and 20 crowns.
Peru	Sol	Silver	.80, 6	Sol.
Portugal	Milreis of 1,000 reis	Gold	1.08	2, 5, and 10 milreis.
Russia	Rouble of 100 copecks	Silver	.64, 5	$\frac{1}{2}$, $\frac{1}{4}$, and 1 rouble.
Spain	Peseta of 100 centimes	Gold and silver.	.19, 3	5, 10, 20, 50, and 100 pesetas.
Sweden	Crown	Gold	.26, 8	10 and 20 crowns.
Switzerland	Franc	Gold and silver.	.19, 3	5, 10, and 20 francs.
Tripoli	Mahbub of 20 piasters	Silver	.72, 7	
Turkey	Piaster	Gold	.04, 4	25, 50, 100, 250, and 500 piasters.
U. S. of Colombia	Peso	Silver	.80, 6	Peso.
Venezuela	Bolivia	Gold and silver.	.19, 3	5, 10, 20, 50, and 100 bolivar.

Page 234 ; Sec. III.

ANNUITIES.

1. An *annuity* is a stated sum of money due and payable annually.

The term *annuity* is also applied to sums payable at any regular intervals ; as semi-annually, quarterly, or monthly.

2. Annuities are classified

1. From the period or time of their continuance, as

a. Certain,—continuing a definite or fixed period of time.

b. Contingent,—continuing an indefinite or uncertain period.

c. Perpetual,—continuing forever.

2. From the date at which they begin, as

a. In possession,—beginning at once.

b. Deferred,—beginning at a future fixed date.

c. In reversion,—beginning on the occurrence of a certain future event.

Ex. An annuity for twenty years to begin at once is *certain* and *in possession* ; to begin at the end of ten years and continue twenty years is *certain* and *deferred* ; for twenty years after the decease of C. D. is *certain* and *in reversion*.

a. A perpetual annuity is commonly called a perpetuity.

b. Contingent annuities and *perpetuities* may also each be in possession, deferred, or in reversion.

3. An annuity is *forborne*, or *in arrear*, if the payments due have not been made.

4. The *amount* or *final value* of an annuity is the sum of all the payments plus interest on each from the time it is due to the termination of the annuity.

5. The *present value* of an annuity in expectancy is the sum of money which, at the given rate of interest, will amount to the final value of the annuity at its termination.

a. The present value of a perpetuity is a sum of which the perpetuity is the interest for one interest period.

b. The present worth or *gross value* of annuities in reversion is computed from the Northampton Tables.

6. The *principal applications of annuities* are leases, life estates, dower rights, reversions, pensions, and life-insurance.

7. The *computations in annuities* are

1. To find the final value of an annuity, or the amount due on an annuity in arrears.

2. To find the present value of an annuity in expectancy.

I. ANNUITIES AT SIMPLE INTEREST.

8. Each payment of an annuity at simple interest consists of the interest on the payment for one interval plus the amount of the next succeeding payment.

Thus, of five payments in arrears, the first bears interest 4 years, the second 3 years, the third 2 years, and the fourth 1 year. Hence,

In annuities at simple interest by arithmetical progression,—

Annuity = first term.

Interest on annuity for one period = common difference.

Number of payments = number of terms.

Final value = sum of the series.

Ex. 1. Find the final value and the present value of an annuity of \$100 for 6 years at 6% simple interest.

PROCESS.

$\$100 \times .06 = \$6$, the interest on the annuity for 1 interval.

$a = \$100$; $d = \$6$; $n = 6$; to find S .

FORMULA. $S = \frac{n \times [2 \times a + (n - 1) \times d]}{2}$. (Art. 402.)

$\frac{6 \times (2 \times \$100 + 5 \times \$6)}{2} = \$690$, the final value.

$\$690 \div 1.36 = \$507.35$ +, the present value. (Art. 531.)

Ex. 2. Find the present value of an annuity for \$80 for 20 yr. at 6% simple interest, if deferred 8 yr.

PROCESS.

$a = \$80$; $d = \$4.80$; $n = 20$; to find S .

FORMULA. $S = \frac{n \times [2 \times a + (n - 1) \times d]}{2}$

$\frac{20 \times (\$160 + \$91.20)}{2} = \$2,512$, the final value.

20 yr. + 8 yr. (deferred) = 28 yr., the term of discount.
 $\$2,512 \div 2.68 = \$937.31$, the present value required.

II. ANNUITIES AT COMPOUND INTEREST.

9. Each payment of an annuity at compound interest is the amount of the preceding payment for one interval.

Thus, of four payments in arrears, the final value of the first payment is its amount for 3 years, of the second is its amount for 2 years, and of the third is its amount for 1 year. Hence,

In annuities at compound interest by geometrical progression,—

Annuity = first term.

1 + rate of interest = ratio.

Number of payments = number of terms.

Final value = sum of the series.

Ex. 1. Find the final value, or amount due, on an annuity of \$150 8 years forborne, at 4% compound interest.

PROCESS.

$a = \$150$; $r = 1.04$; $n = 8$; to find S .

FORMULA. $S = \frac{a \times (r^n - 1)}{r - 1}$. (Art. 412); $r^n = 1.368569$ (page 442);

$\frac{\$150 \times .368569}{.04} = \$1,382 +$, the final value, or amount due.

Ex. 2. Find the final value and the present value of an annuity of \$200 for 20 years, deferred 10 years, at 3½% compound interest.

PROCESS.

$a = \$200$; $r = 1.035$; $n = 20$; to find S .

FORMULA. $S = \frac{a \times (r^n - 1)}{r - 1}$; $r^n = 1.989789$ (page 442);

$\frac{\$200 \times .989789}{.035} = \$5,655.94 -$, the final value.

20 yr. + 10 yr. (deferred) = 30 yr., the term of discount.

2.806794, the per cent of amount for 30 yr. at 3½% (page 442).

$\$5,655.94 \div 2.806794 = \$2,015.09 -$, the present value.

10. The rule in New York State for estimating the value of life estates is the

84TH RULE OF THE SUPREME COURT TO ASCERTAIN THE GROSS SUM IN
PAYMENT OF LIFE ESTATES.

“Whenever a party, as a tenant for life, or by the courtesy, or in dower, is entitled to the annual interest or income of any sum paid into court and invested in permanent securities, such party shall be charged with the expense of investing such sum, and of receiving and paying over the income thereof; but if such party is willing and consents to accept a gross sum in lieu of such annual income or interest for life, the same shall be estimated according to the then value of an annuity at 6% on the principal sum, during the probable life of such person according to the Portsmouth, or

NORTHAMPTON ANNUITY TABLE,

Showing the Value of an Annuity of \$1 at 6%.”

Age	No. of years purchase the Annuity is worth	Age	No. of years purchase the Annuity is worth	Age	No. of years purchase the Annuity is worth	Age	No. of years purchase the Annuity is worth
1	10.107	25	12.063	49	9.563	73	4.781
2	11.724	26	11.992	50	9.417	74	4.565
3	12.848	27	11.917	51	9.273	75	4.354
4	12.769	28	11.841	52	9.129	76	4.154
5	12.962	29	11.763	53	9.980	77	3.952
6	13.156	30	11.682	54	8.827	78	3.742
7	13.275	31	11.598	55	8.670	79	3.514
8	13.337	32	11.512	56	8.509	80	3.281
9	13.335	33	11.423	57	8.343	81	3.156
10	13.285	34	11.331	58	8.173	82	2.926
11	13.212	35	11.236	59	7.999	83	2.713
12	13.130	36	11.137	60	7.820	84	2.551
13	13.044	37	11.035	61	7.637	85	2.402
14	12.953	38	10.929	62	7.449	86	2.266
15	12.857	39	10.819	63	7.253	87	2.138
16	12.755	40	10.705	64	7.052	88	2.031
17	12.655	41	10.589	65	6.841	89	1.882
18	12.562	42	10.473	66	6.625	90	1.689
19	12.477	43	10.356	67	6.405	91	1.422
20	12.398	44	10.235	68	6.179	92	1.186
21	12.329	45	10.110	69	5.949	93	0.806
22	12.265	46	9.990	70	5.716	94	0.518
23	12.200	47	9.848	71	5.479		
24	12.132	48	9.707	72	5.241		

RULE.—*Calculate the interest at 6%, for one year, upon the sum to the income of which the person is entitled. Multiply this interest by the number of years purchase set opposite the person's age in the Table, and the product is the gross value of the life estate of such person in said sum.*

Ex. 1. A widow 51 years of age is entitled to dower in real estate worth \$12,000. What is the present value of her dower right?

PROCESS.

$$\$12,000 \times .06 = \$720, \text{ yearly annuity.}$$

$$\$720 \times 9.273 = \$6,676.56, \text{ the present value required.}$$

Ex. 2. What is the present value of a life annuity of \$300, if the annuitant is 60 years of age?

PROBLEMS.

1. What is the present value of any annuity of \$100, that has 5 years to run, at 6% simple interest?

2. If I deposit \$30 in a savings-bank every 3 months for 3 years, and receive 4% interest, compounded quarterly, how much have I in the bank at the end of the 3 years?

3. Find the final value of an annuity of \$275 a year payable quarterly, for 6 years, at 7% simple interest.

4. Find the present value of \$480 a year payable semi-annually, for 7 years, at 5% per annum, compound interest.

5. A store was rented for \$1,750 a year, but the rent remained unpaid for 5 years. What was the amount then due, money being worth 10%?

6. Mr. H—— purchased a farm for \$7,500, and paid cash down \$3,000. He agreed to pay the remainder in monthly instalments of \$100, but failing to meet this agreement, he paid the entire amount at the end of 3 years. At 7% simple interest, how much did he then pay?

7. A widow 36 years of age has a dower right in $\frac{1}{3}$ of a block of stores that pay \$3,000 a year net. What is the present value of her interest?

8. A soldier's pension of \$8 per month, payable quarterly, is in arrears for 17 years. What sum is he entitled to receive in full payment, interest at 5%?

9. Find the present value of an annuity of \$350 for 9 years, at $4\frac{1}{2}\%$ compound interest.

10. What is the present value of a perpetuity of \$200 at 5% compound interest, if it is deferred 15 years?

11. Find the present value of an annuity of \$360 for 15 years, deferred 10 years, at 6% compound interest.

12. A pension of \$12 per month, payable quarterly, has remained unpaid for 11 yr. 9 mo. How much is due, money being worth $4\frac{1}{2}\%$?

13. The final value of an annuity for 7 years at 6% compound interest, is \$3,183.60. What is the annuity?

14. How much must I invest each year, at 5% compound interest, that I may have \$1,680 invested at the end of 5 years?

15. If I buy a store for \$7,800 and pay for it in 8 equal annual instalments, what is the amount of each payment, interest at 6%?

16. A farm is rented for 999 years at a yearly rent of \$420. What is the present value of the rent, at 6% simple interest?

17. If I can secure an endowment of \$5,000 at the end of 20 years by an annual payment of \$254.50, and money is worth 4% compound interest, how much more do I pay than I receive?

18. If a man, at the age of 32, takes out an endowment policy for \$8,000 payable in 25 years, what is the present value of his payments should he pay throughout the term of his endowment, money being worth $3\frac{1}{2}\%$ compound interest?

PERMUTATIONS AND COMBINATIONS.

1. *Permutations* are all the different orders or arrangements of which any given number of objects are capable.

Thus, the figures 1, 2, 3, may be so arranged as to express six different numbers, as follows: 123; 132; 213; 231; 312; 321.

2. Combinations are the different groups formed from any given number of objects, by selecting a certain number at a time, without regard to arrangement.

Thus, the figures 1, 2, 3, may be taken two at a time in three different combinations, as follows: 12; 13; 23.

3. The principles governing permutations and combinations are developed by algebra. Only rules for computations are given in this work.

RULES.

I. To find the permutations of n objects:—*Find the continued product of the digits in their natural order from 1 to n inclusive.*

II. To find the combinations of n objects taken a at a time:—*Divide the continued product of the digits in their natural order from $n - a + 1$ to n inclusive by the continued product of the digits in their natural order from 1 to a inclusive.*

Ex. 1. Find the possible number of permutations of the 9 digits.

PROCESS.

$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 = 362,880$, the number of permutations required.

Ex. 2. Find the possible number of combinations of 12 objects taken in groups of 4.

PROCESS.

$$n - a + 1 = 12 - 4 + 1 = 9;$$

$$\frac{9 \times 10 \times 11 \times 12}{1 \times 2 \times 3 \times 4} = 495, \text{ the required number of combinations.}$$

Ex. 3. Find the number of different integers of 4 places each, that can be expressed by the 9 digits, without repeating a digit in any number.

PROCESS.

$$n - a + 1 = 9 - 4 + 1 = 6;$$

$$\frac{6 \times 7 \times 8 \times 9}{1 \times 2 \times 3 \times 4} = 126, \text{ the number of combinations of 9 objects taken 4 at a time.}$$

$$1 \times 2 \times 3 \times 4 = 24, \text{ the permutations of each combination.}$$

$$126 \times 24 = 3,024, \text{ the number of integers required.}$$

PROBLEMS.

1. In how many different orders may a class of 13 pupils be arranged?
2. How many different combinations, of 3 colors each, can be made from the 7 colors of the solar spectrum?
3. How many integral numbers can be expressed by the nine digits, using a digit but once in each number?
4. On a shelf in my library are 20 volumes. If they are taken down and replaced at random, what are the probabilities of their being in the same order as before?
5. In a kaleidoscope are 15 pieces of glass, each of a different color. What are the probabilities of a person's ever seeing the same arrangement of colors twice?

[Draw diagrams of all problems before beginning solution.]

MENSURATION INVOLVING RATIO AND PROPORTION.

Page 334; Art. 576.

I. PLANE FIGURES.

CASE I. To find the dimensions of a rectangle or of a triangle, the area and the ratio of the dimensions being given.

1. Ex. 1. The area of a rectangle is 180 sq. ft., and its width is $\frac{4}{5}$ of its length (*i. e.*, its width is to its length as 4 : 5). What are the dimensions?

ANALYSIS.—The accompanying diagram shows that if the sides are divided into 4 and 5 equal parts respectively, and lines are drawn as shown, the area of the rectangle will be divided into $(5 \times 4 =)$ 20 equal squares.

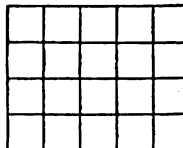
PROCESS.

$$4 \times 5 = 20$$

$$180 \text{ sq. ft.} \div 20 = 9 \text{ sq. ft.}$$

$$\sqrt{9} = 3; \text{ hence, } 3 \text{ ft.}$$

$$4 \times 3 \text{ ft.} = 12 \text{ ft.; and } 5 \times 3 \text{ ft.} = 15 \text{ ft.}$$



The area of the rectangle divided by the number of equal squares equals the area of each square; hence, $180 \text{ sq. ft.} \div 20 = 9 \text{ sq. ft.}$

Extracting the square root of 9, and supplying the linear unit corresponding to

square feet, the result obtained is 3 feet, the unit of partition, *i. e.*, one side of each of the squares.

Multiplying the unit of partition by each of the partitive ratios, I obtain 12 feet, the width of the rectangle, and 15 feet, the length.

Ex. 2. The area of a triangle is 810 sq. rd., and the altitude equals $\frac{2}{3}$ of the base. Required the base and altitude.

OUTLINES OF ANALYSIS.

PROCESS.

- | | |
|---|--|
| (1) Area of a rectangle of equal base and altitude. | (1) $2 \times 810 \text{ sq. rd.} = 1,620 \text{ sq. rd.}$ |
| (2) Product of the partitive ratios. | (2) $9 \times 20 = 180$ |
| (3) Area of each equal square. | (3) $1,620 \text{ sq. rd.} \div 180 = 9 \text{ sq. rd.}$ |
| (4) Unit of partition. | (4) $\sqrt{9} = 3$; hence, 3 rd. |
| (5) Base and altitude. | (5) $20 \times 3 \text{ rd.} = 60 \text{ rd.}$; $9 \times 3 \text{ rd.} = 27 \text{ rd.}$ |

PROBLEMS.

1. The base of a parallelogram is to the altitude as 9 : 5. If the area of the parallelogram is 1,096 sq. rd., what are its dimensions?

2. Find the sides of a rectangular field that contains 14 A. 9.16 sq. ch., and whose length is 5 times its breadth.

3. The base of a triangle is $\frac{2}{3}$ of its altitude, and its area is 532 sq. in. Find the base and the altitude.

4. Find the dimensions of a rectangle whose area is 33 A., and whose length is $2\frac{1}{2}$ times its breadth.

5. The sides of a triangle are in the ratios of 3, 4, and 5, and its area is 1,014 sq. yd. Required the three sides.

CASE II. To find two sides of a right-angled triangle, their ratios and the third side being given.

2. Ex. 1. The hypotenuse of a right-angled triangle is 30 feet, and the ratio of the base to the perpendicular is as 3:4. Required the base and the perpendicular.

ANALYSIS.—I first find the area of the square on the hypotenuse, —900 sq. ft.

I next find the number of equal squares into which the squares on the base and perpendicular are, by the terms of the ratio, supposed to be divided, which is 25. $3 \times 6 \text{ ft.} = 18 \text{ ft.}$; and $4 \times 6 \text{ ft.} = 24 \text{ ft.}$

PROCESS.

$$30 \times 30 \text{ sq. ft.} = 900 \text{ sq. ft.}$$

$$3^2 + 4^2 = 25$$

$$900 \text{ sq. ft.} \div 25 = 36 \text{ sq. ft.}$$

$$\sqrt{36} = 6$$
; hence, 6 ft.

Since the square on the hypotenuse equals the sum of the squares on the base and perpendicular (page 386), the area of the 25 equal squares is 900 sq. ft., and the area of each of them is the quotient of 900 sq. ft. divided by 25, which is 36 sq. ft.

Extracting the square root of 36 and supplying the linear unit corresponding to sq. ft., I have 6 ft., the unit of partition.

Multiplying the unit of partition by each of the ratios, I obtain 18 ft., the base of the triangle, and 24 ft., the perpendicular.

Ex. 2. The base of a right-angled triangle is 36 rods, and the ratio of the other two sides is as 5 : 3. Required the hypotenuse and the perpendicular.

OUTLINES OF ANALYSIS.

PROCESS.

- | | |
|--|---|
| (1) Area of square on base. | (1) $36 \times 36 \text{ sq. rd.} = 1,296 \text{ sq. rd.}$ |
| (2) Difference in number of equal squares, by terms of ratio, on hypotenuse and perpendicular. | (2) $5^2 - 3^2 = 16$ |
| (3) Area of each equal square. | (3) $1,296 \text{ sq. rd.} \div 16 = 81 \text{ sq. rd.}$ |
| (4) Unit of partition. | (4) $\sqrt{81} = 9$; hence, 9 rd. |
| (5) Length of hypotenuse 45 feet, and of perpendicular 27 feet. | (5) $5 \times 9 \text{ ft.} = 45 \text{ ft.}$; and $3 \times 9 \text{ ft.} = 27 \text{ ft.}$ |

PROBLEMS.

6. The perpendicular of a right-angled triangle is 24 chains, and the base is $\frac{4}{3}$ as long as the hypotenuse. Find the two unknown sides.

7. The hypotenuse of a right-angled triangle is 90 feet, and the ratio of the base to the perpendicular is $\frac{4}{3}$. Required the base and the perpendicular.

8. The hypotenuse of a right-angled triangle is 60 rods, and the other two sides are equal. Required the length of each.

9. The base of a right-angled triangle is 36 rods, and the perpendicular is $\frac{4}{7}$ of the sum of the two unknown sides. Find those sides.

10. Find the three sides of a right-angled triangle whose perimeter is 80 feet, and whose base is $\frac{4}{5}$ of its perpendicular.

11. The area of a right-angled triangle is 1,470 sq. yd., and the base is $\frac{4}{5}$ as long as the hypotenuse. Required the three sides of the triangle.

II. SIMILAR PLANE FIGURES.

3. Similar plane figures are plane figures that have the same form.

- a. Similar polygons have each the same number of sides and the same number of angles similarly placed.
- b. All regular polygons of the same number of sides are similar.
- c. All circles are similar.

4. By Geometry are demonstrated the following

IMPORTANT FACTS.

I. *The corresponding angles of similar polygons are equal.*

II. *The corresponding sides and also the corresponding dimensions of similar plane figures are in a common ratio each to each.*

III. *The areas of similar plane figures are in the same ratio as the squares of their corresponding sides or corresponding dimensions.*

IV. *The corresponding sides and also the corresponding dimensions of similar plane figures are in the same ratio as the square roots of their areas.*

CASE I. To find the sides of a polygon similar to one whose sides are given, one side only of the required polygon being known.

5. Ex. The sides of a triangle are 9 ft., 16 ft., and 18 ft. What are the sides of a similar triangle whose side corresponding to the first in order in the given triangle is 27 ft.?

PROCESS.

STEPS IN PROCESS.

(1) and (2) Statements of proportion (4, II).

(3) Solution of proportions.

$$(1) \quad 9 \text{ ft.} : 27 \text{ ft.} :: 16 \text{ ft.} : \text{--- ft.}$$

$$(2) \quad 9 \text{ ft.} : 27 \text{ ft.} :: 18 \text{ ft.} : \text{--- ft.}$$

$$(3) \quad \frac{27 \times 16 \text{ ft.}}{9} = 48 \text{ ft.}; \text{ and } \frac{27 \times 18 \text{ ft.}}{9} = 54 \text{ ft.}$$

Hence, the sides of the required triangle are 27 ft., 48 ft., and 54 ft.

PROBLEMS.

12.		13.		15.	
Given the sides of a	to find the corresponding sides of a similar	Given the sides of a	to find the corresponding sides of a similar	Given the sides of a	to find the corresponding sides of a similar
hexagon	hexagon	triangle	triangle	trapezium	trapezium
$\left\{ \begin{array}{l} 7 \text{ ft.} \\ 11 \text{ ft.} \\ 13 \text{ ft.} \\ 14 \text{ ft.} \\ 17 \text{ ft.} \\ 5 \text{ ft.} \end{array} \right.$	$\left\{ \begin{array}{l} \text{— ft.} \\ 44 \text{ ft.} \\ \text{— ft.} \\ \text{— ft.} \\ \text{— ft.} \\ \text{— ft.} \end{array} \right.$	$\left\{ \begin{array}{l} 9 \text{ ch.} \\ 13 \text{ ch.} \\ 15 \text{ ch.} \end{array} \right.$	$\left\{ \begin{array}{l} 21 \text{ ch.} \\ \text{— ch.} \\ \text{— ch.} \end{array} \right.$	$\left\{ \begin{array}{l} 8 \text{ in.} \\ 13 \text{ in.} \\ 27 \text{ in.} \\ 31 \text{ in.} \end{array} \right.$	$\left\{ \begin{array}{l} \text{—} \\ \text{—} \\ 67\frac{1}{2} \text{ rd.} \\ \text{—} \end{array} \right.$
14.		16.			
triangle	triangle	rectangle	rectangle		
$\left\{ \begin{array}{l} 39 \text{ rd.} \\ 47 \text{ rd.} \\ 71 \text{ rd.} \end{array} \right.$	$\left\{ \begin{array}{l} \text{— ft.} \\ \text{— ft.} \\ 18.2 \text{ ft.} \end{array} \right.$	$\left\{ \begin{array}{l} 9 \text{ ft. } 4 \text{ in.} \\ 31 \text{ ft. } 2 \text{ in.} \end{array} \right.$	$\left\{ \begin{array}{l} 28 \text{ ch. } 35 \text{ l.} \\ \text{—} \end{array} \right.$		

17. The sides of a triangle are 11 rd., 29 rd., and 33 rd. Find the sides of a similar triangle whose perimeter is 803 inches.

18. The base of a right-angled triangle is 24 ft. and the hypotenuse is 51 ft. Find the perimeter of a similar triangle whose perpendicular is 225 rods.

19. The major axis of an ellipse is 20 feet, and the minor axis 12 feet. What is the major axis of a similar ellipse whose minor axis is 33 ch.?

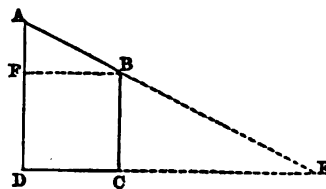
CASE II. To find the sides of a triangle produced from a trapezoid, or the sides of a trapezoid cut from a triangle.

6. IMPORTANT FACTS.

V. A line drawn parallel to either side of a triangle divides the triangle into a trapezoid and a similar triangle.

VI. The oblique sides of a trapezoid produced to a vertex are the sides of a triangle whose base is the longer parallel side of the trapezoid.

7. The sides of the trapezoid $ABCD$ when produced, meet at E and form the triangle AED . The line BC is parallel to the line AD ; hence, the triangle BEC is similar to the triangle AED (6, V); and the line FB drawn paral-



lel to the line D E, cuts off the triangle A B F, which is similar to the triangle A E D (6, V). Therefore, the three triangles shown by the diagram are similar triangles.

The base A D of the triangle A E D is to its altitude D E as the base A F of the similar triangle A B F is to its altitude F B. And

The base A F of the triangle A B F is to its altitude F B as the base B C of the similar triangle B E C is to its altitude C E (4, II).

The figure D F B C is a parallelogram, and the line F D equals the line B C; hence the base A F of the triangle A B F is the difference of the parallel sides of the trapezoid A B C D; also the altitude F B of the triangle is the altitude of the trapezoid. Hence,

8. PRINCIPLE.—*The difference of the parallel sides of a trapezoid is to its altitude as the longer parallel side is to the altitude of the resulting triangle.*

Ex. 1. The parallel sides of a trapezoid are 18 feet and 26 feet, and its altitude is 10 feet. What is the altitude of the triangle formed by producing the oblique sides of the trapezoid to a vertex?

PROCESS.

$$26 \text{ ft.} - 18 \text{ ft.} = 8 \text{ ft.}; 8 \text{ ft.} : 10 \text{ ft.} :: 26 \text{ ft.} : \text{--- ft.}; \frac{10 \times 26 \text{ ft.}}{8} = 32\frac{1}{2} \text{ ft.}$$

Ex. 2. Find the shorter parallel side of the trapezoid formed by a line drawn through a triangle, parallel to and 9 rods from the base, the altitude of the triangle being 27 rods and the base 16 rods.

STEPS IN PROCESS.

(1) Find altitude of similar triangle cut off by line parallel to and 9 rods from the base (6, V).

(2) Statement of proportion (p. 459, II).

(3) Solution of proportion.

Hence, 10 $\frac{1}{2}$ rods, the required side.

PROCESS.

$$(1) \quad 27 \text{ rd.} - 9 \text{ rd.} = 18 \text{ rd.}$$

$$(2) \quad 16 \text{ rd.} : 27 \text{ rd.} :: \text{--- rd.} : 18 \text{ rd.}$$

$$(3) \quad \frac{16 \times 18 \text{ rd.}}{27} = 10\frac{1}{2} \text{ rd.}$$

PROBLEMS.

20. Required the altitude of a triangle from which a trapezoid whose altitude is 8 rods and whose parallel sides are 14 rods and 20 rods, can be cut by a line parallel to the base of the triangle.

21. The altitude of a triangle is 28 rods and the base 30 rods. Required the shorter parallel side of a trapezoid cut from the triangle by a line parallel to and 12 rods from the base.

22. A triangular field whose base is 9 ch. 26 l. and altitude 12 ch. 4 l., is divided by a fence parallel to the base and 8 ch. from the apex. Required the amount of land in each part.

23. What is the area of a trapezoid cut from a triangle whose base is 18 feet and altitude 24 feet, by a line 12 feet long?

24. The area of a triangle whose base is 36 rods, is 648 sq. rd. Required the area of a trapezoid, cut from this triangle by a line 20 rods (perpendicular distance) from the vertex.

CASE III. To find the area of a plane figure from the area of a similar figure, two corresponding sides or corresponding dimensions—or their ratios,—being given.

9. Ex. 1. The area of a triangle whose base is 36 ft. is 490 sq. ft. Required the area of a similar triangle whose base is 24 ft.

STEPS IN PROCESS.

PROCESS.

(1) Statement of proportion
(page 459, III).

(2) Solution of proportion.

(Observe that any number

squared is used twice as a factor.)

$$(1) \ 490 \text{ sq. ft.} : \text{---} \text{ sq. ft.} :: 36^2 : 24^2$$

$$(2) \ \frac{24 \times 24 \times 490 \text{ sq. ft.}}{36 \times 36} = 217\frac{1}{3} \text{ sq. ft.}$$

Hence, $217\frac{1}{3}$ sq. ft., required area.

Ex. 2. The area of a circle 35 inches in diameter is 962.115 square inches. Required the diameter of a circle whose area is 3,017.015 square inches.

PROCESS.

$$(1) \ 35 \text{ in.} : \text{---} \text{ in.} :: \sqrt{962.115 \text{ sq. ft.}} : \sqrt{3,017.015 \text{ sq. ft.}}$$

$$(2) \ 1,225 \text{ in.} : (\text{---})^2 \text{ in.} :: 962.115 \text{ sq. ft.} : 3,017.015 \text{ sq. ft.}$$

$$(3) \ 1,225 \text{ in.} : (\text{---})^2 \text{ in.} :: 962.115 : 3,017.015$$

$$(4) \ \frac{3,017 \times 1,225 \text{ in.}}{962.115} = 3,841.373822 + \text{in.}$$

$$(5) \ \sqrt{3,841.373822 + \text{in.}} = 61.9788 + \text{in.}$$

STEPS IN PROCESS.

- (1) Statement of proportion (page 459, IV).
 - (2) Every term to 2d power (Art. 377, Prop. XI).
 - (3) Second couplet to abstract numbers—divide by 1 sq. ft. (Art. 377, Prop. II).
 - (4) Solution of proportion.
 - (5) Square root of the number of inches in result of proportion.
- Hence, 61.9788 + in., the required diameter.

Ex. 3. The diameters of two circles are in the ratio of 2 to 5, and the area of the larger circle is 38 sq. yd. Required the area of the smaller circle.

STEPS IN PROCESS.

PROCESS.

- (1) Statement of proportion (page 459, III). $38 \text{ sq. yd.} : \text{---} \text{ sq. yd.} :: 5^2 : 2^2$
 - (2) Solution of proportion. $\frac{2 \times 2 \times 38 \text{ sq. yd.}}{5 \times 5} = 6.08 + \text{sq. yd.}$
- Hence, 6.08 + sq. yd., the required area.

PROBLEMS.

25. The area of a triangle whose base is 3 ch. 11 l., is 1 A. 3.25 sq. ch. What is the area of a similar triangle whose base is 5 ch. 91 l.?

26. The ratio of the corresponding sides of two similar rectangles is $\frac{3}{11}$, and the area of the smaller is .35 A. Find the area of the larger rectangle.

27. The areas of two similar pentagons are 186 sq. rd. and 342 sq. rd., and one side of the smaller pentagon is 12 rd. Required the corresponding side of the larger pentagon.

28. An iron cylinder 6 in. in diameter and 1 ft. long, weighs 20 lb. 4.8 oz. Required the weight of an iron cylinder of the same length and $2\frac{1}{2}$ in. in diameter.

29. A regular hexagon is 16 feet on each side, and its area is 665 sq. ft. 15.43 sq. in. Find one side of a regular hexagon whose area is 178.36 sq. rd.

30. The areas of two circles are to each other as 3:13, and the circumference of the larger is 19 rd. 11 ft. 4 in. Find the circumference of the smaller circle.

CASE IV. To divide a triangle or a trapezoid into required parts by a line parallel to one of its sides.

10. Ex. 1. It is desired to cut from a triangular field of 42 acres whose altitude is 27 chains, by a line parallel to the base, 18 acres that shall include the vertex of the triangle. Locate the dividing line.

OUTLINES OF ANALYSIS.

See page 460, V.

(1) Statement of proportion (page 459, IV).

(2) Square of terms (Art. 377, Prop. XI).

(3) Lowest terms of first couplet—divide by 6 A.—(Art. 377, Prop. II).

(4) Solution of proportion.

(5) Square root of result.

(6) Distances of dividing line from vertex and from base of triangle.

PROCESS.

$$(1) \sqrt{42} A. : \sqrt{18} A. :: 27 \text{ ch.} : \text{--- ch.}$$

$$(2) 42 A. : 18 A. :: (27^2) \text{ ch.} : (\text{---}^2) \text{ ch.}$$

$$(3) 7 : 3 :: (27^2) \text{ ch.} : (\text{---}^2) \text{ ch.}$$

$$(4) \frac{3 \times (27 \times 27) \text{ ch.}}{7} = 312.4285 \text{ ch.}$$

$$(5) 312.4285 \text{ ch.} = (17.678 + ^2) \text{ ch.}$$

$$(6) \text{Hence, } 17.678 + \text{ch. from vertex; and } 27 \text{ ch.} - 17.678 + \text{ch.} = 9.322 - \text{ch. from base.}$$

Ex. 2. A triangle is divided into two equal parts by a line parallel to the base. The two oblique sides are 71 ft. and 94 ft. Where will the dividing line intersect these sides?

OUTLINES OF ANALYSIS.

Area of given triangle is to area of similar triangle cut off as 2 : 1.

(1) Statement of proportion—side 71 ft. long (page 459, IV).

(2) Solution of proportion.

(3) Statement of proportion—side 94 ft. long (page 459, IV).

(4) Solution of proportion.

PROCESS.

$$(1) \sqrt{2} : \sqrt{1} :: 71 \text{ ft.} : \text{--- ft.}$$

$$(2) \frac{\sqrt{1} \times 71 \text{ ft.}}{\sqrt{2}} = 50.2 + \text{ft.}$$

$$(3) \sqrt{2} : \sqrt{1} :: 94 \text{ ft.} : \text{--- ft.}$$

$$(4) \frac{\sqrt{1} \times 94 \text{ ft.}}{\sqrt{2}} = 64.4 + \text{ft.}$$

Hence, 50.2+ ft. from vertex on side 71 feet long, and 64.4+ ft. from vertex on side 94 feet long.

Ex. 3. A board is 14 inches wide at one end, 18 inches wide at the other, and 16 feet long. Where shall the board be sawed crosswise to divide it into equal parts?

OUTLINES OF ANALYSIS.

(1) Contents of the board.

(2) Contents of each equal part.

(3) Statement of proportion (p. 461, Prin.).

(4) By alternation (Art. 377, Prop. X).

(5) Lowest integral terms of first couplet—divide by 2 in.,—(Art. 377, Prop. II).

(6) Lower terms (Art. 377, Prop. IV).

(7) Altitude of resulting triangle.

(8) Area of resulting triangle.

(9) Area of resulting triangle less area of part of board cut from base equals area of similar triangle.

PROCESS.

$$\begin{aligned} (1) \quad & \frac{14 + 18}{2} \times \frac{1}{12} \times 16 \text{ sq. ft.} = 21\frac{1}{2} \text{ sq. ft.} \\ (2) \quad & \frac{1}{2} \text{ of } 21\frac{1}{2} \text{ sq. ft.} = 10\frac{3}{4} \text{ sq. ft.} \\ (3) \quad & 4 \text{ in.} : 16 \text{ ft.} :: 18 \text{ in.} : \text{--- ft.} \\ (4) \quad & 4 \text{ in.} : 18 \text{ in.} :: 16 \text{ ft.} : \text{--- ft.} \\ (5) \quad & 2 : 9 :: 16 \text{ ft.} : \text{--- ft.} \\ (6) \quad & 1 : 9 :: 8 \text{ ft.} : \text{--- ft.} \\ (7) \quad & 9 \times 8 \text{ ft.} = 72 \text{ ft.} \\ (8) \quad & \frac{18}{12} \times \frac{72}{2} \text{ sq. ft.} = 54 \text{ sq. ft.} \\ (9) \quad & 54 \text{ sq. ft.} - 10\frac{3}{4} \text{ sq. ft.} = 43\frac{1}{4} \text{ sq. ft.} \\ (10) \quad & (\sqrt{54}) \text{ sq. ft.} : (\sqrt{43\frac{1}{4}}) \text{ sq. ft.} :: 72 \text{ ft.} : \text{--- ft.} \\ (11) \quad & \sqrt{54} : \sqrt{43\frac{1}{4}} :: 72 \text{ ft.} : \text{--- ft.} \\ (12) \quad & 54 : 43\frac{1}{4} :: (72^2) \text{ ft.} : (\text{---}^2) \text{ ft.} \\ (13) \quad & \frac{43\frac{1}{4} \times (72 \times 72) \text{ ft.}}{54} = 4160 \text{ ft.} = (64.5 - \text{---}) \text{ ft.} \\ (14) \quad & 72 \text{ ft.} - 64.5 - \text{ft.} = 7.5 + \text{ft.} \end{aligned}$$

Hence, 72 ft. - 64.5 - ft. = 7.5 + ft., the distance from the wider end of the board.

(10) Statement of proportion (page 459, IV).

(11) Divide terms of first couplet of (10) by 1 sq. ft. (Art. 377, Prop. II).

(12) Terms of (11) raised to 2d power.

(13) Solution of proportion, and result expressed by 2d power of the root.

(14) Length of resulting triangle less length of similar triangle cut off equals the distance from the base of the resulting triangle, i.e., from the wider end of the board.

PROBLEMS.

31. Locate a line parallel to the base of a triangle whose base is 6 ch. 72 l. and altitude 7 ch. 14 l., so that it will divide the triangle into equal parts.

32. Locate a line parallel to the base of a triangle whose base is 112 rods and altitude 92 rods, so that the trapezoid thus formed will contain 15 A.

33. B owns a piece of land in the form of a trapezoid, the parallel sides or ends of which are 9 rods and 20 rods, and are 48 rods apart. He desires to sell from the smaller end three building lots of one half acre each, with dividing lines parallel to the ends of the piece of land. Locate the three lines.

34. The area of a right-angled triangle is 4 acres, and the base is $\frac{1}{2}$ as long as the perpendicular. Locate a line, parallel to the hypotenuse, that shall include $\frac{1}{3}$ of the area of the triangle between it and the hypotenuse.

11. RULES.

I. To find the dimensions of a rectangle or of a triangle from its area and the ratio of its dimensions:—

1. *Divide the area of a rectangle, or twice the area of a triangle, by the product of the partitive ratios of its dimensions.*
2. *Extract the square root of this quotient for the unit of partition.*
3. *Multiply the unit of partition by the ratios.*

II. Hypotenuse and ratios of the base and perpendicular given, to find those sides:—

1. *Divide the square of the hypotenuse by the sum of the squares of the ratios of the required sides.*
2. *Extract the square root of this quotient for the unit of partition.*
3. *Multiply the unit of partition by the ratios.*

III. Base or perpendicular of a right-angled triangle and ratios of the other two sides given, to find those sides:—

1. *Divide the square of the given side by the difference of the squares of the ratios of the required sides.*
2. *Extract the square root of this quotient for the unit of partition.*
3. *Multiply the unit of partition by the ratios.*

IV. To find corresponding sides or corresponding dimensions of similar plane figures:—

1. *Make the given corresponding sides the first couplet of a proportion, and each remaining given side and the unknown corresponding side a second couplet.*
2. *Solve the proportions.*

V. To compute the sides or dimensions of a triangle resulting from producing the sides of a trapezoid to a vertex:—

1. *Make the difference of the parallel sides of the trapezoid and the distance they are apart on the required side or dimension the first couplet of a proportion, and the longer parallel side of the trapezoid and the required side or dimension the second couplet.*

2. *Solve the proportion.*

VI. To divide a triangle into any required parts by lines parallel to any one of its sides:—

1. *Make the square root of the area of the whole triangle and the square root of the area of the triangle cut off by the dividing line the first couplet of a proportion, and a side or dimension of the whole triangle and the unknown corresponding side of the triangle cut off the second couplet.*

2. *Solve the proportion.*

VII. To divide a trapezoid into any required parts, by lines parallel to any one of its sides:—

1. *Find the triangle formed by producing the oblique sides of the trapezoid.*

2. *Divide the resulting triangle according to Rule for dividing triangles.*

PROBLEMS.

35. The area of a right-angled triangle is 480 sq. ft., and the base is to the hypotenuse as 4 : 5. Find the three sides.

36. The perpendicular of a right-angled triangle is 38 feet, and the base is $\frac{3}{4}$ of the hypotenuse. Required the area.

37. The area of an equilateral triangle is 17,320 sq. yd. What is the length of one of its sides?

38. The altitudes of two similar triangles are 16 rods and 53 rods. The area of the larger triangle equals how many times the area of the smaller?

39. A right-angled triangle whose base is to its perpendicular as 9 to 17, has an area of 13 A. 124 sq. rd. What is the area of a similar triangle whose altitude is 18 ch. 34 l.?

The sides of a scalene triangle are 38 rd., 51 rd., and 69 rd., and the triangle is divided into 3 equal parts by lines parallel to the shortest side.

40. Locate the points where the dividing lines intersect the other two sides.

41. Find the length of each of the dividing lines.

42. Find the altitude of each of the three equal parts.

43. I have a circular fish-pond 176 feet in circumference, and I wish to increase its size $\frac{1}{4}$. How wide a circular belt or ring must be excavated?

44. A triangle contains 1,864 square rods, and its sides are in the ratios of 16, 19, and 23. Find the three sides.

45. If a pipe 2 in. in diameter fill a reservoir in 6 h. 14 min., how much time will it take a pipe $4\frac{1}{2}$ in. in diameter to fill the same reservoir, making no allowance for friction or pressure?

46. I divide a trapezoidal field 7 ch. 22 l. on one parallel side, 11 ch. 5 l. on the other, and 13 ch. 77 l. long, by a line parallel to and 8 ch. 20 l. from the shorter parallel side. Required the areas of the two parts.

47. A has a field bounded on one side by a highway, and on the opposite side by a railroad. The fences on the ends of the field are at right angles to the highway, 22 rods and 37 rods long respectively; and the field is 54 rods long on the line of the highway. He sells to the railroad company a strip of this land, of uniform width, across the side next to the railroad, and containing 2.16 acres. Locate the dividing line.

III. SOLIDS.

CASE I. To find the dimensions of a rectangular solid, its volume and the ratios of the dimensions being given.

12. Ex. Required the dimensions of a rectangular solid whose cubic contents are 56,595 cu. ft., and the ratios of whose length, width, and thickness are respectively 11, 5, and 3.

PROCESS.

$$(1) \quad 3 \times 5 \times 11 = 165$$

$$(2) \quad 56,595 \text{ cu. ft.} \div 165 = 343 \text{ cu. ft.}$$

- (3) $\sqrt[3]{343} = 7$; hence, 7 ft.
 (4) $3 \times 7 \text{ ft.} = 21 \text{ ft.}$; $5 \times 7 \text{ ft.} = 35 \text{ ft.}$; $11 \times 7 \text{ ft.} = 77 \text{ ft.}$

OUTLINES OF ANALYSIS.—Apply diagram Ex. 1 (page 456).

- (1) Number of equal cubes in the solid.
- (2) Volume of each equal cube.
- (3) One edge of equal cube, i. e., unit of partition.
- (4) Unit of partition multiplied by the several ratios.

Results.—21 ft., the thickness; 35 ft., the width; and 77 ft., the length.

PROBLEMS.

48. A rectangular solid is 4 times as long and $1\frac{1}{2}$ times as wide as it is thick, and its volume is 82,944 cu. ft. Required its dimensions.

49. A grain bin has a capacity of 3,000 bushels, and its dimensions are in the ratios of 5, 2, and 3. Find its length, width, and depth.

CASE II. To find the pyramid or the cone of which a frustum is a part.

13. Apply the principle relating to triangles produced from trapezoids (page 461) to pyramids or cones produced from frustums.

Ex. A frustum of a cone has a bottom diameter of 8 ft., a top diameter of 6 ft., and an altitude of 5 ft. Required the altitude of the cone.

STEPS IN PROCESS.

- (1) Difference in diameters of the two ends of the frustum.
- (2) Statement of proportion.
- (3) First couplet in (2) divided by 1 ft.
- (4) Solution of proportion. Altitude required.

PROCESS.

- (1) $8 \text{ ft.} - 6 \text{ ft.} = 2 \text{ ft.}$
- (2) $2 \text{ ft.} : 5 \text{ ft.} :: 8 \text{ ft.} : - \text{ft.}$
- (3) $2 : 5 :: 8 \text{ ft.} : - \text{ft.}$
- (4) $\frac{5 \times 8 \text{ ft.}}{2} = 20 \text{ ft.}$

PROBLEMS.

50. Find the altitude of the pyramid from a frustum whose ends are regular hexagons 5 ft. 5 in. and 3 ft. 2 in. on a side respectively, and whose altitude is 7 ft. 9 in.

51. Find the slant height of a cone a frustum of which is 10 ft. 3 in. bottom diameter, 7 ft. top diameter, and 6 ft. 2 in. slant height.

IV. SIMILAR SOLIDS.

14. *Similar solids* are solids that have the same form.

Adapt remarks *a*, *b* (page 459) to similar solids.

15. IMPORTANT FACTS.

VII. *The corresponding dimensions of similar solids are in a common ratio each to each.*

VIII. *The corresponding surfaces of similar solids are similar plane figures.*

IX. *The volumes of similar solids are in the same ratio as the cubes of their corresponding dimensions. Conversely—*

X. *The corresponding dimensions of similar solids are in the same ratio as the cube roots of their volumes.*

CASE I. To find the volume of a solid from the volume of a similar solid, two corresponding dimensions—or their ratios—being given.

16. Ex. 1. A pyramid 8 feet square at the base contains 784 cubic feet. Find the volume of a similar pyramid 15 feet square at the base.

STEPS IN PROCESS.

PROCESS.

(1) Statement of proportion (15, IX). (1) $(8^3) \text{ ft.} : (15^3) \text{ ft.} :: 784 \text{ cu. ft.} : \text{— cu. ft.}$

(2) Solution of proportion. (2) $\frac{15 \times 15 \times 15 \times 784 \text{ cu. ft.}}{8 \times 8 \times 8} = 5,167\frac{3}{4} \text{ cu. ft.}$

Hence, $5,167\frac{3}{4}$ cu. ft., the volume required.

Ex. 2. Two balls of lead weigh respectively $4\frac{1}{2}$ pounds and $121\frac{1}{2}$ pounds. Required the ratios of their diameters.

OUTLINES OF ANALYSIS.

PROCESS.

Since the ratios of the diameters are required, I represent the first term of the ratio by unity.

(1) Statement of proportion (15, X).

(1) $(\sqrt[3]{4\frac{1}{2}}) \text{ lb.} : (\sqrt[3]{121\frac{1}{2}}) \text{ lb.} :: 1 : \text{—}$

(2) $\sqrt[3]{4\frac{1}{2}} : \sqrt[3]{121\frac{1}{2}} :: 1 : \text{—}$

(3) $\sqrt[3]{1} : \sqrt[3]{27} :: 1 : \text{—}$

- (2) First couplet divided by 1 (4) $\frac{3 \times 1}{1} = 3$
 lb. (Art. 377, Prop. II).
 (3) First couplet in (2) divided
 by $\sqrt[3]{4\frac{1}{2}}$ (Art. 377, Prop. II). $1 : 3$
 (4) Solution of proportion. Hence, the diameters are as 1 : 3.

PROBLEMS.

A ball of iron 4 inches in diameter weighs 9 lb. 7 oz.

52. What is the weight of an iron ball 11 inches in diameter?

53. What is the diameter of an iron ball that weighs 1,000 pounds?

54. What part in volume of a 9-inch cube is a 4-inch cube?

55. The top of a pyramid 40 inches high is cut off by a plane parallel to the base and 18 inches from the apex. What part of the pyramid is in the frustum?

56. An oil-can 16 inches in diameter and $11\frac{1}{2}$ inches high, holds 10 gallons. What are the dimensions of a similar can whose capacity is 6 gallons?

CASE II. To divide a pyramid, a cone, or a frustum into required parts by a plane parallel to its base.

17. IMPORTANT FACT XI.—*A plane cutting a pyramid or a cone parallel to its base, divides it into a frustum and a similar pyramid or a similar cone.*

Ex. 1. Divide a pyramid 16 feet high into two equal parts by a plane parallel to its base. Locate the plane.

OUTLINES OF ANALYSIS.

Since the pyramid is to be divided into two equal parts, the volume of the whole pyramid is to the volume of the similar pyramid or segment cut off as 2 : 1; hence,

(1) Statement of proportion (15, X).

(2) Solution of proportion.

Result.—The dividing plane is located 12 ft. 8.3 + in. from the apex; or (16 ft. — 12 ft. 8.3 in. =) 3 ft. 8.7 in. from the base.

PROCESS.

$$(1) \sqrt[3]{2} : \sqrt[3]{1} :: 16 \text{ ft.} : - \text{ft.}$$

$$(2) \frac{\sqrt[3]{1} \times 16 \text{ ft.}}{\sqrt[3]{2}} = 12 \text{ ft. } 8.3 + \text{ in.}$$

Ex. 2. The frustum of a cone is 11 ft. bottom diameter, 9 ft. top diameter, and 4 ft. high. Locate a plane parallel to the base that shall cut off 200 cu. ft. from the top of the frustum.

PROCESS.

- (1) $\frac{1}{3}$ of $4 \times .7854 \times (9^2 + 11^2 + 9 \times 11)$ cu. ft. = 315.2 + cu. ft.
- (2) 2 ft. : 11 ft. :: 4 ft. : — ft.; hence, 22 ft.
- (3) $\frac{1}{3}$ of $22 \times (.7854 \times 11^2)$ cu. ft. = 696.9 cu. ft.
- (4) 696.9 cu. ft. — (315.2 cu. ft. — 200 cu. ft.) = 581.7 cu. ft.
- (5) $(\sqrt[3]{696.9})$ cu. ft. : $(\sqrt[3]{581.7})$ cu. ft. :: 22 ft. : — ft.
- (6) $\sqrt[3]{696.9} : \sqrt[3]{581.7} :: 22 \text{ ft.} : \text{— ft.}$
- (7) $696.9 : 581.7 :: (22^3) \text{ ft.} : (\text{—}^3) \text{ ft.}$
- (8)
$$\frac{581.7 \times (22 \times 22 \times 22) \text{ ft.}}{696.9} = 8,887.9 + \text{ft.}$$
- (9) $8,887.9 + \text{ft.} = (20.71 +)^3 \text{ ft.}$
- (10) $22 \text{ ft.} - 20.71 + \text{ft.} = 1.29 \text{ ft.} = 1 \text{ ft. } 3.48 + \text{in.}$

OUTLINES OF ANALYSIS.

- | | |
|--|---|
| (1) Volume of frustum. | (8) Solution of proportion. |
| (2) Altitude of entire cone. | (9) Result expressed in the cube of the altitude of the segment. |
| (3) Volume of entire cone. | (10) Altitude of the entire cone minus altitude of segment. |
| (4) Volume of segment cut off. | Hence, 1 ft. 3.48 + in., the perpendicular distance of the dividing plane from the base of the frustum. |
| (5) Statement (15, X). | |
| (6) First couplet in (5) reduced (Art. 377, Prop. II). | |
| (7) All terms to the 3d power (Art. 377, Prop. XI). | |

PROBLEMS.

57. A tin pail is 11 inches in diameter at the top, 7 inches in diameter at the bottom, and 8 inches deep. How deep must it be filled with cherries to contain 6 quarts?

58. Divide a stack of hay 16 ft. in diameter and 14 ft. high, with a conical top 12 ft. high, into 3 equal portions by planes parallel to its base. Locate the planes of division.

18. RULES.

I. To find the dimensions of a rectangular solid, from its volume and the ratios of its dimensions :—

1. *Divide the volume by the product of the partitive ratios of its dimensions.*
2. *Extract the cube root of the quotient, for the unit of partition.*
3. *Multiply the unit of partition by the ratios.*

II. To compute the altitude of a pyramid or of a cone from its frustum :—

1. *Make the difference of two corresponding dimensions of the base and top of the frustum and the same dimension of the base one couplet of a proportion, and the altitude of the frustum and the required altitude the other couplet.*
2. *Solve the proportion.*

III. To compute the volume of a solid from the volume of a similar solid and corresponding dimensions of the two solids :—

1. *Make the cubes of the dimensions one couplet of a proportion, and the given volume and the required volume the other couplet.*
2. *Solve the proportion.*

IV. To divide a pyramid or a cone into required parts, by a plane parallel to its base :—

1. *Make the cube root of the volume of the entire pyramid or cone and the volume of the segment cut off by the dividing plane the first couplet of a proportion, and the given dimension of the entire cone and the required corresponding dimension of the segment the other couplet.*
2. *Solve the proportion.*

V. To divide a frustum into required parts, by a plane parallel to its base :—

1. *Compute the dimension of the pyramid or cone of which the frustum is a part.*
2. *Proceed according to Rule IV for dividing pyramids and cones.*

PROBLEMS.

59. The capacity of a rectangular reservoir is 642,000 gallons, and its dimensions are in the ratios of 2, 3, and 5. Find the dimensions.

60. A stack 21 feet high contains 12 tons of straw. Find the height of a similar stack that shall contain 35 tons.

61. If a rope 2 inches in diameter is required to support a ball of iron 3 feet in diameter, what is the diameter of the rope of the same kind required to support a ball of iron 5 feet in diameter?

62. If a cubic foot of marble weighs 162.5 lb., what is the weight of a base for a monument 4 ft. square at the bottom, 3 ft. square at the top, and 2 ft. 10 in. high?

63. Find where a stick of timber 1 ft. 8 in. square at one end, 1 ft. 2 in. square at the other, and 26 ft. long, will balance on a skid or roller.

64. A stick of rosewood 30 ft. long, and 19 in. and 23 in. in diameter at the ends, is to be divided crosswise into three equal parts. Locate the places where it shall be sawed.

65. A cistern is 6 ft. bottom diameter, 8 ft. top diameter, and 7 ft. 2 in. deep. Required the depth of the water when the cistern is $\frac{3}{4}$ full.

66. How high was the work when $\frac{1}{3}$ of the stone in a pyramid 40 feet square on the base and 90 feet high was laid?

67. A block of ebony 13 inches on one edge is $\frac{3}{4}$ of the bulk of a similar block. Required the corresponding edge of the larger block.

68. Find the ratios of the contents of three globes whose diameters are in the ratios of 4, 7, and 9.

RESULTS.

NUMBERS OF PROBLEMS ARE IN BOLD-FACE ITALIC FIGURES; RESULTS ARE IN ROMAN TYPE.

Page

- 51 **1**—61, 67, 71, 73, 79, 83, 89, 97. **2**—101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199. **3**—See Sup., p. 386.
- 53 **1**—2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 5, 5. **2**—2, 2, 2, 2, 3, 3, 13, 101. **3**—2, 2, 2, 3, 3, 5, 19. **4**—2, 2, 3, 3, 3, 17. **5**—2, 2, 5, 5, 5, 5. **6**—2, 5, 5, 5, 7, 23. **7**—5, 7, 7, 17, 17, 47. **8**—2, 3, 5, 5, 11. **9**—3, 3, 3, 5, 7. **10**—2, 3, 5, 7, 19. **11**—3, 5, 7, 17, 19. **12**—3, 3, 5, 191. **13**—2, 5, 53, 113. **14**—2, 2, 2, 4, 967. **15**—3, 3, 5, 7. **16**—2, 3, 7, 19. **17**—2, 5, 5, 7. **18**—3, 13, 19. **19**—5, 7, 7. **20**—29, 41, 43. **21**—61, 109, 997. **22**—11, 23, 103. **23**—5, 31, 751. **24**—2, 3, 3, 7, 7, 11, 13. **25**—2, 2, 3, 3, 3, 5. **26**—2, 2, 3, 3, 7. **27**—2, 5, 5, 691. **28**—5, 7, 11, 13. **29**—17, 53, 101. **30**—Prime. **31**—Prime. **32**—3, 29. **33**—2, 3, 5, 7; 6, 10, 14, 15, 21, 35; 30, 42, 70, 105. **34**—3, 5, 5, 7; 15, 21, 25, 35; 75, 105, 175. **35**—Prime. **36**—3, 3, 149; 9, 447. **37**—3, 5, 5; 15, 25. **38**—2, 2, 2, 3, 5; 4, 6, 10, 15; 8, 12, 20, 30; 24, 40, 60. **39**—2, 2, 2, 3, 3; 4, 6, 9; 8, 12, 18; 16, 24, 36; 48, 72. **40**—2, 2, 2, 5, 5, 5; 4, 10, 25; 8, 20, 50, 125; 40, 100, 250; 200, 500. **41**—3, 3, 3, 3, 5, 13; 9, 15, 39, 65; 27, 45, 117, 195; 135, 351, 585. **42**—3, 7, 59, 73; 21, 177, 219, 413, 511, 4,307; 1,239, 1,533, 12,921, 30,149. **43**—7, 11, 13, 101; 77, 91, 707, 143, 1,111, 1,313; 1,001, 7,777, 9,191, 14,443. **44**—3, 19, 311; 57, 933, 5,909. **45**—2, 3, 3, 7, 11, 13, 37; 6, 14, 22, 26, 74, 9, 21, 33, 39, 111, 77, 91, 259, 143, 407, 481; 18, 42, 66, 78, 222, 154, 182, 518, 286, 314, 962, 63, 99, 117, 333, 231, 273, 777, 169, 1,221, 1,443, 1,001, 2,849, 3,367, 5,291; 126, 198, 234, 666, 462, 546, 1,554, 858, 2,442, 2,886, 2,002, 5,698, 6,734, 10,582, 693, 819, 2,331, 1,287, 3,663, 4,329, 3,003, 8,547, 10,101, 15,873, 37,037; 1,386, 1,638, 4,662, 2,574, 7,326, 8,658, 6,006, 17,094, 20,202, 31,746, 74,074, 9,009, 25,641, 30,303, 47,619, 111,111; 18,018, 51,282, 60,606, 95,238, 222,222, 333,333. **46**—2, 2, 41; 4, 32. **47**—2, 2, 5, 5, 5, 5; 4, 10, 25; 20, 50, 125; 100, 250, 625; 500, 1,250.
- 55 **48**—3. **49**—2. **50**—2, 3, 6. **51**—2, 4, 8, 16. **52**—2, 3; 4, 6; 8, 12; 24. **53**—2, 5; 4, 10; 8, 20; 16, 40; 80. **54**—2, 3, 5, 7; 6, 10, 14, 15, 21, 35; 30, 42, 70, 105; 210. **55**—2, 3, 11; 6, 22, 33; 66.
- 58 **56**—53. **57**—4. **58**—25. **59**—4. **60**—44. **61**—159. **62**—544. **63**—Relatively prime. **64**—800. **65**—2. **66**—101. **67**—41. **68**—13. **69**—23. **70**—Relatively prime. **71**—7. **72**—541. **73**—991. **74**—Relatively prime. **75**—4. **76**—3 rd. **77**—14 ft. **78**—56 fruit-trees. **79**—91 soldiers. **80**—40 yd. **81**—\$12.

Page

- 59 1-8,400. 2-226,800. 3-2,550. 4-67,729. 5-10,368; 20,736 (= 2 × 10,368); 81,104 (= 8 × 10,368).
- 62 6-90. 7-\$126. 8-2,520. 9-720. 10-3,000. 11-900. 12-6,480. 13-88,070. 14-16,071. 15-53,440. 16-184,767,080. 17-2,520.
- 63 18-12¢. 19-2,520 ft. 20-180 A. 21-\$1.50. 22-300 lb.; 75 4-lb. sacks, 60 5-lb., 50 6-lb., 25 12-lb., 12 25-lb. 23-1,400 rd. 24-\$102; 255 chickens, 120 ducks, 68 turkeys. 25-\$1.20. 26-\$210. 27-38 ft. 28-1,766,232 nails. 29-1,680 spikes.
- 65 1-54. 2-1. 3-3. 4-8½. 5-17½ T. 6-35 da. 7-\$132. 8-\$1,350. 9-3,960 bu. 10-65½ da. 11-773,820 bricks. 12-1,440 words.
- 69 1-.50. 2-.800. 3-.30000. 4-.0600000. 5-.810000000. 6-.05700000000. 7-.090000000. 8-.1000000000000. 9-.629.0000000000. 10-25.402000. 11-10.0800. 12-.0103000000000000. 13-.014000000000. 14-810.00. 15-8.10. 16-1.004000.
- 70 17-.0001000000000000; .1000000000000000; .75.0100000000000000; 102.0000000000000000; 1,000.0300000000000000; 5,010.0000000000000000. 18-901.0000000000000; .9010000000000; 8.0000000000000; 8000000000000. 19-.2. 20-.8500. 21-.790. 22-26.0. 23-869.00. 24-105. 25-10.000000; 10.00; 10. 26-.7900; 1,001.0000; 99.0110; 1.4800; .0001. 27-10,010.250000; .010070; .000500; .065500; 91.000000; 50.050000; 1,000,000.000000. 28-.031; 26.500; 80.500. 29-7.2. 30-.750000 A. 31-490,000,000 ten-millionths mi. 32-1,800.00; 19.10; 100.01; 64.00. 33-77.0000; .2000; 15.0000; .1001; 3.0000; 125.0000; 8.3500. 34-1 ten-thousandth. 35-1 millionth. 36-1 millionth. 37-.00909; 900.00900; 900.00009. 38-200,000.00000; .00002; .20200. 39-86,000 mills. 40-1,215 cents. 41-759 mills. 42-1,049.5 cents. 43-500,018 cents. 44-\$9,584.27. 45-\$485.26. 46-93,005 mills. 47-125,625 mills. 48-6,847.5 cents; \$68.475. 49-\$8,005.805.
- 71 1-½; ½; ½. 2-¾; ¾; ¾. 3-⅞; ⅞; ⅞. 4-⅞; ⅞; ⅞. 5-⅞; ⅞; ⅞. 6-⅞; ⅞; ⅞.
- 72 7-⅞. 8-⅞. 9-⅞. 10-⅞. 11-⅞; ⅞; ⅞. 12-⅞; ⅞; ⅞. 13-⅞; ⅞; ⅞. 14-⅞; ⅞; ⅞. 15-⅞; ⅞; ⅞. 16-⅞; ⅞; ⅞. 17-⅞; ⅞; ⅞ sq. yd.
- 73 18-⅞; ⅞; ⅞; ⅞. 19-⅞; ⅞; ⅞; ⅞. 20-⅞; ⅞; ⅞; ⅞. 21-⅞; ⅞; ⅞; ⅞. 22-⅞; ⅞; ⅞; ⅞. 23-⅞; ⅞; ⅞; ⅞. 24-⅞; ⅞; ⅞; ⅞. 25-⅞; ⅞; ⅞; ⅞. 26-⅞; ⅞; ⅞; ⅞. 27-⅞; ⅞; ⅞; ⅞. 28-1547. 29-240. 30-1650. 31-360 yd.
- 74 32-⅞; ⅞; ⅞. 33-⅞; ⅞; ⅞. 34-⅞; ⅞; ⅞. 35-⅞; ⅞; ⅞. 36-⅞; ⅞; ⅞. 37-⅞; ⅞; ⅞. 38-⅞; ⅞; ⅞. 39-⅞; ⅞; ⅞. 40-⅞; ⅞; ⅞.

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- 74 41—1111 A.; 1111 A.; 1111 A.; 1111 A. 42—111. 43—111. 44—11111111. 45—111111. 46—11; 801; 1511. 47—154; 71; 2,43311. 48—7; 9; 16,359. 49—961 yd.; 120211; 641 doz. 50—164811; 2341 wk.; 8511 cu. ft. 51—1,083,358 ft.; 8311 da.; 1,28911 bu.
- 75 52—11; 111, 111. 53—100000; 11111111; 11111. 54—11011011; 11000000; 11111. 55—1111 da.; 11111 yd.; 11111 lb. 56—11111 pk.; 111111 cd.; 1111111 T. 57—11111 ch.; 11111 mi.; 11111. 58—11111; 1111. 59—1111; 11111 ft.; 11111 lb.
- 76 60—111111; 11; 1. 61—11; 111; 111. 62—111; 11; 11. 63—11111; 1111; 1111111. 64—4,101111; 568; 1,000,1111. 65—8,80111 T.; 2221 mi. 66—10511; 4,915; 99,108,1111. 67—3211 A.; 4,641,111 bar.; 621 lb. 68—1111; 111111; 11111. 69—1111111111; 11111111; 1111111. 70—1111 mi.; 11111 A.; 1111111 cu. ft. 71—1111 yr.; 111111 sq. mi.; 1111 gal. 72—1111111. 73—11111111. 74—11; 111; 111. 75—11111111. 76—1111111 bu. 77—1111111 da. 78—1111; 1111; 1111. 79—11111 sq. yd.; 1111 bu.; 1111111 wk.; 1111111 T. 80—11111; 11111; 11111; 11111; 11111. 81—1111; 111111; 111111; 111111; 111111. 82—1111; 1111; 1111. 83—111; 111; 111. 84—1111; 111. 85—111111; 111111; 111111; 111111; 111111. 86—111 bu.; 111 bu.; 11 bu. 87—111 wk.; 111 wk.; 111 wk. 88—111; 111; 111; 111; 111; 111. 89—111 lb.; 111 lb.; 111 lb.; 111 lb. 90—1111; 1111; 11111; 111111. 91—111; 111; 111; 111. 92—111; 111; 111. 93—1111; 1111; 1111; 1111. 94—111. 95—1111.
- 77 1—1; 11; 11; 1. 2—1; 111; 111; 1. 3—11; 1111; 11111; 111111. 4—11; 1111; 11; 1. 5—1111 A.; 1111111 gal.; 1111111 yr. 6—611; 511; 18,111. 7—11; 87,111; 1,111. 8—411; 8,001,111; 10,11111. 9—1,1111 mi.; 224,11111 T. 10—20,11111 cu. ft.; 88,111111 sq. mi.
- 78 11—1.625; .4; .68; .5625. 12—.025; .00925; .328125; .001875. 13—.0000221 mi.; .5625 T.; 1.161; 1.025. 14—.25016; .125875; .000492. 15—15.075; 518.03125; 100.09161. 16—19.9461; 5.5881; 7.075. 17—1,212.75; 24.021 yd.; 82.5625 gal. 18—6.0375; 72.831; 154.6921; 24.7141. 19—111; 11111; 11111. 20—1.875; .15625; .005; .0003125. 21—711; 21,111; 100,111. 22—11 da.; 1 da. 23—11111 ft.; 11111 mi. 24—8.17456. 25—.79161; .57431; .4968111.
- 79 1—606 gi. 2—541,266 in. 3—96,848 d. 4—972 pt. 5—12,848 cu. ft. 6—1,3791. 7—605 qt. 8—153,420'. 9—38,400 sheets. 10—576,000 A. 11—17,641,800 sq. in. 12—26,542 l. 13—552,073 oz. 14—125,280 gr. 15—5,555 gr. 16—15,458'.
- 80 17—175,730,389 sec. 18—594,520 cu. in. 19—834,444 sq. in. 20—144,6701 sq. ft. 21—69,076,850 sq. l. 22—8,253,015 sec. 23—.324 in. 24—.48 rd. 25—72 gr. 26—.56 pt. 27—11 qt. 28—11 min. 29—1111 ft. 30—111 oz. 31—34,084.8 sec. 32—1.84 pt. 33—672 oz. 34—24,360 far. 35—8,329 l. 36—111 pt. 37—3411 pwt. 38—85711 sq. yd. 39—15,624 gi. 40—3,024,84711 sec.

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- 81 41—8 pk. 8 qt. 1.04 pt. 42—2 qt. 1 pt. 2 gi. 43—19 gal. 2 qt. 1 pt. 2 gi. 44—6 da. 6 h. 12 min. 81.68 sec. 45—57 cu. ft. 1,036.8 cu. in. 46—1,728 lb. 8 oz. 47—7' 23.2". 48—5 oz. 16 pwt. 9.6 gr. 49—1,840 lb. 50—.005 gr. 51—81 A. 2 sq. rd. 12 sq. yd. 129.6 sq. in. 52—185 mi. 2 rd. 2 yd. 7.2 in. 53—Cong. 14 O 2 f 7 f 2 m 46.656. 54—£6 8s. 4d. 3.2 far. 55—7s. 8d. 8 far. 56—93 $\frac{1}{2}$ mills. 57—5s. 8d. 8 far. 58—2 da. 1 h. 59—14 h. 24 min. 60—142 A. 35 sq. rd. 16 sq. yd. 7 sq. ft. 36 sq. in. 61—5 cu. ft. 1,357 $\frac{1}{4}$ cu. in. 62—7 A. 146 sq. rd. 20 sq. yd. 1 sq. ft. 72 sq. in. 63—2 cwt. 40 lb. 64—2 rm. 9 quires 8 sheets. 65—17 gt. gro. 6 gro. 8 doz. 66—100 rd. 15 ft. 3 $\frac{1}{4}$ in. 67—18 lb. 7 oz.
- 82 68—285 gal. 69—62 bu. 8 pk. 2 qt. 1 pt. 70—4° 42.6". 71—39 lb. 4 g. 72—6 T. 17 cwt. 6 lb. 6 oz. 73—18 cu. yd. 18 cu. ft. 1,501 cu. in. 74—22 A. 153 sq. rd. 2 sq. yd. 7 sq. ft. 108 sq. in. 75—89 wk. 4 da. 47 min. 20 sec. 76—123 rm. 1 quire 16 sheets. 77—£42. 78—15 mi. 57 ch. 8 rd. 5 l. 79—2 lb. 5 oz. 8 pwt. 18 gr. 80—17 T. 3 cwt. 81—72 gt. gro. 5 gro. 5 doz. 9 buttons.
- 83 82—.0000075 T. 83—.0004°. 84—.025 yd. 85—.636328125 bu. 86—.4968 $\frac{1}{10}$ da. 87—.11 $\frac{1}{10}$ mi. 88—.11 $\frac{1}{10}$ gal. 89—£.11 $\frac{1}{10}$. 90—.11 $\frac{1}{10}$ sq. rd. 91—.11 $\frac{1}{10}$ A.
- 84 92—.8 lb. 93—.5 rd. 94—.6 wk. 95—.49265625 T. 96—£.040625. 97—.78699 $\frac{1}{10}$ cu. yd. 98— $\frac{1}{10}$ ft. 99— $\frac{1}{10}$ rm. 100— $\frac{1}{10}$ bu. 101— $\frac{1}{10}$ hhd. 102— $\frac{1}{10}$ leap-yr. 103— $\frac{1}{10}$ gro. 104—.2248263 $\frac{1}{10}$ lb. 105—.7684 $\frac{1}{10}$ mi. 106—278.75 bu. 107—278 $\frac{1}{10}$ bu.
- 86 108—580,966 in. 109—613,022.75 oz. 110—29,186 gr. 111—1,788 ft. 112—1,157,220 sec. 113—1,656 stat. mi. 114—£8,394.71 $\frac{1}{10}$. 115—1,016.1833 mi. 116—255,744 cu. in. 117—1,185,092.8 oz. 118—501 cu. ft. 119—11,572 gr. 120—11 $\frac{1}{10}$ ft. 121—11 $\frac{1}{10}$ min. 122— $\frac{1}{10}$ lb. 123—.216 sec. 124—.9 doz. 125—12.15'. 126—83 gal. 1 qt. 127—301 sq. mi. 256 A. 106 sq. rd. 7 sq. yd. 2 sq. ft. 48.96 sq. in. 128—4 cu. yd. 19 cu. ft. 998.6 cu. in. 129—3 lb. 4 $\frac{1}{10}$ oz. 130—£49 7s. 8d. 2.4 far. 131—170 da. 8 h. 132—10s. 11d. 1 far. 133—241° 4' 17 $\frac{1}{10}$ ". 134—116 rd. 4 yd. 1 ft. 2.4 in. 135—2 rm. 17 quires 12 sheets. 136—36 gal. 1 pt. 3.2 gi. 137—573 A. 27 sq. yd. 5 sq. ft. 42.048 sq. in. 138—3 mi. 54 rd. 4 yd. 1 ft. 3 $\frac{1}{10}$ in. 139—19 cu. ft. 1,226.88 cu. in. 140—71 bar. 4 lb. 141—2 bar. 26 gal. 3 qt. 1 pt. 1 half-pt. 142—£1,698 16s. 9d. 2.4 far. 143—£2,054 17s. 8d. 2 $\frac{1}{10}$ far. 144—10 A. 144 sq. rd. 24 sq. yd. 4 sq. ft. 145—89 ch. 77 l. 146—850 lb. 5 oz. 147—32' 18.4". 148—234 da. 6 h. 49 min. 149—.00297 $\frac{1}{10}$ da. 150—.8025 lb. 151—.58649 $\frac{1}{10}$ sq. mi. 152—.6985 $\frac{1}{10}$ bar. 153— $\frac{1}{10}$ rd. 154— $\frac{1}{10}$ A. 155— $\frac{1}{10}$ mi. 156— $\frac{1}{10}$ of a half-mile.
- 87 157—.58 $\frac{1}{10}$ lb. 158—\$.43775. 159—£.53125. 160—.42706 $\frac{1}{10}$ cu. yd. 161—.100418526 $\frac{1}{10}$ gro. T. 162— $\frac{1}{10}$ sq. mi. 163— $\frac{1}{10}$ rm. 164— $\frac{1}{10}$ da. 165— $\frac{1}{10}$ Cong. 166— $\frac{1}{10}$ lb. 167—180 $\frac{1}{10}$ °. 168—181,459,828 ft. 169—352 rails. 170—1,806 revolutions. 171—10,825.71065 $\frac{1}{10}$ mi. 172—\$95.35.

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- 87 173— $\frac{1}{4}$ of 8 rd. 174—16 $\frac{1}{4}$ rd. 175—72 A. 106 sq. rd. 181 $\frac{1}{2}$ sq. ft. 176—72,600 hills. 177—5,898.501 lb. 178— $\frac{1}{10}$ of 8 cd. 179—26232 $\frac{1}{2}$ of 8 $\frac{1}{2}$ bu. 180—107 times and 1 $\frac{1}{2}$ pt. rem. 181—17 $\frac{1}{2}$ lb. 182—21 $\frac{1}{2}$ lb.
- 88 183— $\frac{1}{10}$ rm. 184—87,840 mi. 185—36,524 da. 186—4,680 powders. 187—27,777 da. 7 h. 46 min. 40 sec. 188—If a leap-year, 7,862,400 sec.; if a common year, 7,776,000 sec. 189—459,900". 190—1,008 bottles. 191—2,940 T. 192—.00048 $\frac{1}{4}$ mi. 193—22.5 $\frac{1}{4}$ °. 194—.00017 $\frac{1}{4}$ A. 195— $\frac{1}{4}$ bu. 196—.6684 $\frac{1}{4}$ cd. 197—.3875 rm. 198—617 doz. 199—\$3.376. 200—1,016 mi. 42 rd. 10 ft. 9.888 in. 201—£3 12s. 5d. $\frac{1}{4}$ far. 202—2,724125 mi. 203—82 A. 20 sq. rd. 7 sq. yd. 2 sq. ft. 48.96 sq. in.
- 90 1—6° 19' 80". 2—18° 51'. 3—1° 3' 45". 4—106° 16' 15". 5—16° 84' 15". 6—4° 80'. 7—6° 50'. 8—30° 7' 80". 9—108° 30'. 10—4° 54'. 11—46° 15". 12—121° 2' 15". 13—52° 80'. 14—12° 27' E.
- 91 15—45° 26' 30". 16—89° 56' 53". 17—92° 35' 15". 18—10° 36' 21". 19—77° 15". 20—3 h. 9 min. 7 sec. 21—9 min. 42 $\frac{1}{4}$ sec. 22—18 min. 3 sec. 23—1 h. 8 min. 12 sec. 24—38 min. 22 $\frac{1}{4}$ sec. 25—6 h. 81 min. 87 $\frac{1}{4}$ sec. 26—52 min. 1 $\frac{1}{4}$ sec. 27—3 h. 9 min. 18 $\frac{1}{4}$ sec. 28—8 h. 44 min. 29—17 min. 11 $\frac{1}{4}$ sec. 30—58 $\frac{1}{4}$ sec. 31—8 h. 17 min. 17 sec. 32—10 h. 51 min. 6 $\frac{1}{4}$ sec. 33—11 h. 13 min. 28 sec. 34—9 o'clock 52 sec., A.M. 35—33° 20' W. 36—145° 15' E.
- 96 1—109,598. 2—168,735. 3—55,830. 4—1,448,459. 5—1,438,561. 6—120,442. 7—107,246. 8—9,052. 9—106,816.
- 97 10—878,488. 11—1,065,279,377. 12—\$1,620,189,7725. 13—22,465,845761. 14—526,792. 15—878,588,950. 16—25,727,477019. 17—677,906. 18—583,175,741. 19—24,719,719816. 20—676,582. 21—580,570,766. 22—25,517,444166. 23—1,171,805. 24—1,068,449,102. 25—86,381,134513. 26—\$2,725,499,4825. 27—1,380,222. 28—1,948,868,312. 29—\$2,725,499,4825. 30—88,367,95278. 31—1,868,589,95278. 32—1,948,906,679,95278. 33—1,945,236,901,95278. 34—\$4,271,959,615.
- 98 35—727,000,000,008,000,005,097,108,059,087,674,844. 36—7,087,500,028,897,355,76006241.
- 99 37—91 bu. 2 qt. 38—12 yr. 233 da. 23 h. 8 min. 20 sec. 39—Cong. 4 f $\frac{3}{4}$ 4 f $\frac{3}{4}$ 3 m 80. 40—6 mi. 282 rd. 2 yd. 10 in. 41—149 lb. 6 oz. 1 pwt. 18 gr. 42—1,043 A. 84 sq. rd. 43—£2,114 8s. 5d. 3 far. 44—\$18,965.75. 45—3 cir. 8 quad. 59° 30' 47". 46—40 T. 17 cwt. 22 lb. 47—lb. 10 $\frac{3}{4}$ 7 $\frac{3}{4}$ 2. 48—10 $\frac{1}{4}$ grt. gro. 49—468 gal. 1 pt. 3 gi. 50—3 cu. yd. 25 cu. ft. 1,542.2 cu. in. 51—39 yr. 296 da. 1 min. 14 sec.
- 100 52—24. 53—3 $\frac{1}{10}$. 54—25 $\frac{1}{10}$. 55—74 $\frac{1}{10}$. 56—2 $\frac{1}{10}$. 57—580 $\frac{1}{10}$. 58—19 $\frac{1}{10}$. 59—93 $\frac{1}{10}$.
- 101 60—4 $\frac{1}{10}$ mi. 61—\$214 $\frac{1}{10}$. 62—260 $\frac{1}{10}$ A. 63—710 $\frac{1}{10}$ gal. 64—1500 $\frac{1}{10}$. 65—17,010 $\frac{1}{10}$. 66—5,629 $\frac{1}{10}$. 67—684,5184 $\frac{1}{10}$.

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- 102 68—42,281.2 bar. 69—83 $\frac{1}{4}$ yd. 70—\$56.53. 71—1,568 $\frac{1}{4}$ mi. 72—466 $\frac{1}{4}$ A. 73—24 gal. 2 qt. 1 pt. 74—1,061 A. 5 sq. ch. 10 P. 598.125 sq. l. 75—28 $\frac{1}{4}$ cu. yd. 76—85,598 bu. 77—566 bu. 6 qt. 1 pt. 78—\$428.81 $\frac{1}{2}$. 79—1 rm. 6 quires 16 sheets. 80—1,110 $\frac{1}{4}$ cd.
- 108 81—684.875 T. 82—85—C, 7 $\frac{1}{4}$ cd.; D, 6 $\frac{1}{4}$ cd.; E, 7 $\frac{1}{4}$ cd.; total, 22 $\frac{1}{4}$ cd. 86—847.875 bu. 87—\$9.01. 88—55 gal. 89—52 T. 14 cwt. 21.4 lb. 90—88 $\frac{1}{4}$ mi.
- 107 1—440,898. 2—3,420,766. 3—99,901. 4—54896. 5—84685. 6—8125. 7—,00989. 8—149,8175. 9—71375. 10—100.44868. 11—9,666.667. 12—\$71.875. 13—\$12,858.75. 14—\$.125. 15—449.94485. 16—267.45 ft. 17—\$99.625. 18—5,624,411.
- 108 19—48,924.000026. 20—808.08079991. 21—109 rd. 8 yd. 2 ft. 6 in. 22—1 gal. 1 qt. 1 pt. 2 $\frac{1}{2}$ gi. 23—O 4 f 3 8 f 3 8 m 27. 24—1 lb. 7 oz. 2 pwt. 22 gr. 25—lb. 10 5 7 3 6 9 1 gr. 8.
- 109 26—78° 22' 36". 27—41 bu. 8 qt. 1 $\frac{1}{2}$ pt. 28—28 T. 15 cwt. 67 lb. 29—4 A. 119 sq. rd. 4 sq. ft. 86 sq. in. 30—82 cd. 89 cu. ft. 31—27 mi. 59 rd. 1 yd. 1.5 ft. 32—35 sq. mi. 171 A. 2 sq. ch. 2 P. 166 sq. l. 33—5 s. 27° 5' 22". 34—2 yr. 56 da. 15 hr. 22 min. 58 sec. 35—18 cu. yd. 8 cu. ft. 1,846 cu. in. 36—6 yr. 5 mo. 19 da. 37—4 yr. 24 da. 38—15 yr. 1 mo. 12 da. 39—8 rm. 7 quires 18 sheets.
- 110 40— $\frac{1}{4}$. 41— $\frac{1}{4}$. 42— $\frac{1}{4}$. 43— $\frac{1}{4}$. 44—6 $\frac{1}{4}$. 45—30 $\frac{1}{4}$. 46— $\frac{1}{4}$ T. 47—189 $\frac{1}{4}$ mi. 48—20 $\frac{1}{4}$ bu. 49— $\frac{1}{4}$ hhd. 50— $\frac{1}{4}$ T. 51—4 $\frac{1}{4}$ gal. 52—847 $\frac{1}{4}$ mi. 53—11 $\frac{1}{4}$ lb. 54—11 $\frac{1}{4}$ cwt. 55—200 $\frac{1}{4}$ rd. 56—102 $\frac{1}{4}$.
- 111 57—69,599,626. 58—,886281. 59—15,263,508.017215. 60—\$.25. 61—2,340,151 sq. mi. 62—82 sq. rd. 154 sq. ft. 36 sq. in. 63—3 yr. 9 mo. 12 da. 64—218 da. 28 hr. 35 min. 15 sec. 65—658 $\frac{1}{4}$ ft. 66—20 $\frac{1}{4}$ gal.
- 112 67—\$4,301.25. 68—9.998671784. 69—412,836,258,906 $\frac{1}{4}$. 70—27 $\frac{1}{4}$. 71—99 bu. 3 $\frac{1}{4}$ pk. 72—36 yr. 24 da. 8 hr. 15 min. 73—1,240 A. 74—873,991,081,108,038.00294812796. 75—63 yr. 1 mo. 24 da. 76—4 yr. 29 da. 77—18 yr. 10 mo. 2 da. 78—6 mo. 29 da. 79—192 yr. 1 mo. 19 da. 80—1 yr. 9 mo. 12 da.
- 116 1—450,430. 2—475,200. 3—2,445. 4—1,1914. 5—296.64. 6—9,792.3. 7—49,545. 8—\$750. 9—\$4.52. 10—\$35,106.75.
- 117 11—830.4. 12—88,040. 13—849,062,200. 14—26,545,000,000. 15—4,800,700. 16—8,059,000. 17—6,098,040,000. 18—1,205,789,000,000. 19—82,007. 20—71,819,200. 21—8,600.71. 22—72.08. 23—\$16,000. 24—\$3,400. 25—196,000 lb. 26—15,840,000 ft. 27—\$6,000. 28—2,610,388,800,000.
- 118 29—3,528,756. 30—514,584,772. 31—2,135,606,132. 32—54,312,500. 33—13,150,840,500,000. 34—417,266,000. 35—44,944,200,000,000. 36—40,742,260,000. 37—531,225,000. 38—935,757,900.

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- 131 216—1,543.8480. 217—5.570. 218—90,611.27. 219—.00226.
220—3,204.88. 221—21,153.667. 222—8,267.899. 223—.658. 224—.184.
225—.008. 226—107.378.
- 135 1—10,475. 2—394.05. 3—4.8808 $\frac{1}{2}$. 4—.079882. 5—.92025. 6—9,554 $\frac{1}{100}$.
7—48,942 $\frac{1}{2}$. 8—.0120354 $\frac{1}{100}$. 9—1.1494 $\frac{1}{2}$. 10—1,068,156. 11—1,004,041 $\frac{1}{2}$.
12—532.3. 13—.00407 $\frac{1}{100}$. 14—169,000 $\frac{1}{2}$. 15—.0003448 $\frac{1}{100}$. 16—.176 $\frac{1}{100}$.
17—.00000801953125. 18—.00408.
- 136 19—66,524 $\frac{1}{2}$. 20—14,810. 21—141.47 $\frac{1}{100}$. 22—.02229 $\frac{1}{100}$. 23—.075 $\frac{1}{2}$.
24—.001299 $\frac{1}{100}$. 25—492.3 $\frac{1}{2}$. 26—.0003645 $\frac{1}{2}$. 27—4,000. 28—163,277 $\frac{1}{100}$.
29—.6516 $\frac{1}{100}$ mi. 30—2.7368 $\frac{1}{100}$ gal. 31—5,331,369.23 $\frac{1}{2}$ bu. 32—.0006584 $\frac{1}{2}$.
33—6,585.256 $\frac{1}{2}$.
- 137 34—6 sq. mi. 7 sq. rd. 16 sq. yd. 7 sq. ft. 46 $\frac{1}{2}$ sq. in. 35—11 gal. 8 qt. $\frac{1}{2}$ gi.
36—6 T. 5 cwt. 95 $\frac{1}{2}$ lb. 37—3 bu. 2 pk. 6 qt. $\frac{1}{2}$ pt. 38—74 sq. rd. 4 sq. ft.
76 $\frac{1}{2}$ sq. in. 39—18 $\frac{1}{2}$. 40—5 cu. yd. 3 cu. ft. 576 cu. in. 41—21 $\frac{1}{2}$.
42—97 $\frac{1}{100}$. 43—43 $\frac{1}{2}$. 44—21.5 $\frac{1}{2}$ gr. 45—1 $\frac{1}{100}$.
- 138 46— $\frac{1}{100}$. 47— $\frac{1}{100}$. 48—2 $\frac{1}{2}$. 49— $\frac{1}{100}$. 50— $\frac{1}{2}$. 51— $\frac{1}{100}$. 52—2 $\frac{1}{2}$.
53— $\frac{1}{100}$. 54— $\frac{1}{100}$. 55—17 $\frac{1}{100}$. 56— $\frac{1}{2}$. 57— $\frac{1}{100}$. 58— $\frac{1}{100}$.
59— $\frac{1}{100}$. 60—635 $\frac{1}{100}$.
- 139 61—38. 62—1 $\frac{1}{2}$. 63—146 $\frac{1}{100}$. 64— $\frac{1}{100}$. 65—15 $\frac{1}{2}$. 66—6 $\frac{1}{2}$.
67—7,071 $\frac{1}{2}$. 68—1 $\frac{1}{2}$. 69—1 $\frac{1}{2}$. 70—8,342 $\frac{1}{100}$. 71—762 $\frac{1}{2}$ bu. 72—22 $\frac{1}{2}$ gal.
73— $\frac{1}{100}$. 74— $\frac{1}{100}$. 75—9,119 $\frac{1}{100}$.
- 140 76—8 $\frac{1}{2}$. 77—1 $\frac{1}{100}$. 78—184 $\frac{1}{100}$. 79— $\frac{1}{100}$. 80—1 $\frac{1}{2}$. 81—1 $\frac{1}{2}$.
82— $\frac{1}{100}$. 83—2,179 $\frac{1}{100}$. 84— $\frac{1}{100}$. 85—10,471 $\frac{1}{2}$. 86— $\frac{1}{100}$. 87—1 $\frac{1}{100}$.
88— $\frac{1}{100}$. 89— $\frac{1}{100}$.
- 141 90—5,207 $\frac{1}{100}$. 91—.6623 $\frac{1}{100}$. 92—2.0657 $\frac{1}{100}$. 93—.000732684.
94— $\frac{1}{100}$. 95—75. 96—.00004375. 97— $\frac{1}{100}$. 98—.000011764 $\frac{1}{100}$.
99— $\frac{1}{100}$. 100—3.6844 $\frac{1}{100}$. 101—4 $\frac{1}{2}$. 102—125.9674 $\frac{1}{100}$.
103—87,781,068 $\frac{1}{100}$. 104—438.6513 $\frac{1}{100}$. 105—104 sq. rd. 26 sq. yd. 1
sq. ft. 44 $\frac{1}{2}$ sq. in. 106—3 oz. 1 pwt. 6 $\frac{1}{2}$ gr. 107—8 mi. 140 rd. 11 ft. 2 $\frac{1}{2}$ in.
108—3 cu. ft. 915 $\frac{1}{100}$ cu. in. 109—15,539. 110—1 $\frac{1}{2}$. 111—1 $\frac{1}{2}$.
112—16 $\frac{1}{100}$ mi. 113—9 $\frac{1}{100}$ cd. 114—10 $\frac{1}{100}$ cd. 115—11 $\frac{1}{100}$ cd.
- 142 116—\$2,330. 117—8 $\frac{1}{100}$. 118—2 $\frac{1}{100}$. 119—12,734 revolutions.
120—\$7.75. 121— $\frac{1}{100}$. 122—\$4, amt. rec'd per cd. 123—\$2.85 $\frac{1}{2}$,
cost per cd. 124—\$.66 $\frac{1}{2}$. 125—\$.75. 126—36 $\frac{1}{2}$ ft. 127—81 $\frac{1}{2}$.
128—£183 $\frac{1}{100}$. 129—8. 130—967. 131—238. 132—725. 133—4.55.
134—961,430 $\frac{1}{2}$. 135—709.
- 143 136—137 $\frac{1}{2}$. 137—329 $\frac{1}{2}$. 138—7 $\frac{1}{2}$. 139—62.45. 140—4.53.
141—1,375.06. 142—538.6. 143—1,481.472. 144—24.4725. 145—1,183.36.
146—130.9375. 147—21.664. 148—.544.

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 145 149—.008. 150—2.24. 151—100.28. 152—28.838+. 153—.204+. 154—141.771+. 155—1.9097+. 156—.0442+. 157—.0004+. 158—88.41760+.
- 147 1—652.010081. 2—2 $\frac{11111}{11111}$. 3—6.402675. 4—1 $\frac{11111}{11111}$. 5— $\frac{11111}{11111}$. 6— $\frac{11111}{11111}$. 7— $\frac{11111}{11111}$. 8—10 yr. 7 mo. 5 da. 9—7 $\frac{11111}{11111}$. 10—1 $\frac{111}{111}$. 11— $\frac{111}{111}$ greater. 12—\$.04. 13—57,142 $\frac{1}{2}$. 14—.000000160007062929960999999999528. 15—99.99. 16—.90466875. 17—9 rd. 5 yd. 1 ft. 9 in. 18—12 $\frac{11111}{11111}$. 19—1 $\frac{1111}{1111}$. 20—70 yr. 7 mo. 21 da. 21—\$114.17+. 22—106 sq. rd. 12 sq. yd. 4 $\frac{1}{2}$ sq. ft.
- 148 23—.140014. 24—805 $\frac{11111}{11111}$. 25—81 $\frac{1}{2}$. 26—27 $\frac{11111111}{11111111}$. 27—2 $\frac{1}{2}$ gr. 28—188 $\frac{1111}{1111}$. 29—20 $\frac{111}{111}$. 30—6.903. 31—1 $\frac{11111}{11111}$. 32—944 $\frac{1}{2}$. 33—2,803 $\frac{1}{2}$. 34—25 $\frac{1}{2}$. 35—88 $\frac{11111}{11111}$. 36—1,296,000". 37— $\frac{11}{11}$. 38—1,856 $\frac{111}{111}$. 39—65°. 40—4 piles, and 6 cd. 48 cu. ft., over. 41—150 A. 156 sq. rd. 14 sq. yd. 6 sq. ft. 19+ sq. in. 42—20 hhd. and 1,087 $\frac{1}{2}$ lb. remaining. 43—162° 6' 39".
- 149 44—July 25, 1886. 45—June 1, 1885. 46—Aug. 20, 1891. 47—Aug. 13, 1890. 48—Apr. 7, 1888. 49—Apr. 8, 1888. 50—Mar. 5, 1887. 51—Mar. 7, 1887. 52—Aug. 26, 1881. 53—Nov. 10, 1887. 54—Aug. 5. 55—July 11. 56—Oct. 11. 57—May 18. 58—May 1, 1892. 59—Oct. 8. 60—110 da. 61—105 da. 62—185 da. 63—298 da. 64—1 $\frac{1}{2}$. 65—2 $\frac{111111}{111111}$. 66— $\frac{11111111}{11111111}$. 67—1 $\frac{11111}{11111}$. 68— $\frac{111111}{111111}$; 1 $\frac{111111}{111111}$. 69—Result depends on the year. 70—51 or 52. 71—\$11.43. 72—\$1,702.28.
- 150 73—16,063 $\frac{1}{2}$ bu. wheat; 17,971 $\frac{1}{2}$ bu. corn. 74—3 pk. 2.46 $\frac{1}{2}$ qt. 75—\$11.25. 76—81 min. 17 $\frac{1}{2}$ sec. 77—7 gal. 1 qt. 2 gi. 78—122° 26' 45" W. 79—\$95.10+. 80—10. 81—65 A. 50 $\frac{1}{2}$ sq. rd. 82—\$845.58. 83—\$2.54, paid by farmer.
- 151 84—3,037.655 A. 85—9,112.965 A. 86—6379.0755 A. 87—18,225.98 A. 88—7,594.1375 A. 89—303.7655 A. 90—23 $\frac{1}{2}$ gross. 91—\$2.30. 92—66 $\frac{1}{2}$ da. 93—6,250 bu. 94—\$26.72. 95—\$70 $\frac{1}{2}$. 96—\$186.11, gain. 97—54 min. 19 sec. sooner. 98—\$2.63, gain.
- 152 99—\$20,212.50. 100—20 $\frac{11111111}{11111111}$ A. 101—\$2,831.41. 102—A owned $\frac{1}{4}$ of the mill, and B $\frac{3}{4}$. 103—\$496.54 $\frac{1111}{1111}$. 104—9 min. 3 sec. after 6 P.M. 105—45 M. 106— $\frac{1}{5}$ da. 107—779.20 $\frac{1}{4}$ mi. 108—3,619.046 $\frac{11}{11}$ mi. 109—\$257.67. 110—295 $\frac{11}{11}$ bu. 111—\$18.47. 112—\$4.98. 113—\$2.30, 1st man; \$3.83, 2d; \$6.13, 3d; \$6.22, 4th; \$.99, 1st boy; \$.85, 2d; \$3.08, 3d.
- 153 114—84° 88' 45". 115—\$378.065. 116—41 $\frac{11}{11}$ lb. sugar; 10 $\frac{11}{11}$ lb. coffee. 117—Less. 118—421 $\frac{111}{111}$ lb. 119—\$90,250, widow's share; \$10,027 $\frac{1}{2}$, each daughter's share; \$30,083 $\frac{1}{2}$, each son's share. 120—1 hr. 22 min. 31 sec. P.M. 121—\$356.25. 122—42 min. 40 sec. after 2 o'clock A.M., Jan. 1, 1893. 123—42 min. 80 sec. after 8 o'clock P.M., Dec. 31, 1892. 124—\$8.84+, gain. 125—2,946 $\frac{1}{2}$ lb. avoird. 126—3,590 $\frac{1}{2}$ lb. Troy. 127—\$27.50.
- 160 1—159, value of each member. 2—55, value of each member. 3—64, values of members. 4—9, value of each member. 5—455, value of each member. 6—94. 7—158. 8—9 $\frac{1}{2}$. 9—8 $\frac{1}{2}$. 10—134 $\frac{1}{2}$. 11—732. 12—70 $\frac{11}{11}$. 13—24. 14—47. 15—6. 16—6 $\frac{1}{2}$. 17—35, value of each member. 18—106, value of each member. 19—80, value of each member.

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- 162 1—1 lb. 1 oz. 9.96 gr. fine silver. 2—1 lb. 1 oz. 18 pwt. 10.5 gr. fine silver.
3—5 lb. 9 oz. 16 pwt. 18.5 gr. 4—4 lb. 10 oz. 5 pwt. 12½ gr. 5—Five-dollar piece.
- 163 6—\$409.44½.
- 164 7—\$428.75. 8—53.42+.
- 166 12—\$3.96. 13—\$86.42. 14—\$168.05. 15—\$527.14. 16—\$120.29.
17—\$148.88.
- 168 1—726. 2— $\frac{117}{100}$. 3—91 $\frac{1}{100}$. 4—136.662½. 5—212½ da. 6—3,949½ lb.
7—918.267½ gal. 8—43 mi. 277 rd. 18 ft. 6½ in. 9—88½¢ per bu. 10—\$2.17½
per bar. 11—6½¢ per lb. 12—\$.84½ per lb. 13—\$6.90½ per bar.
14—\$301.56½ per head. 15—\$91½ per A. 16—780½.
- 169 17—\$78.44½. 18—\$684. 19—\$304.04½. 20—\$379.95½. 21—52½ lb.
22—\$15.27½. 23—\$.292+. 24—.53½ in. 25—.31½ in. 26—9.69½ in.
27—\$.83½. 28—\$9.89½.
- 170 29-35—Sunday, +11.45½°; Monday, -7.62½°; Tuesday, -16.95½°; Wed-
nesday, -21.66½°; Thursday, -2.41½°; Friday, +18.25°; Saturday, +29.41½°.
36-41—-4.5½° at 2:00 A.M.; -5.17½° at 6:00 A.M.; +.03½° at 10:00 A.M.;
+6.39½° at 2:00 P.M.; +5.2½° at 6:00 P.M.; +2.7½° at 10:00 P.M. 42—+.77½°.
43-21. 44—\$2.14½. 45-50—May, 525½ lb.; June, 575½ lb.; July, 524½ lb.;
Aug., 502½ lb.; Sept., 462½ lb.; Oct., 432½ lb. 51-57—Beef, 921½ lb.; Pork,
435½ lb.; Veal, 501½ lb.; Mutton, 474½ lb.; Lamb, 438½ lb.; Fish, 358½ lb.;
Poultry, 401½ lb. 58-508½ lb. 59—1 ft. 10½ in. 60—29 ft. 4½ in.
- 171 61-65—4½ da. for each of 5 hands; 4½ da. for each of 7 hands; 5½ da. for
each of 12 hands; 7½ da. for each of 32 hands; 6½ da. for each of all the men.
66-8½ hr. 67—\$52.39½. 68—\$.58½. 69—386½ bar. 70—\$2,103.30.
71—\$5.44½. 72—S. 34° 58' 45" E. 73—47° 22' 20½" N. Lat.
- 173 1—\$34,987.87½. 2—\$4.81½. 3—48. 4—82.75 C. ft. 5—214.62 squares.
6—525 doz. 7—3,854.125 M. 8—8.1925 T. 9—220½ gro. 10—57½ grt. gro.
11—\$88.81½. 12—\$.25½. 13—\$2.078½. 14—\$31.98½. 15—\$422.28.
16—\$11.86½. 17—\$127.42½. 18—\$586.61½. 19—\$2.50½. 20—\$564.30.
21—\$13.28½. 22—\$123.75.
- 174 23—\$11.53. 24—\$18.75. 25—\$3. 26—\$.84. 27—\$22.50. 28—\$18.50.
29—\$33. 30—\$21½. 31—6,506½ bu. 32—224½ gal. 33—7.53 C.
34—69.4 C. 35—8.596 M. 36—32.75 M. 37—69 T. 38—17½ T.
39—\$16,818.75. 40—\$18,837. 41—76.16 M. ft. 42—24 doz. 43—524½ lb.
44—\$.43½, price per pr. 45—34.65 M. bricks.
- 175 46—\$27.50 per M. 47—\$5.90½. 48—\$92.81½. 49—\$13.50. 50—\$11.50.
51—\$1.989. 52—750. 53—\$.08½. 54—\$26. 55—\$22. 56—2,985.
57—\$10,986.30. 58—\$12,808+. 59—2,448. 60—\$268.80. 61—\$55.978+.
62—9.164½ T. 63—2,940 lb.

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 177 $1-\frac{1}{4}$ of cost. $2-1\frac{1}{4}$ times cost. $3-\frac{1}{4}$ of cost. $4-2\frac{1}{4}$ times cost. $5-\frac{1}{4}$ of cost. $6-2\frac{1}{4}$ times cost. $7-\frac{1}{4}$. $8-6$ times. $9-\frac{1}{4}$. $10-6\frac{1}{4}$ times. $11-\frac{1}{4}$. $12-\frac{1}{10000}$. $13-4\frac{1}{4}$ times. $14-491\frac{1}{4}$ times. $15-17\frac{1}{4}$ times. $16-\frac{1}{4}$. $17-\frac{1}{4}$. $18-7\frac{1}{4}$ times. $19-\frac{1}{4}$. $20-2\frac{1}{4}$ times. $21-\frac{1}{4}$. $22-3\frac{1}{4}$ times. $23-\frac{1}{4}$. $24-4\frac{1}{4}$ times. $25-2\frac{1}{4}$. $26-\frac{1}{4}$. $27-\frac{1}{4}$. $28-2\frac{1}{4}$. $29-320$ times. $30-50$ times. $31-6\frac{1}{4}$. $32-2\frac{1}{4}$. $33-1\frac{1}{4}$. $34-\frac{1}{4}$. $35-\frac{1}{4}$. $36-\frac{1}{1000}$. $37-9\frac{1}{4}$ times. $38, 39-\frac{1}{4}$, spends; $\frac{1}{4}$, saves. $40, 41-\frac{1}{4}$, selling price; $\frac{1}{4}$, loss.
- 178 $42-\frac{1}{4}$. $43, 44-\frac{1}{4}$, used; $\frac{1}{4}$, left. $45-\frac{1}{4}$. $46-2\frac{1}{4}$. $47-\frac{1}{4}$. $48-\frac{1}{4}$. $49-15$. $50-A, \frac{1}{4}$; B, $\frac{1}{4}$. $51-\frac{1}{4}$. $52-54$ -quantity of wheat is $\frac{1}{4}$ of corn; price of wheat is $\frac{1}{4}$ of price of corn; value of the wheat is $\frac{1}{4}$ of the value of the corn. $55-\frac{1}{10000}$, Oldest; $\frac{1}{10000}$, Second; $\frac{1}{10000}$, Youngest. $56-\frac{1}{10000}$. $57-\frac{1}{4}$. $58-\frac{1}{4}$, First; $\frac{1}{4}$, Second. $59-\frac{1}{4}$, First; $\frac{1}{4}$, Second; $\frac{1}{4}$, Third; $\frac{1}{4}$, Fourth.
- 181 $1-912$. $2-8,420$. $3-1,551$. $4-2,823\frac{1}{4}$. $5-2,460.29$. $6-765\frac{1}{4}$. $7-83$ yd. 1 ft. 3.3 in. $8-900$. $9-\$4,954.40$. $10-1,298$ gal. 2 qt. $11-1.22$. $12-\$2.75$. $13-\$3.41$, First; $\$4.09$, Second. $14-35\text{¢}$. $15-86\frac{1}{4}$ lb.; $113\frac{1}{4}$ lb.
- 182 $16-75\frac{1}{4}$ da. $17-166\frac{1}{4}$ ft. $18-\$144.87\frac{1}{4}$, E; $\$99.12\frac{1}{4}$, F; $\$76.25$, G. $19-11\frac{1}{4}$ bar. $20-\$9,600$, A's; $\$7,200$, B's; $\$12,000$, C's. $21-\$366$. $22-12$ boxes. $23-\$1,117.40$, or $\$1,124.80$ (if a leap-year). $24-\frac{1}{4}$, First; $\frac{1}{4}$, Second. $25-M$'s age, $4\frac{1}{4}$ yr.; N's, $73\frac{1}{4}$. $26-24$. $27-1\text{st}, 21\frac{1}{4}$; $2\text{d}, 101\frac{1}{4}$.
- 183 $28-1\text{st}, 31$; $2\text{d}, 215$. $29-6\frac{1}{4}$ da. $30-42$ persons. $31-1\text{st}, 74$ yd.; $2\text{d}, 49\frac{1}{4}$ yd.; $3\text{d}, 61\frac{1}{4}$ yd. $32-27$ min. $52\frac{1}{4}$ sec. past 2 o'clock P.M. $33-208\frac{1}{4}$ mi. $34-1\text{st}$ received $105\frac{1}{4}$ lb.; $2\text{d}, 93\frac{1}{4}$ lb.; $3\text{d}, 71\frac{1}{4}$ lb. $35-528$ yda. $36-\frac{1}{4}$. $37-1\text{st}$ part, $634\frac{1}{4}$; $2\text{d}, 1,808\frac{1}{4}$; $3\text{d}, 2,617\frac{1}{4}$. $38-R, \frac{1}{4}$; S, $\frac{1}{4}$; and T, $\frac{1}{4}$.
- 184 $39-R, \$246.43$; S, $\$184.82$; T, $\$481.25$. $40-\$10,875$. $41-\$14,500$. $42-6\frac{1}{4}$ gal. $43-\$1.87\frac{1}{4}$, green wood; $\$2.37\frac{1}{4}$, dry wood. $44-350$ trees. $45, 46-D$ earned $\$10.32$; E, $\$51.80$; D earned $\$.60$ per day; E, $\$.3$. $47-W$'s age, 45 yr.; Y's age, 54 yr. $48-\$2,150$. $49-2\frac{1}{4}$ da. $50-914\frac{1}{4}$ A. $51-Third$.
- 188 $1-16,807$. $2-5,832$. $3-78,125$. $4-4,294,967,296$. $5-42,875$. $6-7 \times 7 \times 7 \times 7 \times 7, (7 \times 7) \times (7 \times 7) \times 7, (7 \times 7) \times (7 \times 7 \times 7), (7 \times 7 \times 7 \times 7) \times 7$. $7-4,826,809$. $8-2,401$. $9-28,629,151$. $10-2,460,375$. $11-110,075,814,176$. $12-143.48907$. $13-.0009765625$. $14-214,358,881$. $15-42.94967296$. $16-\frac{1}{10000}$. $17-604\frac{1}{4}$. $18-\frac{1}{100000000}$.
- 193 $19-169$. $20-4,096$. $21-3,249$. $22-8,464$. $23-4,913$. $24-97,336$. $25-29,791$. $26-421,875$. $27-42.25$. $28-12,167$. $29-7396$. $30-110592$. $31-12,996$. $32-10.6929$. $33-453.69$. $34-.020164$. $35-158,990.656$. $36-68,417,929$.
- 195 $1-70$. $2-72$. $3-36$. $4-34$. $5-43$. $6-27$. $7-14.4$. $8-2.8$. $9-78$. $10-25.5$. $11-\frac{1}{4}$. $12-6\frac{1}{4}$. $13-63$. $14-105$. $15-42$. $16-3.5$. $17-35$. $18-60$. $19-63$. $20-21$. $21-\frac{1}{4}$. $22-3\frac{1}{4}$. $23-\frac{1}{4}$. $24-13\frac{1}{4}$.

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- 197 25—2 places, 2. 26—2, 8. 27—8, 8. 28—8, 8. 29—4, 2. 30—4, 6.
 31—2, 1. 32—2, 4. 33—3, 1. 34—4, 1. 35—2, 1. 36—1, 6. 37—5, 1.
 38—4 integral, 2 decimal, 2. 39—3 integral, 2 decimal, 2. 40—2 integral, 2
 decimal, 1. 41—1 integral, 3 decimal, 3.
- 201 2—73. 3—453. 4—2,504. 5—2,797.17+. 6—.0628. 7—.6821+. 8—13.31.
 9—3.153+. 10—6.59+. 11— $\frac{1}{11}$. 12—1.41+. 13—22.62+. 14—575.
 15—.954. 16—67.2. 17—1.778+. 18— $\frac{1}{11}$. 19—3 $\frac{1}{11}$. 20—9.81+. 21—18.105+.
- 202 22—74. 23—6.71. 24—.745. 25— $\frac{1}{11}$. 26—30.886+. 27—29.308+. 28—88.
 29—18.1. 30—.774+. 31—31.622+. 32—.954+. 33—2.857+. 34—1.782+.
- 206 1—14. 2—87. 3—.98. 4—18.1. 5—741. 6—979. 7—7.05. 8—1,001.
 9—2,805. 10— $\frac{1}{11}$. 11— $\frac{1}{11}$. 12—.005. 13—1,844.6. 14—388.008+. 15—6.03.
- 208 16—3,725. 17—44.67+. 18—99. 19—22.2. 20—.0271+. 21—63.57.
 22—1.259+. 23—1.442+. 24—1.709+. 25—1.912+. 26—.306.
 27—95.506+. 28—45.877+. 29—77.250+. 30—.691+. 31—20.588+. 1—531,441.
 2—152,587,890,625. 3—16,777,216. 4—.00000000000000000531441.
 5— $\frac{1}{11}$. 6—656 $\frac{1}{11}$. 7—.064. 8—.6561. 9—1,728. 10—1.728.
 11—.001728. 12—.00001728. 13—17. 14—14th power.
- 209 15—6th power. 16—3d power. 17—4,750.104241. 18—15th power.
 19—5th power. 20—28 rd. 4 ft. 8.232+ in. 21—23 ft. 2.76+ in., square.
 22—127.99+ in., cube. 23—812 rd. 10.65+ ft., square. 24—260.81+ ft., cube.
 25—45 rd. 10+ ft. 26—13 rd. 10.7+ ft. 27—52 rd. 15 ft. 2+ in. 28—48 rd.
 29—61.5¢, or \$6.15. 30—1st, 6.13+ in.; 2d, 12.9+ in. 31—8 ft. 3 in.
 32—20 ft. 9.4+ in. 33—7,042.95+ bu. 34—195. 35—32 ft. 9.8+ in.
- 210 36—356 bar. 37—42 sq. ft. 36 sq. in. 38—2,500. 39—660 cabbages.
 40—23 in. 41—236 rows. 42—49 ft. 7+ in., square. 43—87. 44—\$237.50.
 45—15 ch. 92+ l. 46—48—45 pieces; 2,020 yd.; 45¢. 49—3,200 rd.
- 211 50—12.23+ ft.
- 213 1—3. 2—9. 3—6. 4—9. 5—82. 6—18. 7— $\frac{1}{11}$. 8— $\frac{1}{11}$. 9— $\frac{1}{11}$. 10— $\frac{1}{11}$.
 11— $\frac{1}{11}$. 12— $\frac{1}{11}$. 13— $\frac{1}{11}$. 14—12. 15—8. 16—2.4. 17— $\frac{1}{11}$. 18—4 $\frac{1}{11}$.
- 214 19—1:3. 20—8:1. 21—1:8. 22—6:1. 23—50:1. 24—7:50.
 25—218:242. 26—212:849. 27—767:700. 28—27:108. 29— $\frac{1}{11}$. 30—1 $\frac{1}{11}$.
 31—8 $\frac{1}{11}$. 32—1 $\frac{1}{11}$. 33— $\frac{1}{11}$. 34— $\frac{1}{11}$. 35— $\frac{1}{11}$. 36— $\frac{1}{11}$. 37—65. 38—1 $\frac{1}{11}$.
 39— $\frac{1}{11}$. 40—58 $\frac{1}{11}$. 41—4. 42— $\frac{1}{11}$. 43—50. 44— $\frac{1}{11}$. 45—2,016.
 46—11 $\frac{1}{11}$. 47—7 $\frac{1}{11}$. 48—4. 49— $\frac{1}{11}$. 50—17 $\frac{1}{11}$. 51— $\frac{1}{11}$. 52—11 $\frac{1}{11}$.
 53—4:27. 54—3:5. 55—27:49. 56—8:147. 57—36:7. 58—1:56.
 59—300:7. 60—1,701:512. 61—27:80. 62—1:1. 63—9:7. 64—3,808:55.
 65—2 $\frac{1}{11}$. 66—1 $\frac{1}{11}$. 67— $\frac{1}{11}$. 68—2 $\frac{1}{11}$. 69—1 $\frac{1}{11}$. 70—3. 71— $\frac{1}{11}$. 72— $\frac{1}{11}$.
 73— $\frac{1}{11}$. 74—60 $\frac{1}{11}$. 75— $\frac{1}{11}$. 76—96.

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 215 77-65. 78-187. 79-182. 80-9,875. 81-8. 82-18. 83-.018.
 84-4,189 $\frac{1}{2}$.
- 218 1-45. 2-3. 3-255. 4-39. 5- $\frac{1}{2}$. 6- $\frac{1}{2}$. 7-35 bu. 8-5. 9-1 gal.
 10-259. 11-68. 12-4. 13-21 lb. 14-2 $\frac{1}{2}$. 15-14. 16-1 $\frac{1}{2}$ bu.
 17-4,760 $\frac{1}{2}$. 18-26,203 $\frac{1}{2}$. 19-355 oz. 20-2,000 gr. 21-12 $\frac{1}{2}$.
 22-4 yr. 8 mo. 25 da. 23-\$100.09+.
- 219 24-6. 25-12. 26-6. 27-8. 28- $\frac{1}{2}$. 29-1 $\frac{1}{2}$. 30-1.05. 31-3.8.
 32-1.125. 33-39.831+. 34-32. 35-575. 36-931. 37-704. 38-35.
 39-63. 40-21 $\frac{1}{2}$. 41-12 $\frac{1}{2}$.
- 220 42-40 rods. 43-55 ft.
- 222 44-8 $\frac{1}{10}$ mi. 45-\$1,531 $\frac{1}{2}$. 46-4,050 bu. 47-\$37.75. 48-23 da.
 49-53 ft. 50-\$1,620. 51-\$6.664. 52-72 times. 53-390 bar.
- 223 54-24,897 $\frac{1}{2}$ mi. 55-33 sec. 56-9 min. 1 $\frac{1}{2}$ sec. 57-\$45,000.
 58-\$1,262.85 $\frac{1}{2}$. 59-6 da. 60-40 da. 61-5 $\frac{1}{2}$ mo. 62-26 T. 63-\$5.92.
 64-\$172.80. 65-30 $\frac{1}{10}$ mi. 66-\$24.66 $\frac{1}{2}$. 67- $\frac{1}{10}$ ft.
- 224 68-180. 69-1,296. 70-20. 71-7 $\frac{1}{2}$. 72-1,440. 73-1 $\frac{1}{2}$. 74-288.
 75-33 $\frac{1}{2}$. 76-33 $\frac{1}{10}$. 77-207 $\frac{1}{2}$. 78- $\frac{1}{10}$. 79-.263 $\frac{1}{2}$.
- 226 80-11 $\frac{1}{10}$ ft. 81-128 $\frac{1}{2}$ suits. 82-12 da. 83-68 $\frac{1}{10}$ bu. 84-93 $\frac{1}{2}$ da.
 85-64 men. 86-325,000 gal. 87-2,160 lb. 88-60. 89-\$154.61 $\frac{1}{10}$.
 90-2 $\frac{1}{10}$ da.
- 227 91-\$42.4515. 92-1,058 $\frac{1}{10}$ rm. 93-1 da. 1 hr. 8 min. 49 $\frac{1}{10}$ sec. 94-30 da.
- 232 95-5 and 25. 96-40 and 90. 97-32, 64, and 176. 98-201 $\frac{1}{2}$ and 161 $\frac{1}{2}$.
 99-380, 456, 475, and 798. 100-14, 70, and 490. 101-643 $\frac{1}{2}$, 160 $\frac{1}{2}$, 96 $\frac{1}{2}$,
 and 64 $\frac{1}{2}$. 102-\$2,800, B's share; \$400, C's share. 103-\$360, D's share;
 \$240, E's share. 104-\$540, G's share; \$640, H's share. 105-2,276 $\frac{1}{10}$,
 2,439 $\frac{1}{10}$, and 3,984 $\frac{1}{10}$. 106-1,800, 4,400, and 5,700. 107-104,517 cu. ft.
 oxygen; 393,183 cu. ft. nitrogen. 108-267 $\frac{1}{2}$ lb. oxygen; 83 $\frac{1}{2}$ lb. hydrogen.
 109-219 lb. copper, 54 lb. tin, 18 lb. zinc, 36 lb. lead, and 9 lb. iron. 110-4 $\frac{1}{2}$
 pwt. copper, 9 pwt. silver, and 40 $\frac{1}{2}$ pwt. gold.
- 233 111-\$4,204.06+, each son's share; \$2,522.45-, each daughter's share;
 \$1,261.22+, each grandson's share; \$340.82-, each grand-daughter's share.
 112-\$8,843.77 $\frac{1}{10}$, asylum; \$4,258.11 $\frac{1}{2}$, school; \$6,550.94 $\frac{1}{2}$, library.
 113-\$2,250, D's share; \$3,000, E's; \$3,750, F's. 114-\$2,117.65, Mr.
 Brown; \$1,058.82, Mr. White; \$2,823.53, Mr. Morris. 115-\$1,500, Mr.
 Day's share of the gain; \$1,200, Mr. Ray's share. 116-\$638.98, J's share;
 \$1,006.78, K's; \$1,454.24, L's.
- 237 1-28. 2-35. 3-222.
- 238 4-29. 5-16. 6-13. 7-180. 8-12. 9-78. 10-4 mi. 110 rd. 5 yd.
 11-10 mi. 312 rd. 4 yd.

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- 243 12-768. 13-128. 14-1,020. 15- $\frac{1}{2}$. 16-3. 17-484. 18- $\$10.23\frac{1}{2}$.
19-126, sum. 20-364, sum. 21-9, 27, 81, 243, 729. 22- $\$3,579,139.41$.
23-768, 2,304.
- 244 24-2. 25- $\frac{1}{2}$. 26-82 $\frac{1}{2}$. 27-80 ft. 28-2 $\frac{1}{2}$ hr.
- 246 1-50%. 2-60%. 3-66 $\frac{1}{2}$ %. 4-37 $\frac{1}{2}$ %. 5-25%. 6-28 $\frac{1}{2}$ %. 7-16 $\frac{1}{2}$ %.
8-13 $\frac{1}{2}$ %. 9-3 $\frac{1}{2}$ %. 10-12%. 11-41 $\frac{1}{2}$ %. 12-35%. 13-62 $\frac{1}{2}$ %. 14-17 $\frac{1}{2}$ %.
15-14 $\frac{1}{2}$ %. 16-234 $\frac{1}{2}$ %. 17-43 $\frac{1}{2}$ %. 18-1 $\frac{1}{2}$ %. 19-44%. 20-212 $\frac{1}{2}$ %.
21-72 $\frac{1}{2}$ %. 22-185%. 23-20 $\frac{1}{2}$ %. 24-23 $\frac{1}{2}$ %. 25- $\frac{1}{2}$. 26- $\frac{1}{2}$. 27- $\frac{1}{2}$.
28- $\frac{1}{2}$. 29-1 $\frac{1}{2}$. 30-1 $\frac{1}{2}$. 31- $\frac{1}{2}$. 32- $\frac{1}{2}$. 33- $\frac{1}{2}$. 34- $\frac{1}{2}$.
35- $\frac{1}{2}$. 36- $\frac{1}{2}$. 37- $\frac{1}{2}$. 38- $\frac{1}{2}$.
- 247 1-4-75; 1,190.16; 28.71; .03875. 5-8-187.5; 2,975.4; 71.775; .084375.
9-12-50; 793.44; 19.14; .0225. 13-16-90; 1,428.192; 84.452; .0405.
17-20-935.5 T.; 239.1875 A.; 2,060 mi.; $\$167.57$. 21-24-93.55 T.;
23.91875 A.; 206 mi.; $\$16.757$. 25-28-623 $\frac{1}{2}$ T.; 159 $\frac{1}{2}$ A.; 1373 $\frac{1}{2}$ mi.;
 $\$111.71\frac{1}{2}$. 29-32-28.065 T.; 7.175625 A.; 61.8 mi.; $\$5.0271$. 33-42 pas-
sengers in the drawing-room car; 56 passengers in the day coach; 34 passengers
in the smoking car. 34-26% rye; 2,909.5 bu. barley; 4,301 bu. wheat; 2150.5
bu. oats; 3,289 bu. rye. 35- $\$427.50$.
- 248 36-39-240 bu.; 800 yd.; 780 lb.; 16.50. 40-43-300 bu.; 375 yd.; 912 $\frac{1}{2}$ lb.;
30.62 $\frac{1}{2}$. 44-47-900 bu.; 1,125 yd.; 2787.5 lb.; 61.875. 48-51-1,080 bu.;
1,350 yd.; 3,285 lb.; 74.25. 52-55-91.9125 bar.; 1,871.975 mi.; 2,161.25
T.; 16,688.9 gal. 56-59-42.57 bar.; 867.02 mi.; 1,001 T.; 7,727.28 gal.
60-63-106.425 bar.; 2,167.55 mi.; 2,502.5 T.; 19,318.2 gal. 64-67-120.9375
bar.; 2,463.125 mi.; 2,843.75 T.; 21,952.5 gal. 68-71-49,500 A.; 1,910.4 da.;
37,680; 83.736. 72-75-8,250 A.; 318.4 da.; 6,280; 13.956. 76-79-1,443.76
A.; 55.72 da.; 1,099; 3.4423. 80-83-1,856.25 A.; 71.64 da.; 1,413; 3.1401.
84-87-.09417375; 2,970; 85,775; $\$36.0495$. 88-91-.8189125; 9,900; 119,250;
 $\$120.165$. 92-95-.05231875; 1,650; 19,875; $\$20.0275$. 96-99-.1395 $\frac{1}{2}$; 4,400;
53,000; $\$53.40\frac{1}{2}$. 100-103- $\$312.609875$; $\$396.1958125$; $\$24,164.6640625$;
 $\$294.8859375$. 104-107- $\$721.40625$; $\$914.296875$; $\$55,764.609375$; $\$680.390625$.
108-111- $\$7.695$; $\$9.7525$; $\$594.8225$; $\$7.2575$. 112-115- $\$3.8475$;
 $\$4.87625$; $\$297.41125$; $\$3.62875$. 116-119- $\$26.4415625$; $\$20.62075$;
 $\$149.69185$; $\$268.67421875$. 120-123- $\$92.54546875$; $\$72.172625$; $\$523.919725$;
 $\$940.359765625$. 124-127- $\$63.45975$; $\$49.4898$; $\$359.25924$; $\$644.818125$.
128-131- $\$12,713.10325$; $\$9,914.4566$; $\$71,971.60108$; $\$129,178.564875$.
- 249 132-135-20%; 12 $\frac{1}{2}$ %; 4 $\frac{1}{2}$ %; 6 $\frac{1}{2}$ %. 136-139-40%; 25%; 9 $\frac{1}{2}$ %; 12 $\frac{1}{2}$ %.
140-143-70%; 43 $\frac{1}{2}$ %; 16 $\frac{1}{2}$ %; 22 $\frac{1}{2}$ %. 144-147-103 $\frac{1}{2}$ %; 64 $\frac{1}{2}$ %; 24 $\frac{1}{2}$ %; 33 $\frac{1}{2}$ %.
148-151-200%; 39 $\frac{1}{2}$ %; 58 $\frac{1}{2}$ %; 8 $\frac{1}{2}$ %. 152-155-132 $\frac{1}{2}$ %; 26 $\frac{1}{2}$ %; 39 $\frac{1}{2}$ %;
5 $\frac{1}{2}$ %. 156-159-91 $\frac{1}{2}$ %; 18 $\frac{1}{2}$ %; 27 $\frac{1}{2}$ %; 4 $\frac{1}{2}$ %. 160-163-3187 $\frac{1}{2}$ %; 630%;
937 $\frac{1}{2}$ %; 143 $\frac{1}{2}$ %. 164-14 $\frac{1}{2}$ %. 165-5%. 166-21 $\frac{1}{2}$ %. 167-2 $\frac{1}{2}$ %.
168-4%. 169-A's standing, 76 $\frac{1}{2}$ %; B's, 71 $\frac{1}{2}$ %; C's, 52 $\frac{1}{2}$ %; D's, 70 $\frac{1}{2}$ %;
E's, 94 $\frac{1}{2}$ %; F's, 84 $\frac{1}{2}$ %. 170-75%. 171-40%. 172-65%. 173-48%.
174-263 $\frac{1}{2}$ %. 175-860%. 176-27 $\frac{1}{2}$ %.

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250 177—2,640. 178—1,584. 179—1,320. 180—792. 181—3,168.
182-185—504; 86 $\frac{1}{2}$; 16,800; 5,625. 186-189—10,520; 1,801 $\frac{1}{2}$; 350,666 $\frac{1}{2}$;
117,410 $\frac{1}{2}$. 190-193—400.82; 68 $\frac{1}{2}$; 13,344; 4,467 $\frac{1}{2}$. 194-197—7 $\frac{1}{2}$; 1 $\frac{1}{2}$ $\frac{1}{2}$;
245 $\frac{1}{2}$; 82 $\frac{1}{2}$. 198—\$1,856 $\frac{1}{2}$. 199—1,766 $\frac{1}{2}$ lb. 200—20 $\frac{1}{2}$ bu. 201—238 $\frac{1}{2}$ s.
202—1041 $\frac{1}{2}$ bu. 203—\$8,000. 204—\$6,400. 205, 206—109.8; 70.2.
207, 208—145,803.125; 33,646.875. 209, 210—.51 $\frac{1}{2}$ $\frac{1}{2}$; .50 $\frac{1}{2}$ $\frac{1}{2}$.
211, 212—\$95.8381 $\frac{1}{2}$; \$84.1218 $\frac{1}{2}$. 213, 214—2 gal. 2 qt. 1 pt. 2.4 gi.;
1 gal. 1 qt. 1.6 gi.
- 251 215, 216—95.92; 80.08. 217, 218—12 $\frac{1}{2}$; 6 $\frac{1}{2}$. 219, 220—\$19.4264 $\frac{1}{2}$;
\$19.0735 $\frac{1}{2}$. 221—\$10,088 $\frac{1}{2}$. 222—473. 223—\$16,146. 224—11,454.
225-227—8,920; 21,155 $\frac{1}{2}$ $\frac{1}{2}$; 18,599 $\frac{1}{2}$. 228-230—\$1,518.06 $\frac{1}{2}$;
\$3,600.43 $\frac{1}{2}$ $\frac{1}{2}$; \$3,185.81 $\frac{1}{2}$. 231-233—263 bu. 2 pk. 2 qt. 1 $\frac{1}{2}$ pt.; 625 bu.
5 qt. $\frac{1}{2}$ $\frac{1}{2}$ pt.; 549 bu. 2 pk. 3 qt. 1 $\frac{1}{2}$ pt. 234-236—201 $\frac{1}{2}$; 156 $\frac{1}{2}$; 137 $\frac{1}{2}$ $\frac{1}{2}$.
237-239—3,183 $\frac{1}{2}$ lb.; 2,474 $\frac{1}{2}$ lb.; 2,178 $\frac{1}{2}$ lb. 240-242—£56 18s. 2d. 3 $\frac{1}{2}$ far.;
£44 4s. 6d. 3 $\frac{1}{2}$ far.; £38 18s. 8d. $\frac{1}{2}$ $\frac{1}{2}$ far. 243-4,048 qt. 244—9,500 men.
245—3,840.
- 252 246—\$25, cost of chair; \$100, cost of desk. 247—\$22,800.
248-250—45 $\frac{1}{2}$ $\frac{1}{2}$; 4 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$; 678 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$. 251-253—91 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$; 884 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$; 928 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$.
254-256—79 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$; 29 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$; 864 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$. 257-259—84 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$; 21 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$;
55 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$. 260-262—39 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$; 26 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$; 58 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$. 263-265—19 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$;
2 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$; 45 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$. 266—3 $\frac{1}{2}$. 267—20%. 268—16 $\frac{1}{2}$. 269—25%.
- 255 1—\$157.75. 2—4 $\frac{1}{2}$. 3—\$11,014.28 $\frac{1}{2}$. 4—\$1,750. 5—4 $\frac{1}{2}$. 6—\$2,765.41.
7—\$991.01. 8—7 $\frac{1}{2}$. 9—\$42,260.94. 10—\$25.50. 11—\$59.50. 12—40
sheep. 13—\$25, commission. 14—\$7.50. 15—\$150, collected.
- 256 16—\$.625. 17—20%. 18—\$5.68 $\frac{1}{2}$. 19—\$14.01. 20—2 $\frac{1}{2}$. 21—\$6,875.
22—\$.15 $\frac{1}{2}$. 23—16 $\frac{1}{2}$. 24—\$19,507.50. 25—2%. 26—2%. 27—2 $\frac{1}{2}$.
28—2 $\frac{1}{2}$ $\frac{1}{2}$. 29—\$371.40. 30—\$119.70. 31—\$84. 32—\$6.80.
- 257 33—\$.05. 34—15%. 35—\$67.50. 36—\$.08. 37—66 $\frac{1}{2}$ %, gain.
38—\$10.875. 39—\$.87 $\frac{1}{2}$. 40—20%, loss. 41—\$2,148.43 $\frac{1}{2}$. 42—150%.
43—\$26.07, cost. 44—47 $\frac{1}{2}$ $\frac{1}{2}$ more on the oranges. 45—\$25. 46—\$150.
47—14 $\frac{1}{2}$ %.
- 258 48—\$65, premium; \$3,185, sum covered. 49—1 $\frac{1}{2}$ %; \$4,814.06 $\frac{1}{2}$, sum covered.
50—\$12,500, valuation; \$46.87 $\frac{1}{2}$, premium. 51—\$14,758.75, valuation; 1 $\frac{1}{2}$ $\frac{1}{2}$ %.
52—\$17,850, valuation; \$17,693.81 $\frac{1}{2}$, sum covered. 53— $\frac{1}{2}$ %; \$412.50, premium.
54—\$136.76. 55—\$11,581.68. 56—\$6,577.78. 57—1 $\frac{1}{2}$ %.
- 260 58—\$3,376. 59—\$6,885. 60—10 $\frac{1}{2}$ $\frac{1}{2}$ %. 61—8 $\frac{1}{2}$ $\frac{1}{2}$ %. 62—120. 63—58 $\frac{1}{2}$ $\frac{1}{2}$.
64—\$1,620. 65—\$18,745. 66—\$7,748.44. 67—\$50,406.25. 68—\$72,638.
69—\$137,531.25. 70—3 $\frac{1}{2}$ $\frac{1}{2}$ %. 71—70. 72—6%. 73—11 $\frac{1}{2}$ $\frac{1}{2}$ %. 74—128 $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$.
75—6%. 76—\$1,953. 77—\$2,125.50. 78—\$5,700. 79—Mich. Cent. 6's, $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ %.
80—112 $\frac{1}{2}$ % of the par value. 81—69 $\frac{1}{2}$ %. 82—\$13,840. 83—39. 84—7 $\frac{1}{2}$ $\frac{1}{2}$ %.
85—\$278.44. 86—6 $\frac{1}{2}$ %. 87—84 $\frac{1}{2}$ % of the par value.

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- 263 88—\$1,140.16. 89—5 $\frac{1}{11}$ %. 90—\$1,261,025.78. 91—8 $\frac{1}{11}$ %%.
- 264 92—\$4.46. 93—\$1.29. 94—\$41.78. 95—\$15.55. 96—\$.60. 97—\$1.06.
 98—\$52.41. 99—\$11.56. 100—\$9.01. 101—\$63.46. 102—Tax: on
 \$12,272,—\$37.18; on \$925,—\$2.81; on \$6,130,—\$18.57; on \$17,780,—\$58.85;
 on \$480,—\$1.45; on \$19,050,—\$57.70; on \$7,800,—\$23.62; on \$5,500,—\$16.66;
 on \$26,840,—\$81.80; on \$51,700,—\$156.61; on \$2,500,—\$7.57; on \$4,875,—
 \$18.26; on \$121,900,—\$369.27; on \$64,250,—\$194.63; on \$637,—\$1.93; on
 \$288,—\$.87; on \$18,620,—\$56.40; on \$82,800,—\$249.31; on \$14,440,—\$43.74;
 on \$1,970,—\$5.97; on \$8,300,—\$10; on \$2,160,—\$6.54; on \$11,045,—\$33.46;
 on \$520,—\$1.57; on \$7,200,—\$21.81; on \$110,480,—\$334.67; on \$4,175,—
 \$12.65; on \$25,800,—\$78.15; on \$8,330,—\$25.23; on \$72,650,—\$220.07; on
 \$9,875,—\$29.91; on \$4,370,—\$18.24. 103—Tax: on \$12,272,—\$26.11; on
 \$925,—\$1.96; on \$6,130,—\$18.08; on \$17,780,—\$37.82; on \$480,—\$1.02; on
 \$19,050,—\$40.62; on \$7,800,—\$16.59; on \$5,500,—\$11.70; on \$26,840,—\$57.09;
 on \$51,700,—\$109.98; on \$2,500,—\$5.31; on \$4,875,—\$9.31; on \$121,900,—
 \$259.31; on \$64,250,—\$136.68; on \$637,—\$1.35; on \$288,—\$.62; on \$18,620,—
 \$39.61; on \$82,800,—\$175.07; on \$14,440,—\$30.72; on \$1,970,—\$4.19;
 on \$8,300,—\$7.02; on \$2,160,—\$4.59; on \$11,045,—\$23.50; on \$520,—\$1.10;
 on \$7,200,—\$15.32; on \$110,480,—\$235.01; on \$4,175,—\$8.88; on \$25,800,—
 \$54.88; on \$8,330,—\$17.72; on \$72,650,—\$154.54; on \$9,875,—\$21; on
 \$4,870,—\$9.30.
- 266 104—2,080 mi. by rail; 620.1 mi. by steamer; 549.9 mi. by stage. 105—\$7,805.
 106—2 $\frac{1}{11}$ %. 107—5¢. 108—\$38.42 $\frac{1}{2}$. 109—\$195. 110—\$33.47.
 111—\$565.17. 112—\$69.52. 113—4 $\frac{1}{11}$ %. 114—360 acres. 115—\$2,605.57.
 116—19%. 117—14 $\frac{1}{2}$ %. 118—\$17,463.66. 119—5,115 $\frac{1}{11}$ yd. 120—\$87,200.
 121—\$5,200. 122—\$2,400.
- 267 123—\$185.24. 124—\$1,478.60. 125—\$252.28. 126—\$51.40.
 127—\$382.05. 128—\$354.29. 129—71 $\frac{1}{2}$. 130—3 $\frac{1}{11}$ %. 131—2 $\frac{1}{11}$ %.
 132—5 $\frac{1}{11}$ %. 133—25%. 134—10 $\frac{1}{11}$ %, loss. 135—70% of the par value.
 136—15 $\frac{1}{11}$ %. 137—\$74,508.94.
- 268 138—\$.345. 139—\$.196. 140—\$.323. 141—\$.07 $\frac{1}{2}$. 142—\$.75.
 143—\$.69. 144—73 $\frac{1}{2}$. 145—140%. 146—20% and 2 fives gives 2 $\frac{1}{2}$ %
 greater net price. 147—12% stock is $\frac{1}{11}$ % better. 148—20%. 149—22 $\frac{1}{11}$ %.
 150—\$578.91. 151—\$251.81. 152—\$441.09. 153—\$386.85.
 154—\$11,398.625, net proceeds; \$18,375, brokerage. 155—\$3,200.
 156—\$4,139 $\frac{1}{2}$. 157—\$18.17. 158—\$15. 159—\$2,105.47.
- 269 160—80%. 161—10% stocks, 4 $\frac{1}{11}$ % greater. 162—36 $\frac{1}{11}$ ¢. 163—\$1.40 $\frac{1}{11}$ ¢.
 164—20¢. 165—\$.40. 166—\$2.23 $\frac{1}{11}$ ¢. 167—\$.58 $\frac{1}{11}$ ¢. 168—\$450.
 169—4 $\frac{1}{11}$ %. 170—322.85¢. 171—51 bonds, and \$522 $\frac{1}{2}$ remaining.
 172—\$340.54. 173—\$349.27. 174—\$541.87. 175—10,000 bar.
 176—\$4,978.12 $\frac{1}{2}$. 177—\$1,434.16, made.

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- 270 178 — \$74,955 $\frac{1}{2}$. 179 — 12 $\frac{1}{2}$ % loss. 180 — \$500. 181 — \$1.53 $\frac{1}{2}$. 182 — \$5,116.87 $\frac{1}{2}$. 183 — \$1,836. 184 — \$3,268.12 $\frac{1}{2}$. 185 — \$879.94. 186 — \$48.09. 187 — 59 $\frac{1}{2}$ %. 188 — 41 $\frac{1}{2}$ %. 189 — 43 $\frac{1}{10}$ %. 190 — 53 $\frac{1}{2}$ %. 191 — 55%. 192 — 50 $\frac{1}{2}$ %. 193 — \$201.60, net loss. 194 — \$126.60, less than original income. 195 — \$6.75.
- 271 196 — 1 $\frac{1}{2}$ %. 197 — 337 $\frac{1}{2}$ M. 198 — \$675.50. 199 — \$17,718.75. 200 — \$32,045. 201 — \$27,459.25. 202 — \$98,414.50. 203 — \$82,896.50. 204 — \$25,205.81 $\frac{1}{2}$. 205 — 400. 206 — \$477.40 $\frac{1}{2}$. 207 — \$68.61. 208 — 3 $\frac{1}{10}$ %. 209 — \$37.76, A's tax; \$16.51, B's; \$3.80, C's; \$53.67, Star Shoe Mfg. Co.'s; \$153.87, First National Bank's; \$302.45, I. C. RR's. 210 — \$128.29, less.
- 274 1 — .30. 2 — .35. 3 — .21 $\frac{1}{2}$. 4 — .25 $\frac{1}{2}$. 5 — .18 $\frac{1}{2}$. 6 — .17. 7 — .05 $\frac{1}{2}$. 8 — .049. 9 — 11 — \$58.23; \$20.64; \$1.86. 12 — 14 — \$1,808.375; \$639.29; \$41.98. 15 — 17 — \$491.25; \$174.15; \$11.44. 18 — 20 — \$2.62; \$5.24; \$3.82. 21 — 23 — \$35.74; \$71.48; \$52.18. 24 — 26 — \$7.60; \$15.19; \$11.09. 27 — \$409.68. 28 — \$22,564.27. 29 — \$20.11. 30 — \$7,570.83. 31 — \$196.88. 32 — \$10,222.13. 33 — \$28.28. 34 — \$1,199,481.04. 35 — \$7,104.14. 36 — \$3,180.82.
- 275 37, 38 — .002 $\frac{1}{2}$; .001 $\frac{1}{2}$. 39, 40 — .01 $\frac{1}{2}$; .006 $\frac{1}{2}$. 41, 42 — .02 $\frac{1}{2}$; .03 $\frac{1}{2}$. 43, 44 — .01 $\frac{1}{2}$; .02 $\frac{1}{2}$. 45, 46 — .003 $\frac{1}{2}$; .002 $\frac{1}{2}$. 47, 48 — .004 $\frac{1}{2}$; .003 $\frac{1}{2}$. 49 — 51 — \$1.73; \$2.07; \$2.76. 52 — 54 — \$15.92; \$19.11; \$25.48. 55 — 57 — \$.79; \$.95; \$1.27. 58 — 60 — \$.34; \$.88; \$.12. 61 — 63 — \$10.49; \$25.24; \$3.69. 64 — 66 — \$64.89; \$156.09; \$22.80. 67 — \$3.82. 68 — \$20.01. 69 — \$331.71. 70 — \$128.14. 71 — \$6.34. 72 — \$2.68.
- 276 73 — \$3,812. 74 — \$97.15. 75 — \$471.91. 76 — \$639.51. 77 — \$4,923.12. 78 — \$68.37. 79 — \$722.21. 80 — \$187.92.
- 278 81 — 83 — .3081896; .38423387; .54597967. 84 — 86 — .1701659; .209574979; .29022464. 87 — 89 — .1019 $\frac{1}{2}$; .1247808; .170688329. 90 — \$1,021.03. 91 — \$1,169.78. 92 — \$3,641.07. 93 — \$488.43. 94 — \$355.53. 95 — \$5,644.58. 96 — \$98.72. 97 — \$14.74. 98 — \$79.11. 99 — \$184.59. 100 — \$83,420.74.
- 279 101, 102 — \$56.14, interest; \$336.84, amount. 103, 104 — 2 yr. 6 mo. 17 da.; \$160.47, interest. 105, 106 — \$870.64, principal; 6%. 107, 108 — \$11,149.06, principal; \$1,857.40, interest. 109, 110 — 3%; \$578.24, amount. 111, 112 — 5 $\frac{1}{2}$ %; \$6.15+. 113, 114 — 5 mo.; \$27.757, amount. 115, 116 — \$529.51, principal; \$547.94, amount. 117, 118 — \$21,439.98, principal; 8 mo. 20 da. 119, 120 — 1 yr. 4 mo. 23 da.; \$8.18, interest. 121 — \$424.92. 122 — 2 yr. 3 mo. 7 da. 123 — \$425.14.
- 280 124 — \$669.76. 125 — \$269.56. 126 — 2 yr. 1 mo. 20 da. 127 — \$2,540.81. 128 — 7%. 129 — 5%. 130 — 6.9+%. 131 — 7 mo. 27 da. 132 — 241 da. 133 — 7 mo. 27 da. 134 — 7 mo. 24 da. 135 — 16 $\frac{1}{2}$ yr. 136 — 14 yr. 2 mo. 13 da.
- 287 1 — \$168.24. 3 — \$267.80. 4 — \$2,110.84. 5 — \$921.45.

Page

- 288 6—\$259.75. 7—\$874.18. 8—\$1,816.22. 9—\$150.97. 10—\$361.51.
- 293 11—June 4/7. 12—July 5/8. 13—Sept. 18/21. 14—Sept. 1/4. 15—Feb. 28 or 29/Mar. 8. 16—Sept. 26/29. 17—Dec. 29/Jan. 1. 18—Jan. 25/28. 19—June 14/17. 20—July 1/4, succeeding year. 21—\$16.73. 22—\$3.86. 23—\$6.02. 24—\$997.47. 25—\$219.81. 26—\$395.80.
- 294 27—\$37.44; \$1,508.31. 28—\$38.76; \$806.91. 29—\$387.07; \$18,692.70. 30—\$163.29; \$3,961.96. 31—\$650.93; \$7,007.12. 32—\$607.81; \$22,769.53. 33—\$511.90. 34—\$1,785.51. 35—\$231.77. 36—\$8,432.32. 37—\$3,226.14. 38—\$796.96. 39—\$2,053.04. 40—\$964.54. 41—\$10,072.18. 42—\$1,495.78. 43—\$25,025.03. 44—\$641.08. 45—\$551.71.
- 297 46—\$633.94. 47—\$167.86. 48—\$923.52. 49—\$1,812.71. 50—\$2,806.72. 51—\$3,868.10. 52—\$212.46. 53—\$339.58. 54—\$317.04. 55—\$299.77. 56—\$862.07. 57—\$1,500.51. 58—\$989.58. 59—\$651.79. 60—\$913.14. 61—\$2,120.90. 62—\$3,408.70. 63—\$725.26. 64—\$3,724.65. 65—\$1,005.03. 66—\$2,935.83. 67—\$7,892.58. 68—\$2,975.29. 69—\$2,443.75.
- 299 70—\$1,227.60. 71—\$1,628.18. 72—\$2,678. 73—\$308.10. 74—£418 8s. 8d. 3 + far. 75—2,481.6 francs. 76—3,741 $\frac{1}{11}$ marks. 77—\$3,864.02. 78—\$1,654.64. 79—\$1,203. 80—\$564.59.
- 301 81—\$789.81. 82—\$82.72. 83—\$123.02. 84—\$2.49. 85—\$10.23. 86—\$.95. 87—\$633.56, present worth; \$50.69, discount. 88—\$3,574.79, present worth; \$235.21, true discount. 89—\$648.83, present worth; \$11.17, true discount. 90—\$1,097.70, present worth; \$28.05, true discount.
- 303 1—\$162.16.
- 304 2—\$227.32. 3—\$25.87.
- 306 4—65 da. 5—7 mo. 15 da. 6—Jan. 6 of the next year. 7—Mar. 23.
- 307 8—Feb. 18. 9—Aug. 3. 10—July 15. 11—Sept. 27/30.
- 308 12—Aug. 13. 13—July 19. 14—Sept. 30. 15—Sept. 30.
- 309 16—Jan. 31, 1888.
- 311 17—Aug. 8, 1886.
- 312 18—\$79.46, Aug. 15, 1888. 19—\$298.65, Feb. 24, 1888.
- 313 20—Mar. 17, 1888. 21—\$279.95.
- 315 22—\$191.49. 23—\$219.07.
- 316 24—\$353.96.
- 317 25—\$10,183.62, due July 8, 1896.
- 318 26—\$1,536.93, due Sept. 11, 1886. 27—\$3,110.47, due Dec. 10, 1888.
- 321 1—Brown's share of the capital, \$43,081.14; Smith's, \$60,944.06; Holt's, \$97,720.64; Perry's, \$99,832.16.

- Page
 322 2—M's share of the capital, \$60,897.84; N's, \$64,541.80; P's, \$96,811.96; Q's, \$102,016.90. 3—First partner's share, \$1,818; second, \$1,062.87; third, \$1,199.18. 4—D's share, \$4,602.60; E's, \$6,903.90; F's, \$3,835.50. 5—A's share, \$17,358.80; B's share, \$16,866.86; C's share, \$16,738.84. 6—Brooks, \$2,931.46; Ross, \$2,708.54.
- 323 7—Widow, \$25,768.03; each son, \$15,460.82; each daughter, \$11,043.44. 8—H invested \$27,084.75; I, \$19,915.25; J gained \$3,188.30. 9—\$348.21, each original partner's loss; \$325, first new partner's loss; \$278.57, second new partner's loss. 10—\$75.45, first; \$132.62, second; \$151.93, third. 11—\$18,173.06, A's interest; \$14,457.97, B's; \$14,583.74, C's.
- 324 1—\$6,775.84. 3—\$47,015. 4—2 yr. 6 mo. 5—\$46,833½.
- 325 6—\$1.65. 7—\$590.46. 8—\$5,787.12. 9—1 yr. 1 mo. 15 da. 10—\$412.50. 11—\$1,318.79. 12—1 yr. 2 mo. 6 da. 13—20% 14—\$1,088.54. 15—\$3,787.40. 16—\$876.24. 17—4½%. 18—\$182½. 20-23—\$5.42½, bank discount; \$344.57½, proceeds: \$6.33, bank discount; \$348.67, proceeds: \$8.34, bank discount; \$341.66, proceeds: \$9.78, bank discount; \$340.27, proceeds. 24-27—\$110.48, bank discount; \$7,017.52, proceeds: \$128.90, bank discount; \$6,999.10, proceeds: \$169.88, bank discount; \$6,958.12, proceeds: \$198.20, bank discount; \$6,929.80, proceeds.
- 326 28-31—\$136.74; \$192.90; \$135.63; \$191.36. 32-35—\$59.33; \$83.70; \$58.85; \$83.03. 36—\$224.55. 37—The cash offer, \$23.32 better. 38—\$852.79. 39—\$3.70. 40—4 mo. 28 da. 41—\$512.95. 42—\$2,426.82. 43—\$8,460.76. 44—\$520.16. 45—\$790.20. 46—\$1,014.20. 47—\$5.27, discount; \$334.73, proceeds.
- 327 48—\$647.01. 49—\$1,110.63. 50—\$2,140.89. 51—14 yr. 10 mo. 17 da. 52—\$9.08. 53—\$1.19. 54—\$11.55. 55—\$15.44; \$31.89½; \$.69; \$.28½—total value, \$47.81. 56—\$12.60 favor S. 57—\$789.76.
- 328 58—\$640.89. 59—\$644.97. 60—\$644.97. 61—\$641.96. 62—\$221.18. 63—\$55.62. 64—\$634.09. 65—\$24.73. 66—\$260.71½. 67-71—At,—1%, 100 yr.; 4%, 25 yr.; 6%, 16½ yr.; 7%, 14½ yr.; 3½%, 28½ yr. 72—6 mo. 1 da. 73—22 yr. 2 mo. 20 da. 74—\$180.97.
- 329 75—\$112.35+. 76—\$2,100.06. 77—\$122.90. 78—\$616.50. 79—\$588.60. 80—\$6,181.95. 81—W's amount \$8.31 in excess of R's. 82—\$10,707.82. 83—16 yr.; and \$18.61, last payment. 84—\$6,930.21. 85—\$1,429.90.
- 330 86—\$954.99. 87—\$3,959.70. 88—\$2,250. 89—\$527.
- 336 1—24 ft. 2—60 ch. 3—16 ft. 4—24 rd.
- 337 5—42.638+ in. 6—249.8+ rd. 7—36 ft. 7.82+ in. 8—The corner is a square corner.

Page

- 338 9—1 A. 95 sq. rd. 10—29 sq. rd. 21 sq. yd. 7 sq. ft. 108 sq. in. 11—1 sq. yd. 2 sq. ft. 25½ sq. in. 12—1,780 sq. mi. 240 A. 13—6 A. 80 sq. rd. 19 sq. yd. 8 sq. ft. 84.56 sq. in. 14—17 A. 80 sq. rd. 27 sq. yd. 2 sq. ft. 8.6 sq. in. 15—10 sq. rd. 22 sq. yd. 5 sq. ft. 72 sq. in. 16—20 sq. mi. 48 A. 17—96 sq. in. 18—11 sq. ft. 182 sq. in. 19—64 A. 35 sq. rd. 20—344 sq. rd. 24 sq. yd. 8 sq. ft. 118.04 sq. in. 21—83 sq. rd. 7 sq. yd. 6 sq. ft. 43.2 sq. in. 22—26 ft. 23—229 rd. 3 yd. 2 ft. 9½ in. 24—11 ft. 1¼ in. 25—442 rd. 1 yd. 2 ft. 5¼ in. 26—17 rd. 2 yd. 5¼ in.
- 339 27—5 sq. rd. 10 sq. yd. 8 sq. ft. 108 sq. in. 28—6 sq. ft. 124 sq. in. 29—25 sq. yd. 6 sq. ft. 36 sq. in. 30—2 A. 159 sq. rd. 22 sq. yd. 4 sq. ft. 79.776 sq. in. 31—35 A. 143 sq. rd. 28 sq. yd. 8 sq. ft. 180.5 sq. in. 32—35 sq. rd. 4 sq. yd. 7 sq. ft. 45 sq. in. 33—1 sq. rd. 8 sq. yd. 5 sq. ft. 128 + sq. in. 34—3 A. 25.0805 + sq. rd. 35—1 sq. rd. 13 sq. yd. 5 sq. ft. 14.83 + sq. in. 36—9 rd. 37—80 yd. 1 ft. 4½ in. 38—14 ch. 27.8 + l. 39—17 ft. 8½ in. 40—5½ in. 41—12 ch. 82.11½ l. 42—79 rd. 8 ft. 10½ in. 43—27 ch. 61¼ l.
- 340 44—25 bu. 3 pk. 3 qt. 1¼ pt. 45—11 ch. 67.7 + l. 46—31 ch. 87½ l.
- 341 47—4 sq. ft. 114 sq. in. 48—1 sq. rd. 6 sq. yd. 7 sq. ft. 108 sq. in. 49—22 sq. mi. 2 A. 80 sq. rd. 50—276 sq. yd. 5 sq. ft. 135.9 sq. in. 51—789 sq. rd. 24 sq. yd. 6 sq. ft. 108 sq. in. 52—2 sq. rd. 15 sq. yd. 6 sq. ft. 24 sq. in. 53—1 sq. rd. 20 sq. yd. 7 sq. ft. 1.98 sq. in. 54—25 A. 42 sq. rd. 55—22 rd. 56—14½ sq. ft. 57—7½ in. 58—\$17.72.
- 342 59—12 A. 125 sq. rd. 80 sq. yd. 108 sq. in. 60—16 A. 118 sq. rd. 18 sq. yd. 1.7 + sq. ft.
- 344 61—62 ft. 10 + in. 62—7 ft. 10.2 + in. 63—24.08234 + rd. 64—27.46 ft. 65—15.91545 + mi. 66—137 ch. 47.6416 l. 67—170 rd. 1 yd. 1 ft. 5 + in. 68—84 yd. 9.448 in.
- 345 69—5 ft. ¼ in. 70—11 ft. 6½ in. 71—14 in. 72—2,639 ft. 11.89 + in.
- 346 73—3 sq. ft. 20.89 sq. in. 74—50 A. 148 sq. rd. 80 + sq. yd. 75—3 A. 50 sq. rd. 28 sq. yd. 1 sq. ft. 48.4 + sq. in. 76—20 sq. yd. 4 sq. ft. 94.5 + sq. in. 77—6 sq. yd. 2 sq. ft. 136 + sq. in. 78—91 A. 29 sq. yd. 14.2 + sq. in. 79—1 sq. mi. 199 A. 98 sq. rd. 19 sq. yd. 92.7 + sq. in. 80—1 sq. rd. 4 sq. yd. 5 sq. ft. 134 + sq. in. 81—8 sq. ft. 79.63 + sq. in. 82—50 sq. rd. 8 sq. yd. 44.58 + sq. in. 83—1 sq. rd. 4 sq. yd. 5 sq. ft. 132 sq. in.
- 347 84—1.128877 + mi. 85—6.076 + in. 86—57.3477 + rd. 87—21.5662 + yd. 88—51 ft. 8.8 + in. 89—3 ft. 6.5 + in. 90—219.912 sq. in. 91—169.6464 sq. rd. 92—18.037 + in. 93—4 ft. 10.56 + in. 94—7.91 + rd., minor axis; 18.8425 + rd., major axis.
- 348 95—1 ft. 11.83 + in. 96—7 ft. 9 + in. 97—8.4 + in.

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349	98—27,489 sq. ft. 99—854.5153 sq. rd. 100—9.2566 + A. 101—12 A. 26.78669 sq. rd.
350	102—\$255.47. 103—38 ft. 8 $\frac{10}{107}$ in. 104—\$171.95. 105—15 $\frac{1}{4}$ ft. 106—27.349 + ft. 108—17 ft. 109—19 $\frac{1}{4}$ in. 110—1,097.37 + sq. ft. 111—\$28.44.
351	112—14.42 + ft. 113—1.414 + mi. 114—2 A. 54.7 sq. rd.
352	5—80 A. 6—120 A. 7—200 A.
353	1—65 sq. ft. 120 sq. in. 2—425 sq. ft. 6 sq. in. 3—149 sq. ft. 11.98 + sq. in. 4—7 in. 5—1 sq. ft. 56.78 + sq. in.
354	6—79 $\frac{1}{2}$ cu. ft. 7—\$41.60. 8—\$12.88. 9—\$190.40. 10—1,436 $\frac{1}{2}$ gal. 11—3 ft. 7 $\frac{1}{4}$ in. 12—8.5 + in. 13—10.7 + in.
355	14—\$2.08. 15—407.3 + ft. 16—738.94 + sq. in. 17—527.84 + sq. ft. 18—3 $\frac{1}{4}$ sq. ft. 19—16.76 + ft.
356	21—3.18 + qt. 22—93.47 + bar. 23—935 + gal. 24—3 ft. 6 in.
358	25—89 sq. ft. 6.84 + sq. in. 26—121.5 sq. ft. 27—\$180.005. 28—17 $\frac{1}{4}$ bu. 29—108.9 + cu. in. 30—142.33 + cu. ft. 31—190.038 + bu.
360	32—6.943 + sq. yd. 33—7,083.287 + lb.
361	34—76.377 + bar. 35—No. 36—13.3 + liquid qts. 37—125 $\frac{1}{4}$ cu. ft. 38—129.248 + cu. ft.
362	39—846.68 + sq. ft. 40—2,816.61584 cu. ft. 41—998.2 sq. in. 42—2,962.032 + cu. in. 43—6 ft. 9 + in. 44—96 sq. ft. 45—25 board ft. 46—\$80.64.
363	47—\$16.31. 48—3 $\frac{1}{4}$ cd. 49—62,640 lb. 50—200 $\frac{1}{4}$ board ft. 51—250 shingles. 52—4,066 $\frac{1}{2}$ lb. 53—5.5 + in. 54—96,923 bricks. 56—8.1 + in. 57—162 + sq. in. 58—7,375.6 + sq. ft. 59—48,000 cu. ft. 60—92.19 + ft. 61—94.339 + ft. 62—3,214 $\frac{1}{2}$ gross tons.
364	63—2,203 $\frac{1}{4}$ sq. ft. 64—2,203 $\frac{1}{4}$ sq. ft. 65—\$31.17. 66—14,985 + shingles. 67—1,625 board ft. 68—\$119.48. 69—28 $\frac{1}{4}$ rolls. 70—\$7.80. 71—42 $\frac{1}{2}$ perch.
383	1—53.8868 rd. 2—26.4552 lb. 3—9.8424 yd. 4—60.155 pk. 5—12.4274 mi. 6—20.8656 ^{Ks} 7—4.7574 ^{HL} 8—3.6853885 ^L 9—71.1115 $\frac{1}{4}$ ^{Ha} 10—7.874 ^{dm} 11—148.0556 ^{Km} 12—500.179632 yd. 13—62.3963 $\frac{1}{4}$ ^{Ks} 14—2.4763 $\frac{1}{2}$ ^{Ks} 15—46.6599 $\frac{1}{2}$ ^{HL} 16—245.5322 ^m 17—148.6427 ^{Km} 18—238.872 $\frac{1}{4}$ ^{stern} 19—34.1333 $\frac{1}{4}$ $\frac{1}{4}$ ^{Ha} 20—23.70356 $\frac{1}{4}$ ^{HL} 21—241,395. 22—\$2,257.14. 23—\$562.28. 24—12 suits, and 6,6192 ^m remaining. 25—29 ^{HL} 6 ^L 8 ^d 4 ^{ml} 26—39 sq. Km. 78 ^{Ha} 52 ^a 22 sq. m. 73 sq. dm. 46 sq. cm. 75 sq. mm.

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389 1-2,970. 2- $\frac{11}{11}$. 3-1,568. 4-60. 5- $\frac{11}{11}$. 6- $\frac{11}{11}$. 7- $\frac{11}{11}$.
8- $\frac{11}{11}$. 9- $\frac{11}{11}$. 10- $\frac{11}{11}$.

391 1- $\frac{11}{11}$, $\frac{11}{11}$, $\frac{11}{11}$, $\frac{11}{11}$, $\frac{11}{11}$, $\frac{11}{11}$, $\frac{11}{11}$. 2- $\frac{11}{11}$, $\frac{11}{11}$, $\frac{11}{11}$, $\frac{11}{11}$, $\frac{11}{11}$, $\frac{11}{11}$, $\frac{11}{11}$. 3- $\frac{11}{11}$, $\frac{11}{11}$, $\frac{11}{11}$, $\frac{11}{11}$, $\frac{11}{11}$, $\frac{11}{11}$, $\frac{11}{11}$.
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7- $\frac{10}{17}$, $\frac{9}{16}$, $\frac{9}{15}$. 8- $\frac{19}{20}$, $\frac{18}{19}$, $\frac{17}{18}$. 9- $\frac{4}{19}$, $\frac{4}{18}$, $\frac{4}{17}$. 10- $\frac{10}{32}$, $\frac{10}{33}$, $\frac{10}{34}$.
11- $\frac{24}{49}$, $\frac{25}{50}$, $\frac{25}{51}$. 12- $\frac{5}{21}$, $\frac{5}{22}$, $\frac{5}{23}$.

393 1st: 1-4. 2-3. 5-5. 8-4. 16-5.

2d: 14-101. 17-33. 18-463.

3d: 3-2, 17. 4-2, 7. 6-1, 7. 7-1, 189. 9-1, 97. 10-1, 3. 11-1, 41.
12-2, 9. 13-4, 3. 15-3, 161.

395 1- $\frac{1}{1}$. 2- $\frac{1}{1}$. 3- $\frac{1}{1}$. 4- $\frac{1}{1}$. 5- $\frac{1}{1}$. 6- $\frac{1}{1}$. 7- $\frac{1}{1}$. 8- $\frac{1}{1}$. 9- $\frac{1}{1}$.
10-1. 11-1. 12- $\frac{1}{1}$. 13- $\frac{1}{1}$. 14- $\frac{1}{1}$. 15- $\frac{1}{1}$. 16- $\frac{1}{1}$.
17- $\frac{1}{1}$. 18- $\frac{1}{1}$. 19- $\frac{1}{1}$. 20- $\frac{1}{1}$.

397 1-8.021712530. 2-211.4197631. 3-4.18649931659832. 4-2.91251433.
5-58.3244608. 6-88.835890. 7-1.996417. 8-35.8412. 9-.01078997088898.
10-.561. 11-17 $\frac{1}{1}$. 12-28 $\frac{1}{1}$. 13-16 $\frac{1}{1}$. 14- $\frac{1}{1}$. 15-1 $\frac{1}{1}$.
1-428 T. 2-35 T. 3-924 T. 4-2585 T. 5-7.6 T. 6-14.256 T.
7-2.7045 T. 8-1.8525 T.

398 1-£.85. 2-£.488 $\frac{1}{2}$. 3-£.670 $\frac{1}{2}$. 4-£.575. 5-£2.1875. 6-£17.591 $\frac{1}{2}$.
7-£204.525. 8-£193.054 $\frac{1}{2}$.

1-96,597. 2-8,094. 3-239,430. 4-233,818. 5-238,728. 6-91756.

399 1-5,624. 2-3,016. 3-609. 4-9,021. 5-4,225. 6-221. 7-2,009.
8-7,216.

1-9,373. 2-10,176. 3-998,980. 4-994,976. 5-99,989,958. 6-9,737.

400 1-10,918. 2-11,445. 3-1,011,023. 4-100,110,018. 5-9,016. 6-988,035.
7-288. 8-9,021.

1-233 $\frac{1}{1}$. 2-39 $\frac{1}{1}$. 3-116 $\frac{1}{1}$. 4-124 $\frac{1}{1}$. 5-165 $\frac{1}{1}$. 6-97 $\frac{1}{1}$. 7-248 $\frac{1}{1}$.
8-163 $\frac{1}{1}$.

401 1-42 $\frac{1}{1}$. 2-159 $\frac{1}{1}$. 3-753 $\frac{1}{1}$. 4-275 $\frac{1}{1}$. 5-108 $\frac{1}{1}$. 6-70 $\frac{1}{1}$. 7-133 $\frac{1}{1}$.
8-30 $\frac{1}{1}$.

1-80 $\frac{1}{1}$. 2-63 $\frac{1}{1}$. 3-35 $\frac{1}{1}$. 4-168 $\frac{1}{1}$. 5-675 $\frac{1}{1}$.

1-121. 2-204. 3-57. 4-408. 5-532. 6-307.

402 1-832. 2-1,725. 3-1,728. 4-4,128. 5-172,368. 6-169,950.
7-160,223. 8-514,598.

403 1-9,801. 2-99,980,001. 3-7,225. 4-3,025. 5-4,096. 6-6,724.
7-8,649. 8-182 $\frac{1}{1}$. 9-90 $\frac{1}{1}$. 10-1,369. 11-2,025. 12-420 $\frac{1}{1}$.

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 406 1-6. 2-4. 3-1. 4-3. 5-7. 6-3. 7-8. 8-6. 9-4. 10-0.
 407 11-5. 12-1.
 410 1-41 $\frac{1}{2}$. 2-6,284 $\frac{1}{2}$. 3-86 $\frac{1}{2}$. 4-88,818 $\frac{1}{2}$. 5-92 $\frac{1}{2}$. 6-41,828 $\frac{1}{2}$.
 7-3,77 $\frac{1}{2}$. 8-4,880 $\frac{1}{2}$. 9-4,828 $\frac{1}{2}$.
 411 1-.625-. 2-.875+. 3-.944-. 4-.81+.
 413 1-56.08921+. 2-986.81283+. 3-.54554+. 4-20.77223+. 5-.59068+.
 6-4.42254+. 7-.82583+. 8-14.75758+. 9-2.23606+. 10-2.22398+.
 11-.63280+. 12-2.62818+.
 414 1-19. 2-23. 3-31. 4-4. 5-1.8. 6-20.04+. 7-1.55+. 8-1.205+.
 419 9-67. 10-146. 11-739. 12-219. 13-1,482. 14-25.
 425 1-1 lb. of 6¢, 1 lb. of 7¢, and 2 lb. of 9¢. 2-31 $\frac{1}{2}$ gal. of 20¢, and 68 of 15¢.
 3-1 lb. of 24¢, 2 lb. of 30¢, 2 lb. of 33¢, and 1 lb. of 42¢. 4-2 at 4 for one cent, 15
 at 2 for one cent, and 3 at 1 for four cents. 5-3 $\frac{1}{2}$ lb. gold; $\frac{1}{2}$ lb. silver.
 6-4 $\frac{1}{2}$ cu. in. gold; 1 $\frac{1}{2}$ cu. in. silver. 7-\$985.37, gold; \$9.84, silver.
- | | | | | | | | | | | |
|----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-----------|
| | Sheep | Sheep | Sheep | Sheep | Sheep | Sheep | Sheep | Sheep | Sheep | |
| 8- | 6 | 9 | 3 | 12 | 9 | 15 | 14 | 10 | 11 | @ \$2.50; |
| | 18 | 9 | 27 | 12 | 15 | 9 | 10 | 14 | 13 | " 2.80; |
| | 20 | 30 | 10 | 14 | 17 | 11 | 12 | 16 | 15 | " 3.15; |
| | 8 | 4 | 12 | 14 | 11 | 17 | 16 | 12 | 13 | " 3.45. |
- 9-16 gal.
 431 1-\$136.85. 2-\$169.15. 3-\$225.25. 4-\$159.80. 5-\$2,696.50.
 6-\$960.25.
 432 7-72 yr. and 71 yr. 8-\$1,456.
 433 1-\$6,180.52, realized by Goodrich & Co. Ship and Freight contribute
 \$1,445.82; Goodrich & Co., \$789.48; Burke & Malone, \$430.56; Miller Bros.,
 \$563.60; Jones, Howe & Co., \$968.76;—10 $\frac{1}{2}$ %.
 434 2-Ship and Freight contribute \$4,295.66; Williams & Co., \$1,421.30; Briggs,
 Sewell & Co., \$1,941.47; Gates & Brown, \$732.63; M. L. Blair, \$1,655.74.
 439 1-\$405.52. 2-9%. 3-\$5,853. 4-\$6,386. 5-\$703.12 $\frac{1}{2}$. 6-\$682.18.
 7-\$358.02. 8-\$2,500.87 $\frac{1}{2}$. 9-\$8.47.
 440 10-\$612.46. 11-\$911.88. 12-\$2,495.45. 13-25%. 14-\$1,801 $\frac{1}{2}$.
 453 1-\$480.77. 2-\$380.475. 3-\$1,982.06. 4-\$3,805.82. 5-\$10,500.
 6-\$4,841.25. 7-\$11,137.
 454 8-\$2,359.80. 9-\$2,544.08. 10-\$1,924.07. 11-\$1,952.37. 12-\$2,184.84.
 13-\$379.28. 14-\$304.04. 15-\$1,256.08. 16-\$213,026.01. 17-\$2,578.52.
 18-\$5,231.84.
 456 1-6,227,020,800. 2-35. 3-986,409. 4-1 time in 2,432,902,008,176,640,000
 times. 5-1 time in 1,307,674,368,000 times.

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- 457 1—44.41617 + rd., and 24.67865 + rd. 2—27.809 + ch., and 5.4618 + ch. 3—39.9 + in., altitude; 26.6 + in., base. 4—118.656 + rd., length; 44.496 + rd., breadth. 5—89 yd., 52 yd., and 65 yd.
- 458 6—84.292 + ch., hypotenuse; 24.494 + ch., base. 7—84.29 + ft., perpendicular; 31.6 + ft., base. 8—42.426 + rd. 9—16.5 + rd., perpendicular; 39.6 + rd., hypotenuse. 10—16 ft., base; 30 ft., perpendicular; 34 ft., hypotenuse. 11—85 yd., base; 84 yd., perpendicular; 91 yd., hypotenuse.
- 460 12—28 ft., 44 ft., 52 ft., 56 ft., 68 ft., and 20 ft. 13—21 ch., 30½ ch., and 35 ch. 14—9½ ft., 12½ ft., and 18½ ft. 15—20 rd., 82½ rd., 67½ rd., and 77½ rd. 16—28 ch. 35 l., and 94 ch. 66½ l. 17—121 in., 319 in., and 363 in. 18—600 rd. 19—55 ch.
- 461 20—26½ rd.
- 462 21—17½ rd. 22—2 A. 4.61½ sq. ch., and 3 A. 1.1889½ sq. ch. 23—120 sq. ft. 24—448 sq. rd.
- 463 25—4 A. 7.84869 + sq. ch. 26—4.70½ A. 27—16.271 + rd. 28—3.52½ lb. 29—186.712 + ft. 30—9 rd. 7 ft. 6.5 + in.
- 465 31—2 ch. 9.1 + l. from base. 32—24.76 + rd. from base.
- 466 33—8.06 + rd., 14.937 + rd., and 21.03 + rd. from shorter parallel side. 34—24 rd. and 32 rd. from vertex of right angle.
- 467 35—26.88 + ft., altitude; 35.77 + ft., base; and 44.72 + ft., hypotenuse. 36—292.068 + sq. ft. 37—199.99 + yd. 38—10.97 +. 39—8 A. 144.56 + sq. rd.
- 468 40—39.44 + rd. and 41.64 + rd. on shortest side; and 39.837 + rd. and 56.838 + rd. on longest side from vertex of angle opposite base. 41—21.93 + rd. and 31.02 + rd. 42—9.198 + rd., 11.99 + rd., 28.95 + rd. 43—6.29 + ft. 44—56.32 + rd., 66.89 + rd., 80.97 + rd. 45—1 hr. 13 min. 52½ sec. 46—5 A. 7.28 + sq. ch. and 6 A. 8.552 sq. ch. 47—6.4 rd. on each parallel side.
- 469 48—24 ft. thick, 86 ft. wide, and 96 ft. long. 49—25 ft. long, 10 ft. wide, and 15 ft. deep. 50—18 ft. 7½ in. 51—19 ft. 5½ in.
- 471 52—196 lb. 4½ oz. 53—1 ft. 6.9 + in. 54—½. 55—½. 56—13.5 - in., diameter; 9.7 - in., height.
- 472 57—6.7 + in. 58—6 ft. and 12 ft. from base.
- 474 59—28 ft. 4.68 + in., 42 ft. 7.02 + in., and 70 ft. 11.7 in. 60—30 + ft. 61—4.8 + in. 62—5.678½ lb. 63—10 ft. 9 + in. from large end. 64—8 ft. 9.7 + in. and 18 ft. 8.1 + in. from large end. 65—5 ft. 8.5 + in. 66—11.3 + ft. 67—27 + in. 68—64:343:729.

